nature portfolio

Peer Review File

Localized Dissipation in Linear Moiré Heat Transport

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This file contains all reviewer reports in order by version, followed by all author rebuttals in order by version.

Version 0:

Reviewer comments:

Reviewer #1

(Remarks to the Author)
The manuscript studies the moiré properties of a bilayer conductive system and poses the question of whether nonlinearity or advection, which was considered in previous works in the literature, is really needed. The goal of the paper is to utilize purely linear and static conductive systems to explore flat-band-related phenomena.
Each "unit cell" in a monolayer of epoxy is composed of a group of 4 alternating regions with different thermal conductivities. Two such staggered square lattice monolayers are then put in contact with each other and with a heat source. Depending on the commensurability or noncommensurability of the twist angle between the layers and the symmetry axes of the square lattices, different unit cells are visible in the temperature profiles. These patterns are interpreted in terms of localized and delocalized heat flow, associated with flat-band physics.

My main concern is that the manuscript applies the same conceptual approach previously reported by the authors in ref. [19], but now in a static conductive medium. While the platform changed, the observed phenomena (localization-to-delocalization), control mechanism (twist angle being Pythagorean or not), and interpretation (flat band in moiré) are the same as the prior work. Also, the manuscript justifies the use of linear systems as a key novelty, contrasting with nonlinear approaches. However, this question about the possibility of flat band engineering avoiding nonlinearities is somewhat largely confirming the expectations based on prior theoretical work on moiré in linear systems in condensed matter analogues (e.g., Ref. [15]; Huang et al, Scientific Reports 6, 32546 (2016); Kariyado & Vishwanath, Phys. Rev. Research 1, 033076 (2019); Gao et al., Phys. Rev. A 108, 013513 (2023)).

Moreover, the manuscript suggests (in Fig. 1ab and related discussion) that the presence of nonlinearity leads to "easier" flat-band formation compared to linear systems, giving the 1D SSH model as an example. This is misleading because numerous linear systems, such as Lieb/kagome/dice lattices, host exact or quasi-flat bands via lattice geometry and interference (see e.g. Leykam et al., Advances in Physics: X 3, 1473052 (2018)). Flat bands are also deeply connected to Wannier orbital localization and the quantum metric, well-defined in linear models. Notably, the presence of nonlinearities often destroys flat band localization phenomena (see e.g. Törmä et al., Nature Review Physics 4, 528 (2022)).

Given these points, I don't think there is a particular motivation for focusing on linear moiré systems as an exception rather than the norm. Linear flat-band systems are extensively studied, also in moiré setups.

While this paper could be of interest to the subfield exploring thermal metamaterials and classical analogues of quantum phenomena, it is unclear what new physical insight emerges uniquely in this bilayer conductive system with spatially engineered diffusivity, which cannot be anticipated from previous studies, including that of the same authors. Without a more compelling case for novelty or general impact, I do not recommend publication in Nature Communications.

Reviewer #2

(Remarks to the Author)
In the present work, the authors create both commensurate and incommensurate moiré patterns, each governed by two distinct modulated wavevectors controlling global periodicity and local unit-cell structure. They show that aperiodic thermal localization emerges in incommensurate quasicrystals with the transition threshold linked to the emergent lattice constant. More specifically, they report diffusive moiré superlattices and reveal the moiré flat bands for temperature field transition between localization and delocalization in purely dissipative conduction. Furthermore, they demonstrate that modulating the twisted angles results in experimental visualizations exhibiting moiré patterns due to inhomogeneous temperature

distributions, which arise from the newly emerging periodicity at corresponding twisted angles.

In conclusion, the authors demonstrate a general strategy for controlling diffusive conduction within the interfacial superlattice by modulating the twist angles and excited energy of the incident temperature wave. The critical energy input for inducing localization-to-delocalization transitions depends on the lattice constant at corresponding non-Pythagorean angles.

The results provide a pathway for geometrically programmable dissipation and non-equilibrium control in heat and mass transport. The manuscript contains essential findings for the field. However, before making a final decision, I have several questions and suggestions for the authors.

1. In Fig. 3g, it would be helpful to include an error bar.

2. Similarly, in Figures 4h and 4i, adding error bars would enhance the clarity of the results.

3. The authors mention that “Significant degeneracies occur with non-flat bands in the commensurable superlattice at 36.87°and 22.62°, while flat bands appear in the incommensurable superlattice at 30°.” Could the authors elaborate on the significance of the degeneracies in the delocalization of the commensurable superlattice?

4. In Fig. 4i, localization in the incommensurable superlattices is attributed to the presence of flat bands. However, below a certain threshold, the incommensurable superlattices are delocalized despite the existence of flat bands. What phenomenon accounts for this delocalization?

5. Fig. 4h presents a threshold that does not seem to be related to localization or delocalization. What does this threshold signify, and what is its physical origin?

Reviewer #3

(Remarks to the Author)
In this paper the authors exploit the spatially inhomogeneous thermal conductivity of a bilayer material to achieve Moiré physics in heat transport. The article is well written and scientifically sound. The data are well presented and the logic of the paper is easy to follow. Experiments show a clear signal that matches the theory.

My main perplexity is about the central claim of the paper and thus about the real novelty of the presented results. In the introduction the authors claim that the crucial novelty of their work is the ability to observe Moiré physics in linear systems. In the introduction I do not see any reference related to other works in this sense. Are they the first to rise such a question? Has this question been previously addressed by the community? I feel that a more comprensive introduction is needed in this sense.

Along this line. I feel that from a theoretical perspective is not very clear what is linear and what is not, here. First of all, it is not clear from the paper what is the model considered. The equation of motions are linear in what? The situation is further misleading when talking about dissipative systems. Dissipative phenomena in many cases lead to nonlinear equation of motion (in a mean-field sense). It is thus of fundamental importance here to clarify here these concepts.

Version 1:

Reviewer comments:

Reviewer #1

(Remarks to the Author)
I appreciate the authors’ detailed responses to my concerns and the other Reviewers.
The discussion around Fig.1 is now fairer to acknowledge the class of flat-band systems that rely on interference effects, providing an intuitive picture of an alternative mechanism for band flattening when geometric frustration is not possible.
The clarifications on what is meant by “linear” in this work are also helpful. I now better understand the distinction the authors draw between prior wave-based systems, relying on interference and nonlinear interactions, and their present system, where the governing diffusion equation is strictly linear and momentum-free. The authors also addressed my concern about this work being incremental to their previous one, as the latter required nonlinear fluid dynamics that is missing here.

