nature portfolio

Peer Review File

Quantum electrodynamics of photonic time crystals

Corresponding Author: Professor Bumki Min

This file contains all reviewer reports in order by version, followed by all author rebuttals in order by version.

Version 0:

Reviewer comments:

Reviewer #1

(Remarks to the Author)
First, I would like to thank the authors for preparing this theoretical manuscript about the quantum electrodynamics model of photonic time crystals, which discusses the momentum bandgap (and the transition from outside the gap to the gap) from a quantum perspective as well as the dynamics of the quantum Rabi model in the photonic time crystals. The technically sound results of the manuscript are important for the community of photonics of time-varying media, and, in particular, for researchers who are interested in quantum aspects of those media (these media have been studied very well since 2009 from a classical picture. Now, logically, the next step is to investigate them in the realm of quantum optics/electrodynamics). However, I read the manuscript carefully, and I see that there are 2 main problems. 1) I strongly feel that the manuscript was prepared in a "Letter" format. Indeed, there are different results which were collected for the reader in only 5 pages in a compact form. Thus, my feeling is that this type of writing or format has dramatically decreased the clarity of the manuscript. Even for me, who is working in this area, it took considerable time and was hard to follow. For example, I should always move to the Appendix, read it precisely, and then come back to the main text again in order to finally understand what the authors want to explain. I highly recommend that the authors discuss the equations explicitly and "in detail" to make them completely clear to the reader. It is worth noting that the problem is not only the equations (which, in my view, have not been explained adequately). Besides equations, the main text is not clear due to a lack of enough explanation. As I mentioned, I think this is probably because the manuscript has been written like a Letter. 2) Another concern is about the novelty of the work mentioned by the authors in the abstract and the introduction. Is the level of novelty of this theoretical work enough in order to publish in Nature Communications? Of course, these results (particularly the last part about the quantum Rabi model) are new; however, the quantum theory of photonic temporal interfaces, photonic time crystals, and even the time-varying media in general is not new, and, honestly, there are different works published in this area. From this point of view, in opposite to what the title of the current manuscript suggests (as well as the abstract and introduction), the topic of the manuscript is not new. The title, abstract, and introduction need to be revised, taking into account all the proper references which studied temporal interfaces, photonic temporal crystals, and time-varying media from a quantum perspective.

As a conclusion, I suggest that the authors submit this theoretical manuscript as a "Letter" to a "technical" journal. From my view and experience, it is not suitable for publication in Nature Communications.

Reviewer #2

(Remarks to the Author)
The authors present an analysis of the photonic states in a photonic time crystal. They describe an insightful Floquet-space with lattice sites. The dynamics is discussed in terms of on-site potentials and hopping terms which are later in the manuscript coupled to atomic states. They find that the Hermitian quantum formulation of the results in a delocalization of the wavefunction in the Floquet-space when the system is parametrically resonant. They characterize this localization regimes with the Lyapunov exponent and energy-level spacing statistics. I have three aspects of the current manuscript that I believe should be addressed before publication.

First, this work is both timely and interesting for the growing community of Floquet photonics. However, there is an expansive body of work on the Dynamic Casimir effect which considers similar Hamiltonians and effects. For example see, Dodonov, “Fifty Years of the Dynamical Casimir Effect.” and references within for a glimpse of the field. An example of a similar system can be found in Romualdo, Hackl, and Yokomizo, “Entanglement Production in the Dynamical Casimir Effect at Parametric Resonance.’, where dynamically driven cavity is considered and the Lyapunov exponent is discussed. Unfortunately, the authors have not referenced any literature from this field. This manuscript would be much better received and attract a much broader audience if it could be related to the well-established field.

Secondly, the central question of the manuscript involves relating the non-Hermitian Hamiltonian of classical models to a Hermitian quantum description. My understanding is it lies in the localization-delocalization transition of the wavefunction in the synthetic Floquet space. The discussion regarding this comparison is sparce and not well highlighted and should be expanded.

Finally, for the delocalized state, the Lyapunov exponent expressed as the acosh〖(Z)〗. Therefore, the delocalization should increase with Z. Since Z is proportional to the inverse of the modulation amplitude α_c, this suggests that the delocalization occurs more for zero modulation. That is counter to the classical description where the amplification/imaginary eigenvalue grows with increasing permittivity modulation.