Overall, despite these clarifications address my main concerns, I still feel the work is an extension of moiré flat bands into diffusive systems, rather than a new paradigm for flat bands. I suggest the authors adjust the manuscript to moderate claims (e.g. last sentence of the abstract, and line 78) focusing on the specific novelty of realizing moiré physics in a linear, momentum-free diffusive system. With this revision, I would find the manuscript suitable for publication.

Reviewer #2

(Remarks to the Author)
The authors' answers to my questions satisfactorily clarify some aspects of the work. They improved the work by addressing

the reviewers' criticisms and, in my opinion, clearly highlighted the manuscript's innovation and relevance, which was the main point raised by both other reviewers. With all the explanations and improvements developed by the authors, I believe the work has all the necessary features to be accepted in a high-level journal. Thus, I recommend this version of the manuscript for publication in Nature Communications.

Reviewer #3

(Remarks to the Author)
The authors have provided exhaustive and satisfactory responses to all the comments raised in the first report. I therefore recommend the publication of the paper in its present form.

Response to the Reviewers' Comments on the manuscript [NCOMMS-25-58105] entitled " Localized Dissipation in Linear Moiré Heat Transport " submitted to *Nature Communications*

We would like to thank the editors and reviewers for their careful reviewing of our work and their relevant comments. These comments are quite valuable, which have allowed us to further improve the manuscript from the aspects of descriptions. Considering these comments, we have further clarified the novelties and core findings of revealing moiré physics in purely linear diffusion without real-space momentum. The changes in the revised manuscript are marked in red. In the following, we provide a point-to-point response to all reviewers' questions and comments. We do hope that the editors and reviewers may appreciate our extended efforts to improve the paper following their suggestions, and we also hope our efforts and clarifications may convince the reviewers of the significant advances and the importance of our current work in exhibiting the moiré physics and the corresponding properties in the momentum-free linear diffusion, which are beyond the scope of conventional territory in solid state physics and further broaden the understanding of moiré physics in the dissipation of physical quantities.

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Reviewer 1:

The manuscript studies the moiré properties of a bilayer conductive system and poses the question of whether nonlinearity or advection, which was considered in previous works in the literature, is really needed. The goal of the paper is to utilize purely linear and static conductive systems to explore flat-band-related phenomena.

Response: We thank the reviewer for the summary of our work. We appreciate the reviewer's attention to the role of nonlinearity or advection in realizing moiré properties. As indicated by the reviewer, the main strategy of our current work is to utilize purely linear and static conductive systems to explore flat-band-related phenomena in thermal diffusion. That is, the targets of this work are not to discuss the advantages of nonlinearity or linearity in creating the moiré flat bands and related behaviors. Instead, we have revealed additional phenomena that were not observed in previous works, including the dual modulated wavevectors controlling localization, the asymmetric commensurate and incommensurate regimes centered on $\pi/4$, correspondence between spatial and conductive components in linear moiré systems, as well as the moiré behaviors in linear and momentum-free thermal system. In the following responses, we would like to provide detailed illustrations on the related concerns.

Each "unit cell" in a monolayer of epoxy is composed of a group of 4 alternating regions with different thermal conductivities. Two such staggered square lattice monolayers are then put in contact with each other and with a heat source. Depending on the commensurability or noncommensurability of the twist angle between the layers and the symmetry axes of the square lattices, different unit cells are visible in the temperature profiles. These patterns are interpreted in terms of localized and delocalized heat flow, associated with flat-band physics.

Response: We are thankful to the reviewer for providing the above summary of our current work.

My main concern is that the manuscript applies the same conceptual approach previously reported by the authors in ref. [19], but now in a static conductive medium. While the platform changed, the observed phenomena (localization-to-delocalization), control mechanism (twist angle being Pythagorean or not), and interpretation (flat band in moiré) are the same as the prior work.

Response: We respectfully thank the reviewer for this important point. While our previous study (Ref. [19]) indeed introduced the concept of hydrodynamic moiré superlattices, the present work is fundamentally different in both the physical mechanism and scientific implications. Below, we provide some typical evidence from four aspects:

1. The linearity and nonlinearity of the physical insane

For our previous works about hydrodynamic moiré superlattice, the system relied on nonlinear advection and flow dynamics to generate moiré localization, requiring active fluid motion and external pumping. That is, **the moiré behaviors are created based on the nonlinear nature of the fluid dynamic**, which is enabled by the advective terms in momentum equation.

In stark contrast, the current work demonstrates moiré-induced localization in a purely linear and static conduction medium, where **transport is governed only by Laplacian diffusion without any advective or nonlinear terms and exhibits as decay**. For instance, the pure conductive system does not possess momentum-based transport, which is also quite different from the cases in scalar wave dynamics with real-space propagation. In this case, our current work not only possesses different regimes (linearity or nonlinearity), but also reveals the moiré behaviors in momentum-free systems without real-space field propagations like the ones in classical wave dynamics.

2. Mechanistic Distinction

For our previous works about hydrodynamic moiré superlattice, the localization emerges from velocity field interference induced by nonlinear vorticity transport.

In our current work, localization is achieved via **engineered spatial conductivity gradients** that act as effective “momentum surrogates”, enabling scalar interactions in an otherwise momentum-free diffusive medium. This represents a qualitatively different mechanism for generating moiré patterns, i.e., generalized potential fields arising from anisotropic, scalar, and linear conduction, not fluid vorticity in our previous work (Ref. 19) or strong correlation/interaction in condensed matters (*Nat. Rev. Mater.* 6, 201-206, 2021) with nonlinearity in interactions and dynamical properties. **The general correspondence between spatial and conductive components in linear moiré systems is also revealed**.

3. Control Parameters

While both works explore the commensurate/incommensurate distinction (Pythagorean vs. non-Pythagorean angles), the present system reveals **new phenomena specific to conduction beyond the critical linear regime**, such as:

- **The dual modulated wavevectors controlling localization**. The key advancement lies in uncovering two distinct moiré wavevectors in our system, including one global wavevector that governs the entire periodicity of the emergent moiré superlattice, and a modulated wavevector that determines the local unit-cell structure within each superlattice. This dual modulation directly governs the propagation and localization behavior of thermal fields, and provides a powerful framework to classify the asymmetric commensurate and incommensurate regimes centered on $\pi/4$. Note that, such asymmetrical properties (Figs. 1e, 2e, 2f and 4g of the main content and Fig. S2 of the Supplementary Materials) around have not been revealed in previous analogues of photonics and electronics as well as our hydrodynamic work.

- **The dependence of localization thresholds on the moiré lattice constant in diffusion-only media.** We experimentally identify an input energy threshold for triggering thermal localization-delocalization transitions, dependent on the quasi-crystalline symmetry of the twist angle. This provides a geometric handle to program dissipation landscapes, creating a framework for controlling non-equilibrium heat and mass transport in purely linear systems without advections.

- The **Fourier number scaling** that quantifies conduction-storage competition (Fig. 2f), which has no analogue in the hydrodynamic case. Besides, the direct experimental realization using **solid epoxy-based bilayers**, which contrasts sharply with flow-driven systems. Conceptual Advancement

4. Conceptual difference

The present study establishes a new paradigm of moiré physics in static linear and dissipative systems, where field couplings arise without relying on nonlinearity in hydrodynamics (particle interactions in condensed matter). This significantly broadens the scope of moiré science from momentum-carrying media to passive diffusive transport, opening pathways toward programmable dissipation in solid-state thermal, chemical, and biological systems.

Therefore, the **governing physics, implementation, and implications** differ fundamentally. The current manuscript should thus be viewed not as a mere extension, but as the **first demonstration of moiré physics and flat-band localization in a purely static and linear conduction regime**.

Also, the manuscript justifies the use of linear systems as a key novelty, contrasting with nonlinear approaches. However, this question about the possibility of flat band engineering avoiding nonlinearities is somewhat largely confirming the expectations based on prior theoretical work on moiré in linear systems in condensed matter analogues (e.g., Ref. [15]; Huang et al, Scientific Reports 6, 32546 (2016); Kariyado & Vishwanath, Phys. Rev. Research 1, 033076 (2019); Gao et al., Phys. Rev. A 108, 013513 (2023)).

Response: We sincerely thank the reviewer for pointing out these theoretical works in condensed matter analogues. At the very beginning, we should emphasize that we have not denied the efforts of linearity in discovering moiré behaviors in the context of electronic and photonic lattice with tight-binding approximation through non-interacting assumption. In these systems, effective linear interactions employed in these systems induce moiré-like behaviors via their **average potentials following mean-field theory in electronic interactions and critical wave coherence in wave dynamics**. That is, **fine-tuned conditions for approximating linear system or real-space vectors** are required in enabling the emergence of associated localized modes and effective bands with effective linearity. However, our work differs in several fundamental aspects as below:

1. Fundamental difference between diffusive and wave-based linear systems:

The works pointed out the reviewer primarily concern **photonics** (Ref. [15]; Huang et al, Scientific Reports 6, 32546 (2016); Gao et al., Phys. Rev. A 108, 013513 (2023)) and **electronic interactions** (Phys. Rev. Research 1, 033076 (2019)), where flat bands arise from lattice interference within momentum-carrying frameworks. Furthermore, we must emphasize that the corresponding behaviors observed in these theoretical works are also realized via the nonlinear essences of the considered systems.

In the above photonic analogues (Ref. [15]; Huang et al, Scientific Reports 6, 32546 (2016); Gao et al., Phys. Rev. A 108, 013513 (2023)), **the entire systems are established based on nonlinear Schrödinger equation (** $i\frac{\partial\psi}{\partial z} = -\frac{1}{2}\nabla^2\psi + \frac{E_0}{1+I(\mathbf{r})}$ **), which can be directly evidenced by their governing functions** (see Eq. (1) in Ref. [15]; Eq. (2) in *Sci. Rep.* 6, 32546 (2016); and Eq. (1) in *Phys. Rev. A* 108, 013513 (2023)). The utilizations of effective Hamiltonians for describing the band models are created via approximating the nonlinear couplings to their corresponding spatial-dependent averages, while their real interactions for supporting the systems are nonlinear. **Note that, the term** $I(\mathbf{r}) = |p_1V(\mathbf{r}) + p_2V(S\mathbf{r})|^2$ **in these governing functions is a typical nonlinear term denoting the optical intensity**, and details of the nonlinear interactions and potential soliton governed by such function can be found in the pioneering work of *Phys. Rev. E* 66, 046602 (2002). Besides, some typical nonlinear behaviors have been discussed in these works, such as the Kerr-nonlinearity in *Sci. Rep.* 6, 32546 (2016).

For the electronic analogue (Phys. Rev. Research 1, 033076 (2019)), while the formalism in Flat band in twisted bilayer Bravais lattices is linear (the Hamiltonian is bilinear in fermion operators), the underlying physics it represents is **inherently nonlinear**. The interlayer coupling potential $V(\mathbf{r})$ arises from orbital overlap and interference effects that are themselves outcomes of complex many-body couplings. In Phys. Rev. Research 1, 033076 (2019), this is replaced by an **effective averaged field** $|V(\mathbf{r})|$, so that the nonlinear interactions between electrons and orbitals are encoded only as a fixed potential landscape in the form of single-particle Hamiltonian with non-interacting behaviors. In other words, the model should be understood as a **mean-field approximation** of a system that is naturally nonlinear at the microscopic level. This is why the authors themselves emphasize at the end that:

"*In the presence of interactions, this can provide a promising route to realizing a range of models from quasi-one dimensional coupled wires to square lattice Hubbard models with staggered orbitals, or the Kugel-Khomskii spin orbital models, with a high degree of tunability and control*". Thus, the theory is linear in form of non-interacting model, but it captures only a simplified slice of a fundamentally nonlinear interacting problem.

In contrast, our system operates in a **purely diffusive regime** with **no intrinsic momentum or wave dynamics**. That is, conduction is governed strictly by Laplacian diffusion with purely linear interactions. Demonstrating moiré flat bands without underlying wavelike dispersion and real-space propagation is not a straightforward extrapolation of those analogues.

2. Unexplored question in pure diffusion:

While the **implementing strategy of mean-field approximation** suggested linear moiré flat bands in condensed matter theories, it was **unclear whether diffusion-only system without momentum**, lacking interference in the usual sense and nonlinear interactions between particles in electronics, could support such phenomena.