A few minor points for clarity:
In the text description of Figure 1d, a blue dotted line is referenced. Perhaps the authors mean the purple band?
The pop-out figures in 4c and 4b do not have labels on the axes of color scales.

Version 1:

Reviewer comments:

Reviewer #1

(Remarks to the Author)
I would like to thank the authors for their careful revisions and modifications. The manuscript has been substantially improved and is now clearer, particularly through the inclusion of the necessary information that addressed my main concern in the previous version. I believe the paper is now suitable for publication in Nature Communications. In addition, I recommend that, alongside references 30 and 31, the authors also cite the following works before the publication:

1) Dr. Mendonca: Theory of Photon Acceleration (CRC Press, Boca Raton, FL, 2000),

2) Dr. Setälä, at al: Quantum state engineering and photon statistics at electromagnetic time interfaces, Physical Review Research 7, 013120 (2025),

3) Dr. Liberal, et al: Quantum vacuum amplification in time-varying media with arbitrary temporal profiles, arXiv:2312.13315.

Thank you.

Reviewer #2

(Remarks to the Author)
The revision provided by the Authors significantly clarifies the results presented. I especially appreciate the review and contrast of this manuscript with prior work. After reviewing the revisions, I do recommend this for publication in Nature Communications. I will note, that there are a few minor grammatical errors in the added text and encourage the Authors to review carefully each sentence for clarity and meaning.

In response to the First Referee

"First, I would like to thank the authors for preparing this theoretical manuscript about the quantum electrodynamics model of photonic time crystals, which discusses the momentum bandgap (and the transition from outside the gap to the gap) from a quantum perspective as well as the dynamics of the quantum Rabi model in the photonic time crystals. The technically sound results of the manuscript are important for the community of photonics of time-varying media, and, in particular, for researchers who are interested in quantum aspects of those media (these media have been studied very well since 2009 from a classical picture. Now, logically, the next step is to investigate them in the realm of quantum optics/electrodynamics)."

Our response: We sincerely thank the referee for the encouraging remarks. We are encouraged by the referee's appreciation of our quantum electrodynamical treatment of photonic time crystals (PTCs), including both the momentum gap transition and the quantum Rabi dynamics. This confirmation reinforces our conviction that the manuscript will benefit specialists in time-varying photonics who wish to move beyond classical models, as well as the broader quantum-optics community.

In the following sections we address each of the referee's suggestions in detail and explain how the revised text improves clarity, highlights the novelty of our results, and positions our work within the existing literature.

"However, I read the manuscript carefully, and I see that there are 2 main problems. 1) I strongly feel that the manuscript was prepared in a "Letter" format. Indeed, there are different results which were collected for the reader in only 5 pages in a compact form. Thus, my feeling is that this type of writing or format has dramatically decreased the clarity of the manuscript. Even for me, who is working in this area, it took considerable time and was hard to follow. For example, I should always move to the Appendix, read it precisely, and then come back to the main text again in order to finally understand what the authors want to explain. I highly recommend that the authors discuss the equations explicitly and "in detail" to make them completely clear to the reader. It is worth noting that the problem is not only the equations (which, in my view, have not been explained adequately). Besides equations, the main text is not clear due to a lack of enough explanation. As I mentioned, I think this is probably because the manuscript has been written like a Letter."

Our response: We sincerely appreciate the referee's thoughtful comments regarding the format and clarity. We fully agree that the earlier Letter-style draft placed an undue burden on readers by requiring frequent reference to the appendices. To address this, we have substantially expanded the narrative, by approximately 33.5 percent.

A further literature survey, expanding the references from 52 to 75, situates our results within recent developments in quantum optics and dynamical Casimir research, while a new Appendix I offers a focused bridge to that field. The appendices now serve as algebraic back-ups or numerical checks. Every addition is highlighted in blue for ease of inspection. Because these revisions place the essential derivations, explanatory prose and visual aids exactly where the reader needs them, we believe the manuscript is now self-contained and readily accessible to both specialists and non-specialists, while the appendices remain available for those who wish to examine the fine detail. We hope the referee will find that the clarity issues have been fully resolved and that the paper now reads with ease.