Our results (Figs. 2 ~ 4) directly answer this question experimentally: moiré localization emerges in **static, linear, dissipative conduction** where the "interference" originates from engineered conductivity gradients rather than wave superposition and interacting particles.

3. New mechanism:

Instead of tight-binding Hamiltonians with hopping parameters via **mean-field approximation**, our mechanism is based on **engineered spatial conductivity fields** that mimic generalized velocities in a momentum-free environment. This enables a **new class of moiré physics** in momentum-free regime distinct from wave-based analogues, with unique features such as direct linear interactions through conductive couplings without approximations and Fourier-number scaling of localization (Fig. 2f).

4. Experimental realization in dissipative conduction:

To our knowledge, this is the **first experimental demonstration of moiré flat bands in a purely diffusive medium**. While theoretical works anticipated linear moiré flat bands in condensed matter under the assumption of non-interacting particles and **mean-field approximation**, no prior study has realized them in **passive solid-state conduction**, where neither momentum, interference, nor nonlinearity is present.

In the cases of the above aspects, our work should not be viewed as a mere confirmation of prior linear theories under the assumption of non-interacting particles and **mean-field approximation**, but as a **conceptual leap from wave-based analogues to dissipative diffusion**. It demonstrates that moiré flat-band physics extends even to systems where interference is not expected, thereby opening a new paradigm for programmable dissipation in thermal and chemical transport.

Moreover, the manuscript suggests (in Fig. 1ab and related discussion) that the presence of nonlinearity leads to "easier" flat-band formation compared to linear systems, giving the 1D SSH model as an example. This is misleading because numerous linear systems, such as Lieb/kagome/dice lattices, host exact or quasi-flat bands via lattice geometry and interference (see e.g. Leykam et al., Advances in Physics: X 3, 1473052 (2018)). Flat bands are also deeply connected to Wannier orbital localization and the quantum metric, well-defined in linear models. Notably, the presence of nonlinearities often destroys flat band localization phenomena (see e.g. Törmä et al., Nature Review Physics 4, 528 (2022)).

Response: We thank the reviewer for providing this comment and we agree that flat bands could emerge in linear systems particularly possessing **geometrically frustrated lattices** such as Lieb, Kagome, and Dice lattices, where **destructive interference enforces band flattening**. Note that, the flat bands in Lieb, Kagome, and Dice lattices originate from different mechanisms (interference) rather the moiré physics considered in our current work.

We should emphasize our intention in Figs. 1a and b was not to claim that nonlinearity is necessary for flat-band formation, but rather to provide an intuitive contrast between the possibility of flattening bands in a well-established model without geometric frustration when nonlinear couplings are introduced. Thus, the SSH example was chosen purely for pedagogical illustration and should not be interpreted as a general statement about linear systems. Our related discussion about these figures emphasize that nonlinearity can further flatten the existing bands in a well-established lattice without requiring additional site configurations or special spatial symmetries.

To clarify, in the revised manuscript we have made the following revision:

1. **Explicitly acknowledge** that some linear lattices (e.g., Lieb/kagome/dice) naturally host flat bands due to geometric interference.
2. **Reframe Fig. 1 discussion** to emphasize that our comparison is specific to the model geometric frustration, which does not host exact flat bands in its linear form, and that nonlinearity in this context can artificially flatten bands.
3. **Stress our contribution**: the novelty of our work is precisely that we achieve moiré-induced flat bands in a **linear diffusive conduction system** without invoking either geometric frustration or nonlinearity.

The related revision can be found in the Introduction and the caption of Fig. 1:

"We are aware that some toy models, including Lieb, Kagome, and Dice lattices, could also support flat bands without moiré physics due to the intrinsic geometric interference or specifical spatial symmetry. However for the cases without geometric frustrations like a SSH model (Figs. 1**a** and **b**), the introduction of nonlinearity could help further flatten the existing bands without requiring additional site configurations or special spatial symmetries (Figs. 1**a** and **b**)."

"**a** and **b** present the schematic band structures of a typical linear and nonlinear 1D SSH models without geometric frustrations possessing the same intracell and intercell hopping. Notably, the introduction of nonlinearity in **b** could artificially flatten the existing bands without additional implementations to create geometric-dependent flat bands with intrinsic geometric interference like Lieb, Kagome, and Dice lattices."

"To compensate for the absence of intrinsic nonlinear coupling and geometric frustration in lattice structure,"

For the concern on "destroying flat band localization phenomena with nonlinearity" mentioned by the reviewer, we have to expressed that this work (Törmä et al., Nature Review Physics 4, 528 (2022)) seems to present the opposite viewpoint. In their work, the authors first utilized the non-interacting model to establish the effective band to connect the Wannier orbital localization and the quantum metric. Then, they separated to discuss the role of interactions (nonlinearity). The main conclusion related to the interacting model (nonlinearity) includes:

(1) In trivial flat bands (exponentially localized Wannier orbitals), even interactions cannot enable transport.

(2) In *interference-based* flat bands (nontrivial), interactions **distort destructive interference** and actually **enable particles to move**.

(3) Bound states always exist in flat bands, and they **disperse due to interactions**, giving Cooper pairs finite effective mass when the quantum metric is nonzero.

In this case, the cited review present that (i) nonlinearity creates dispersive bound states in otherwise dispersion less bands; (ii) nonlinearity is able to unlock transport when combined with nontrivial quantum geometry; and (iii) nonlinearity allows superfluidity proportional to the Brillouin-zone integral of the quantum metric. Some evidences can be directly extracted from Törmä et al., Nature Review Physics 4, 528 (2022) as below:

- "*Non-interacting particles can be immobile either if their Wannier functions are exponentially localized at the lattice sites, or due to destructive interference. In the former case, also interacting particles will be localized, while in the latter, interactions distort the interference and the particles can move due to the finite overlap of the Wannier functions.*"

  This shows that interactions (nonlinearities) do not destroy localization, but can unlock mobility through overlapping Wannier functions.

- "*In the limit of infinite effective mass, that is in a flat band, a bound state is always present for any small value of the attractive interaction strength… the bound state can disperse as a result of interactions.*"

  This statement indicates that nonlinearities do not eliminate flat-band physics; rather, they induce dispersion.

- "*This result shows that Cooper pairs can have a finite effective mass and thus support transport if the flat band quantum metric is nonzero.*"

Here, quantum geometry ensures finite effective mass and superfluid transport via interactions, directly contradicting the claim that nonlinearities destroy localization.

Overall, Törmä et al. emphasize that nonlinear terms (interactions) such as attractive Hubbard interactions, Coulomb repulsion, and RVB-type pairing do not destroy flat band localization. Instead, they either leave trivial flat bands localized or promote mobility in nontrivial flat bands, together with trivial and nontrivial quantum geometries. Note that, the abovementioned interacting terms are inherently nonlinear since they involve self-consistency and produce nonlinear effective potentials.

Given these points, I don't think there is a particular motivation for focusing on linear moiré systems as an exception rather than the norm. Linear flat-band systems are extensively studied, also in moiré setups.

Response: We are thankful to the reviewer for providing this comment. We would like to indicate that the linear flat-band diffusion in our current work is not an extensive study of the well-established moiré setups in other territories (condensed matter, photonics, acoustics, and hydrodynamics), where interference within momentum-carrying wave models naturally produces flat bands. However, our motivation does not lie in treating "linear moiré systems" as exceptional in that broader sense. Instead, it lies in addressing previously unresolved questions, i.e., **the unexpected moiré behaviors and unique motivation governed by Laplacian conduction in purely diffusive and momentum-free systems**. In this regard, our system is fundamentally different from the linear wave-based models where flat bands have become the norm.

Overall, the **motivation of our work** is to extend this paradigm into the realm of **diffusive, and momentum-free transport**, which is fundamentally distinct from previously effective linear cases.

This establishes linear moiré flat bands not as the "norm," but as a **newly accessible regime in dissipative diffusion**, opening opportunities for programmable dissipation in thermal, chemical, and biological systems.

While this paper could be of interest to the subfield exploring thermal metamaterials and classical analogues of quantum phenomena, it is unclear what new physical insight emerges uniquely in this bilayer conductive system with spatially engineered diffusivity, which cannot be anticipated from previous studies, including that of the same authors. Without a more compelling case for novelty or general impact, I do not recommend publication in Nature Communications.

Response: We are thankful to the reviewer for providing the above comments. We have made detailed illustrations on these concerns. Instead of the previous analogues in condensed matter/photonics/hydrodynamics governed by interacting models or non-interacting models under the mean-field approximation, our current work paves the way of discovering linearly diffusive and momentum-free transport framed by Laplace's equation. To the best of our knowledge, both the asymmetric moiré localized phases and dual modulated wavevectors are revealed in the conductive system, which have not been observed in other analogues. Our current work provides a framework for general diffusive process without real-space momentums via direct linear couplings to implement moiré manipulations. The critical relevance between pure conduction and effective band theory is revealed through the dimensionless number of Fo. We do hope our extended efforts could address the reviewer's concerns.

Reviewer 2:

In the present work, the authors create both commensurate and incommensurate moiré patterns, each governed by two distinct modulated wavevectors controlling global periodicity and local unit-cell structure. They show that aperiodic thermal localization emerges in incommensurate quasicrystals with the transition threshold linked to the emergent lattice constant. More specifically, they report diffusive moiré superlattices and reveal the moiré flat bands for temperature field transition between localization and delocalization in purely dissipative conduction. Furthermore, they demonstrate that modulating the twisted angles results in experimental visualizations exhibiting moiré patterns due to inhomogeneous temperature distributions, which arise from the newly emerging periodicity at corresponding twisted angles.

Response: We are thankful to the reviewer for providing the comprehensive summary of our work, and we also appreciate the endorsement on the connection between the diffusive moiré system and the corresponding dual modulated wavevectors.

In conclusion, the authors demonstrate a general strategy for controlling diffusive conduction within the interfacial superlattice by modulating the twist angles and excited energy of the incident temperature wave. The critical energy input for inducing localization-to-delocalization transitions depends on the lattice constant at corresponding non-Pythagorean angles.

Response: We are thankful to the reviewer for the summary of the main findings. We appreciate the reviewer's recognition that the emergence of a critical energy threshold tied to the moiré lattice constant. This observation establishes the lattice constant as a new control knob for programmable dissipation in static linear systems, which we believe will inspire further exploration across thermal, chemical, and biological diffusion platforms.

The results provide a pathway for geometrically programmable dissipation and non-equilibrium control in heat and mass transport. The manuscript contains essential findings for the field. However, before making a final decision, I have several questions and suggestions for the authors.

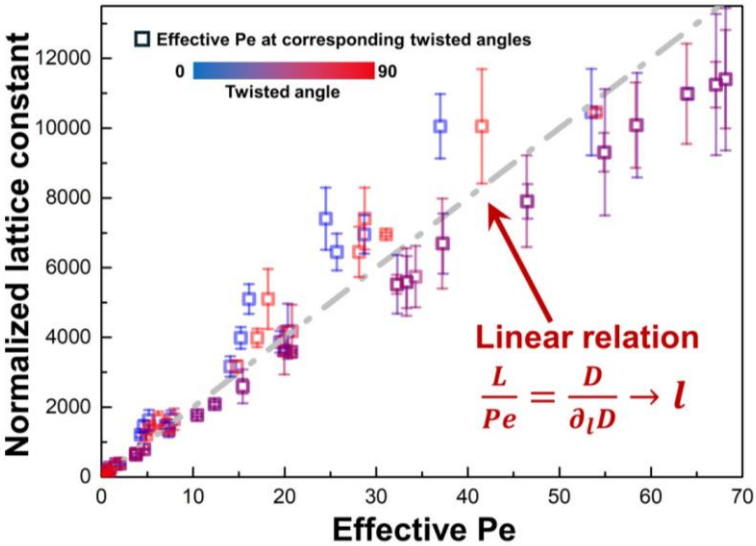
Response: We sincerely thank the reviewer for the thoughtful and constructive feedback, as well as for recognizing that our results provide a pathway toward geometrically programmable dissipation and non-equilibrium control in heat and mass transport. We are especially grateful for the reviewer's positive assessment that the manuscript contains essential findings for the field.

Below, we carefully address each of the reviewer's questions and suggestions point by point. We believe these clarifications and revisions significantly strengthen the manuscript.