2) Another concern is about the novelty of the work mentioned by the authors in the abstract and the introduction. Is the level of novelty of this theoretical work enough in order to publish in Nature Communications? Of course, these results (particularly the last part about the quantum Rabi model) are new; however, the quantum theory of photonic temporal interfaces, photonic time crystals, and even the time-varying media in general is not new, and, honestly, there are different works published in this area. From this point of view, in opposite to what the title of the current manuscript suggests (as well as the abstract and introduction), the topic of the manuscript is not new. The title, abstract, and introduction need to be revised, taking into account all the proper references which studied temporal interfaces, photonic temporal crystals, and time-varying media from a quantum perspective."

Our response: We thank the referee for questioning the novelty of our work; the referee's comment prompted a comprehensive review of the quantum literature on temporal interfaces, PTCs, and the dynamical Casimir effect (DCE). We have now incorporated the key theoretical papers, from the pioneering moving-mirror analysis to the most recent quantum treatments of time-varying media, so that our contribution is situated with precision.

This expanded context clarifies that, although quantum treatments of time-modulated systems do exist, our manuscript establishes three advances not previously available: (i) an exact analytic criterion identifying the momentum-gap edge as a localization-to-delocalization quantum phase transition, applicable to all eigenstates rather than only the photonic vacuum; (ii) a two-dimensional Floquet-photonic lattice framework that naturally incorporates a dipole-coupled two-level atom; and (iii) the prediction that such an atom irreversibly relaxes to a half-and-half mixed state when embedded in the momentum gap—a phenomenon absent from earlier classical or quantum models.

To guide readers, the revised introduction now cites the foundational quantum PTC works by Lyubarov *et al.*, Horsley *et al.*, and Sustaeta *et al.*, along with the DCE analyses of Dodonov and co-workers, and then explains precisely how our results extend these studies. We have also added a new Appendix I that provides a focused comparison with DCE theory and summarizes the extensive quantum-optics literature incorporated into our survey. We believe these changes address the referee's concerns: the novelty of our work is now explicit, fully referenced, and framed for a broad readership. We have incorporated the suggested references and revised the introduction accordingly, and we hope the referee will find the revised manuscript both clearer and more compelling. Below we summarize our literature survey:

- The quantum theory of the electromagnetic field confined by a moving boundary was first formulated by Moore [1] in the Heisenberg picture. Its Schrödinger-picture description, together with an effective Hamiltonian, was later developed in Ref. [2]. The time-varying optical cavity has since attracted significant attention because it enables photon generation through resonant amplification of vacuum fluctuations, $\omega_{\text{photon}} = \frac{1}{2}\,\omega_{\text{boundary}}$ [3, 4]. This phenomenon is known as the *dynamical Casimir effect* (DCE); comprehensive reviews are available in Refs. [5, 6].
- The quantum theory of photonic time crystals (QPTCs) [7, 8, 9] was developed from their classical counterparts. In this setting, the permittivity of the medium is homogeneous in space but varies in time, $\epsilon = \epsilon(t)$, and the system is not necessarily confined within a cavity. The QPTC Hamiltonian was derived in Ref. [7] by quantizing Maxwell's equations with time-dependent permittivity, in close parallel to the approach used for the DCE [2]. Consequently, these two frameworks share important features, such as the exponential growth of photon energy within the momentum gap.
- Ref. [9] and Ref. [10], from the PTC and DCE communities respectively, are most relevant to the first part of our manuscript, namely the quantum theory of a PTC without atomic coupling. Using distinct theoretical approaches, both works analyzed the transi-

tion between amplified photonic states and Bloch-like oscillations, which corresponds to the localized-to-delocalized quantum phase transition in the Floquet–photonic synthetic space considered here.