1. In Fig. 3g, it would be helpful to include an error bar.

Response: We thank the reviewer for providing this suggestion. The error bar has been inserted in Fig. 3g, while the revision is as below:

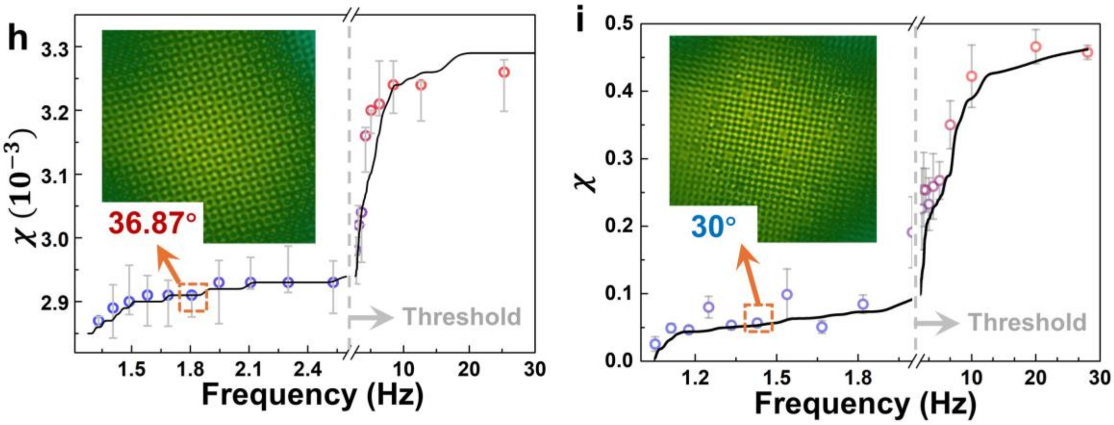
Revised Fig. 3g:



2. Similarly, in Figures 4h and 4i, adding error bars would enhance the clarity of the results.

Response: We thank the reviewer for providing this suggestion. The error bars have been inserted in Figs. 4h and 4i. The revisions are as below:

Revised Figs. 4h and 4i:



3. The authors mention that "Significant degeneracies occur with non-flat bands in the commensurable superlattice at 36.87°and 22.62°, while flat bands appear in the incommensurable superlattice at 30°." Could the authors elaborate on the significance of the degeneracies in the delocalization of the commensurable superlattice?

Response: We thank the reviewer for this insightful question. The degeneracies observed in the effective band structures of the commensurate moiré superlattices (36.87° and 22.62°) play a crucial role in facilitating delocalized transport.

First of all, these degeneracies enable multiple propagation channels for temperature waves. From Figs. 4**a** and **b**, such degeneracies correspond to multiple eigenmodes sharing the same effective frequency. In this case, these degenerate states provide additional channels for energy transport, enhancing the capacity of the superlattice to support extended temperature fields across the interface.

Secondly, these degeneracies further suppress the localized behaviors. Due to the overlaps of several non-flat bands, the effective group velocity the temperature waves remain finite over a wide range of thermal modes. Such redundancy prevents confinement to localized hot spots, leading instead to robust delocalization of the thermal field in commensurate configurations as experimentally observed in Figs. 4**d** and **e**.

In stark contrast to the commensurate cases, the degeneracies are absent and the bands flatten in the incommensurate superlattice at 30°, yielding vanishing group velocity and thus strong localization. This contrast highlights that band degeneracies in commensurate moiré systems act as conduits for transport under translational symmetry, whereas their absence in incommensurate quasicrystals enables localization.

We have clarified this point in the revised manuscript, and the related revision is as below:

In Sec. "Moiré interactions in linear heat transport":

"Specifically, these degeneracies provide multiple propagation channels for temperature transport under translational symmetry. The presence of several degenerate modes allows energy to distribute across parallel pathways, thereby reinforcing the extended character of the temperature field. Moreover, the overlaps between non-flat bands ensure that the effective group velocity remains finite over a broad range of thermal modes. This redundancy suppresses the possibility of confinement into localized hot spots and instead results in robust delocalization."

4. In Fig. 4i, localization in the incommensurable superlattices is attributed to the presence of flat bands. However, below a certain threshold, the incommensurable superlattices are delocalized despite the existence of flat bands. What phenomenon accounts for this delocalization?

Response: We thank the reviewer for providing this point. The apparent delocalization observed in incommensurate superlattices at low excitation frequencies arises from the fact that flat bands alone are not sufficient to enforce localization. That is, the excitation must also provide enough energy to effectively couple into those modes. Specifically at very low incident frequencies (below the threshold in Fig. 4i), the effective excitation wavelength of the imposed temperature wave is much larger than the moiré lattice constant. In this case, the system effectively perceives an averaged homogeneous medium, and the incident energy primarily couples into extended diffusive modes rather than the localized flat-band modes. As a result, delocalization dominates despite the formal presence of flat bands in the band structure.

It is worth noting that, only when the excitation frequency exceeds the critical value, i.e., the incident wavenumber k_z satisfies Eq. (4) and Supplementary Note 5, the incident energy resonantly couple into the flat-band modes. At this point, the group velocity suppression of flat bands manifests physically as strong localization, as observed around 30°.

Thus, the delocalization at low frequencies is not inconsistent with the existence of flat bands; rather, it reflects a mismatch between the excitation scale and the moiré lattice constant. In other words, flat bands provide the potential for localization, but sufficient excitation energy is required to access them. Below that threshold, the system behaves diffusively with extended modes.

We have revised the manuscript to clarify that localization in incommensurate moiré lattices requires both the presence of flat bands *and* sufficient excitation to activate them, which explains the low-frequency delocalization observed in Fig. 4i. The related revision is as below:

In Sec. "Moiré interactions in linear heat transport":

"In this case, the system effectively perceives an averaged homogeneous medium without localized behaviors at a non-Pythagorean angle, once the corresponding wavelength of the imposed temperature wave with low incident frequency is much larger than the moiré lattice constant (the insert of Fig. 4**i**). In this regime, the excitation primarily couples into extended diffusive modes, and delocalization dominates. Once the excitation frequency exceeds the threshold set by the moiré lattice constant, resonant coupling into flat-band modes occurs, and their vanishing group velocity manifests physically as strong localization (as shown in Fig. 4**f**). Thus, localization in incommensurate moiré lattices requires both the existence of flat bands and sufficient excitation to activate them."

5. Fig. 4h presents a threshold that does not seem to be related to localization or delocalization. What does this threshold signify, and what is its physical origin?

Response: We thank the reviewer for drawing attention to this point. The threshold observed in Fig. 4h does not represent a localization-delocalization transition (as in Fig. 4i for incommensurate lattices), but instead reflects the frequency-dependent efficiency of energy coupling into extended modes of the commensurate superlattice.

We first illustrate its physical origin of the threshold. Note that, the "threshold" is strongly connected to the effective lattice constant of the superlattice satisfying the critical wavenumber condition of Supplementary Note 5. That is, the "threshold" in all cases present the characteristic frequency of the superlattice itself under specific lattice constant. In this scenario, the "threshold" indicates the relation between the energies of superlattice at the characteristic frequency and the incident temperature wave.

Following the above aspect, the band structures remain non-flat (Figs. 4a and b) in commensurate moiré lattices (Pythagorean twist angles), and thus transport is always delocalized. However, as the input frequency of the temperature wave increases, the incident energy progressively excites higher-order modes. This decreases the mode extension degree, reflected by a monotonic increase in the integral form factor χ. In this case, the apparent "threshold" in Fig. 4h therefore marks the point where **the excitation frequency becomes comparable to the characteristic frequency associated with the moiré lattice constant**. Unlike the incommensurate case, this threshold is **not tied to a localization transition**, since localization is absent in commensurate lattices. Instead, it should be understood as a **crossover frequency for delocalization efficiency**, i.e., a measure of how strongly the external excitation couples into the extended modes supported by the periodic moiré lattice. Note that, the values of integral form factors of the commensurate cases are far lower than the ones of incommensurate one, thus only indicating the non-localized behaviors in commensurate cases.

To avoid confusion, we have revised the main text (discussion of Fig. 4h) to explicitly note that this threshold signifies the **onset of stronger mode excitation in delocalized commensurate lattices**, and not a localization–delocalization boundary. The related revision is as below:

At the end of the Sec. “Moiré interactions in linear heat transport”:

“Note that, the frequency response in Fig. 4**h** exhibits an apparent threshold. This threshold does not correspond to a localization-delocalization transition, but rather to the characteristic frequency associated with the moiré lattice constant (see Supplementary Note 5). Below this frequency, the excitation predominantly couples into lower-order extended modes, resulting in the mode extension with weak temperature field intensity. Once the incident frequency approaches the characteristic scale of the superlattice, higher order delocalized modes are activated, producing a monotonic increase in the integral form factor with strong intensity. The observed threshold in Fig. 4**h** should therefore be understood as a crossover point for enhanced delocalization efficiency, i.e., a measure of how strongly the external excitation couples into the extended modes of the commensurate superlattice, rather than as a localization-delocalization boundary (which is only present in the incommensurate case, Fig. 4**i**).”

Reviewer 3:

In this paper the authors exploit the spatially inhomogeneous thermal conductivity of a bilayer material to achieve Moiré physics in heat transport. The article is well written and scientifically sound. The data are well presented and the logic of the paper is easy to follow. Experiments show a clear signal that matches the theory.

Response: We sincerely thank the reviewer for the very positive assessment of our work. We are pleased that the reviewer found the manuscript to be well written, scientifically sound, and logically structured, and that the experimental results clearly support the theoretical predictions. We deeply appreciate this encouraging feedback.

My main perplexity is about the central claim of the paper and thus about the real novelty of the presented results. In the introduction the authors claim that the crucial novelty of their work is the ability to observe Moiré physics in linear systems. In the introduction I do not see any reference related to other works in this sense. Are they the first to rise such a question? Has this question been previously addressed by the community? I feel that a more comprehensive introduction is needed in this sense.

Response: We thank the reviewer for raising this concern, which helps us clarify the novelty of our work in a broader context. To the best of our knowledge, the specific question of whether moiré-induced flat bands and localization can occur in **purely linear diffusive** and **momentum-free** systems has not been raised or addressed in the literature.

In the Introduction, we have reviewed some pioneering and representative works about the moiré physics in condense matter, van der Waals heterostructures, and optical lattices. In these cases, flat bands emerge from interference in momentum-carrying modes, while the hopping and couplings of these systems are governed by electronic interactions or nonlinear Schrödinger equation. In this case, though these systems can be approximated to the non-interacting Hamiltonian via averaging the corresponding nonlinear couplings based on mean-field strategy, the intrinsic physical driving force for generating the moiré behaviors origin from interactions possessing apparent nonlinearity.

We are aware that some recent efforts in photonic lattices and stacked acoustic crystals have demonstrated rotationally symmetric moiré superlattices (*Nat. Photonics* 18, 224-229, 2024) and dispersion modulation (*Nat. Commun.* 16, 1988, 2025) under linear coupling conditions. However, these works still focus on discovering the moiré physics in classical wave dynamics, whose real-space momentum naturally enables the photonic/acoustic band model via their critical wave coherences.

As a result, these works still leave the open problem of motivating the moiré physics in passive, linear and momentum-free settings, where the effective band model is ill-defined without momentum space. That is, the case of purely diffusive and momentum-free conduction systems is fundamentally different. Diffusion is governed by Laplacian operators with no wave interference, and thus it was not clear whether moiré flat bands and corresponding physics could emerge in such media. To our knowledge, no prior work has explicitly raised or answered this question.

In our current work, we first demonstrate that moiré flat bands and localization can indeed be realized in a static, linear, dissipative conduction system by exploiting spatially engineered conductivity gradients. This provides a new paradigm distinct from both (i) nonlinear/advective systems (our prior hydrodynamic work) and (ii) nonlinear or linear wave-based systems with real momentum space (condensed matter or photonic analogues).

To clarify this point and further complete the related literature review, we have summarized the above contents in the revision as below:

In the introduction:

"A common thread among these advances is their reliance on nonlinear interactions of versatile platforms to unlock moiré-induced band modifications and topological responses [1-9]. Specifically, these corresponding moiré-induced behaviors arise from interference effects in momentum-carrying modes, where the underlying dynamics are described by effective tight-binding Hamiltonians for the ideal case of non-interacting model or nonlinear Schrödinger-type equations for interacting model [19]. While such models can often be approximated in a mean-field sense as linear, the intrinsic physical mechanisms responsible for band formation, such as electronic correlations [20] and Kerr nonlinearity [18, 21], still rely on the presence of momentum space and, frequently, on nonlinear coupling. Though some recent works demonstrated rotationally symmetric moiré superlattices [22] and dispersion modulation [23] with linear coupling frameworks in classical wave dynamics, these works still possess real-space momentum for supporting well-defined band structures governed by non-interacting momentum-carrying model [19]."