Through our careful literature survey, we have clarified how our work differs from previous studies, and we highlight below the key points in which it advances the field:

- The critical behavior of the momentum-gap transition is characterized by mapping the QPTC Hamiltonian onto a Floquet–photonic synthetic lattice and interpreting the transition from a quantum-transport perspective, namely as the localization-to-delocalization of eigenstates. While a similar viewpoint was mentioned conceptually in Ref. [10], we crystallize it in a 1-dimensional synthetic space by deriving an analytic expression for the localization length, which diverges with critical exponent $\nu = 1$ as the system approaches the momentum-gap edge. Furthermore, using a transfer-matrix method, we demonstrate that this localization-to-delocalization transition applies to *all* eigenstates of the QPTC Hamiltonian, not only to the photonic vacuum.
- We report for the first time the unconventional quantum Rabi dynamics coupled to a QPTC. Our effective description of the QPTC Hamiltonian in the Floquet–photonic synthetic space naturally extends to include a two-level atom whose electric dipole couples to the field. Notably, the coupling is proportional to the time-periodic $\epsilon^{-1}(t)$, and the Floquet formalism provides an ideal framework for studying the quantum electrodynamics of PTCs. In this setting, the previously 1-dimensional localization-to-delocalization transition extends to a 2-dimensional Floquet–photonic synthetic lattice, where atomic transitions mediate the coupling between *wires* of different total momentum.
- Notably, we find that amplified vacuum fluctuations in the QPTC drive the atomic system into a half-and-half mixed state, $\rho_{\rm at}(t \to \infty) = \frac{1}{2}|g\rangle\langle g| + \frac{1}{2}|e\rangle\langle e|$. This behavior can be understood intuitively from the effective Hilbert space, i.e., the 2-dimensional atom–Floquet–photonic synthetic space shown in Fig. 4a, where the dynamics of quantum electrodynamics are constrained by the energy gap, and the delocalization of eigenstates leads to equal population of the $|e\rangle$ and $|g\rangle$ states. Remarkably, this relaxation toward the maximally mixed state occurs irrespective of the initial condition $\rho_{\rm at}(t{=}0)$, provided the atom is coupled to a QPTC within the momentum gap.

Regarding the title of our manuscript, we emphasize that quantum electrodynamics (QED) is the quantum theory describing the interaction of charged particles with the electromagnetic field, allowing for the creation and annihilation of both charges and photons. Since our work explicitly treats atomic state transitions induced by coupling to a quantum PTC ($\sigma_x = \sigma^+ + \sigma^-$), it is appropriate to frame our study within the scope of QED. By contrast, a quantum theory of a PTC without treating the atomic part quantum mechanically, but the time-modulated permittivity and a light emitter are treated as classical objects, would be semi-classical electrodynamics (light is quantized, matter is classical).

To further sharpen this distinction, we have deliberately removed the keyword "cavity" throughout the manuscript. In a PTC, the total momentum δn_k is a good quantum number, and atomic transitions mediate the hybridization between quantum states that differ by one momentum quantum, $\hat{H}_{\rm int} \sim \epsilon^{-1}(t)\, \sigma_x \otimes i\big(\hat{a}_k^\dagger - \hat{a}_{-k}\big) + \text{h.c.}$. As a result, the Hilbert space is spanned jointly by the total momentum δn_k and the Floquet–photonic states, forming the two-dimensional synthetic lattice depicted in Fig. 4a. In contrast, for a mirror-bounded cavity there is no distinction between $|k\rangle$ and $|-k\rangle$ photon states, and the Hilbert space instead decomposes into even and odd photon-number sectors, which are effectively one-dimensional. These clarifications have been implemented consistently in the title, abstract, and main text.

"As a conclusion, I suggest that the authors submit this theoretical manuscript as a "Letter" to a "technical" journal. From my view and experience, it is not suitable for publication in

Nature Communications."

Our response: We thank the referee for raising the question of novelty and venue. In revision we conducted a comprehensive literature review, rewrote the Introduction, and broadened the bibliography so that prior quantum studies of temporal interfaces, photonic time crystals (PTCs), and the dynamical Casimir effect (DCE) are explicitly acknowledged.