"This raises a fundamental question of whether moiré physics be realized in a purely static, momentum-free, and linear thermal conduction system, without invoking advection or interacting nonlinearity? The answer is not obvious. The features of inherent diffusion, lack of intrinsic momentum space and coherent interference, and the linear Laplacian operators in thermal conduction shed dim light on interference-like phenomena and flat-band formation for interacting models."

Along this line. I feel that from a theoretical perspective is not very clear what is linear and what is not, here. First of all, it is not clear from the paper what is the model considered. The equation of motions are linear in what? The situation is further misleading when talking about dissipative systems. Dissipative phenomena in many cases lead to nonlinear equation of motion (in a mean-field sense). It is thus of fundamental importance here to clarify these concepts.

Response: We thank the reviewer for pointing out this concern. We have illustrated related details in Supplementary Note 2 about the governing function (Eq. S4) and linear temperature field. In order to further clarify these points, we first start with the model that we considered.

In our current work, the governing function is the heat diffusion equation, i.e., $\frac{\partial T}{\partial t} = \nabla \cdot (\kappa(\mathbf{r})\nabla T)$, where T is the temperature field and $\kappa(\mathbf{r})$ is the spatially modulated conductivity. This model is a **linear partial differential equation in temperature (T)**, since both time evolution and spatial diffusion are governed by operators that act linearly on the temperature field. In this case, the principle of superposition holds, while the modulation of $\kappa(\mathbf{r})$ changes the coefficients in the equation but does **not introduce nonlinear terms in temperature (T)**.

For the concern on dissipative systems, we agree with the reviewer that many dissipative phenomena (e.g., nonlinear optics, reaction-diffusion systems, hydrodynamic turbulence) lead to nonlinear equations of motion. However, in our case, dissipation is purely passive conduction with no feedback or nonlinear dependence of $\kappa(\mathbf{r})$ and T. Thus, the system remains strictly linear without real-space momentum and exhibits in-situ decay without motions.

To further clarify these concepts, we have added some summarized contents to the main content explicitly stating the governing diffusion equation in a **linear, purely diffusive conduction system**, defining in what sense it is linear, and contrasting it with nonlinear dissipative models. The related revisions are as below:

In the section of "Linear moiré system and dual modulated wavevectors":

"Note that, we consider a pure heat conduction process governed by the spatial-dependent heat diffusion equation, i.e., $\nabla \cdot \left(\kappa(\mathbf{r})\nabla T(\mathbf{r},t)\right) - \rho c \frac{\partial T(\mathbf{r},t)}{\partial t} = 0$, where $T(\mathbf{r},t)$ and $\kappa(\mathbf{r})$ are respectively the temperature field and spatially modulated conductivity. This model is linear in the temperature field and satisfies superposition principles. The spatial inhomogeneity $\kappa(\mathbf{r})$ modifies the coefficients of the diffusion operator but does not introduce nonlinear terms in $T(\mathbf{r},t)$. In stark contrast, nonlinear dissipative systems with momentum feature feedback mechanisms, such as reaction-diffusion models, nonlinear optics, and hydrodynamics, where the field dynamics depend nonlinearly on the field amplitude itself."

Response to the Reviewers' Comments on the manuscript [NCOMMS-25-58105A-Z] entitled " Localized Dissipation in Linear Moiré Heat Transport " submitted to *Nature Communications*

We would like to thank the editors and reviewers for their careful reviewing of our work and their relevant comments. These comments are quite valuable, which have allowed us to further improve the manuscript from the aspects of descriptions. We have further clarified the platform of linear and momentum-free diffusion in the last sentence of the abstract. The changes in the revised manuscript are marked in red. In the following, we provide a point-to-point response to all reviewers' questions and comments. We do hope that the editors and reviewers may appreciate our extended efforts to improve the paper following their suggestions.

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Reviewer 1:

I appreciate the authors' detailed responses to my concerns and the other Reviewers. The discussion around Fig.1 is now fairer to acknowledge the class of flat-band systems that rely on interference effects, providing an intuitive picture of an alternative mechanism for band flattening when geometric frustration is not possible. The clarifications on what is meant by "linear" in this work are also helpful. I now better understand the distinction the authors draw between prior wave-based systems, relying on interference and nonlinear interactions, and their present system, where the governing diffusion equation is strictly linear and momentum-free. The authors also addressed my concern about this work being incremental to their previous one, as the latter required nonlinear fluid dynamics that is missing here.

Response: We thank the reviewer for the recognition of our revisions and the external efforts.

Overall, despite these clarifications address my main concerns, I still feel the work is an extension of moiré flat bands into diffusive systems, rather than a new paradigm for flat bands. I suggest the authors adjust the manuscript to moderate claims (e.g. last sentence of the abstract, and line 78) focusing on the specific novelty of realizing moiré physics in a linear, momentum-free diffusive system. With this revision, I would find the manuscript suitable for publication.

Response: We thank the reviewer for providing this suggestion. We have revised the descriptions of the last sentence of the abstract accordingly. The revision is as below:

In the abstract:

"Our results establish a paradigm for implementing moiré physics in a static, linear, and momentum-free diffusive system, offering a route to geometrically programmable non-equilibrium control in heat and mass transport. "

Reviewer 2:

The authors' answers to my questions satisfactorily clarify some aspects of the work. They improved the work by addressing the reviewers' criticisms and, in my opinion, clearly highlighted the manuscript's innovation and relevance, which was the main point raised by both other reviewers. With all the explanations and improvements developed by the authors, I believe the work has all the necessary features to be accepted in a high-level journal. Thus, I recommend this version of the manuscript for publication in Nature Communications.

Response: We thank the reviewer for the recognition of our revisions and the external efforts.

Reviewer 3:

The authors have provided exhaustive and satisfactory responses to all the comments raised in the first report. I therefore recommend the publication of the paper in its present form.

Response: We thank the reviewer for the recognition of our revisions and the external efforts.