The revised manuscript advances the field in three ways that we believe meet Nature Communications' criteria of conceptual advance with broad relevance:

- From the classical momentum gap transition to quantum criticality: By formulating time-modulated media in a Floquet framework, we show analytically (in the thermodynamic limit) that the classical momentum gap corresponds to a localization–delocalization quantum phase transition. We locate the gap edge in closed form and show that it organizes the entire spectrum, not only vacuum fluctuations. This reframing links photonic time modulation to well-studied critical phenomena and should interest both quantum optics and condensed-matter communities.
- QED in a 2D Floquet–photonic synthetic lattice: We elevate the model to a two-dimensional Floquet–photonic synthetic lattice and carry out a full QED treatment of a two-level emitter embedded in a PTC, uniting methods from cavity QED and Floquet engineering. This mapping opens a route to emulate quantum-transport experiments in optical platforms and predicts an unusual, linear growth of tunneling across the synthetic lattice. which is a behavior not captured by prior classical or single-mode quantum treatments.
- A new dissipation channel in the momentum gap: We predict, and verify numerically, that an atom tuned into the momentum gap relaxes to a maximally mixed state, driven by amplified vacuum fluctuations. Furthermore, we show that the *dissipation* takes place to the mixed state even when the atom is prepared at the ground state. To our knowledge this dissipation pathway has not been reported in earlier classical or quantum models and suggests testable, out-of-equilibrium signatures for near-term photonic and circuit-QED platforms.

These points are flagged in the revised Introduction, developed in the main text, and summarized in the Discussion (new and revised text highlighted in blue). Because the contribution integrates (i) a general Floquet re-interpretation with exact analysis, (ii) a multi-dimensional synthetic-lattice construction with QED, and (iii) a new, experimentally relevant relaxation mechanism, we believe the scope exceeds what can be communicated in a brief technical Letter. We therefore respectfully maintain that the work is suitable for *Nature Communications*.

In response to the Second Referee

Reviewer 2 (Remarks to the Author):

"The authors present an analysis of the photonic states in a photonic time crystal. They describe an insightful Floquet-space with lattice sites. The dynamics is discussed in terms of on-site potentials and hopping terms which are later in the manuscript coupled to atomic states. They find that the Hermitian quantum formulation of the results in a delocalization of the wavefunction in the Floquet-space when the system is parametrically resonant. They characterize this localization regimes with the Lyapunov exponent and energy-level spacing statistics. I have three aspects of the current manuscript that I believe should be addressed before publication."

Our response: We thank the referee for the concise summary of our Floquet-photonic space approach and for the three concrete requests. We address each point in turn and indicate the corresponding changes in the manuscript with section, figure, and page/line references.

"First, this work is both timely and interesting for the growing community of Floquet photonics. However, there is an expansive body of work on the Dynamic Casimir effect which considers similar Hamiltonians and effects. For example see, Dodonov, "Fifty Years of the Dynamical Casimir Effect." and references within for a glimpse of the field. An example of a similar system can be found in Romualdo, Hackl, and Yokomizo, "Entanglement Production in the Dynamical Casimir Effect at Parametric Resonance.', where dynamically driven cavity is considered and the Lyapunov exponent is discussed. Unfortunately, the authors have not referenced any literature from this field. This manuscript would be much better received and attract a much broader audience if it be related to the well-established field."

Our response: We thank the referee for drawing attention to the connections between our work and the dynamical Casimir effect (DCE). In response, we conducted a comprehensive literature survey, which confirmed that many of the questions investigated in the photonic temporal crystal (PTC) community intersect with those addressed in DCE research. Accordingly, we have incorporated the key DCE papers into our reference list and introduced explicit comparisons throughout the revised manuscript (see Introduction and Appendix I). Below, we provide a concise overview of the newly cited studies and explain how our results build upon and extend them:

- The quantization of the PTC was presented in Ref. [7], and its procedure closely parallels that employed for the DCE [2]. Nevertheless, several differences distinguish the two: (i) Cavity-free quantization versus moving-mirror DCE: A PTC generally operates without an optical cavity, so the medium remains translationally invariant. Consequently, the counter-propagating modes $|k\rangle$ and $|-k\rangle$ are distinct but degenerate in frequency. The conserved quantity is the total momentum $P = \hbar kc(n_k - n_{-k})$, where $|n_k, n_{-k}\rangle$ denotes a Fock state. This conservation law allows us to restrict the dynamics to the subset of Fock states represented as the one-dimensional "wires" in Fig. 1 that share the same P. (ii) Expansion of the Hilbert space when atoms are added: When atomic states are coupled to photons, nearest-neighbor *wires* are coupled, and the Hilbert space expands from one to two dimensions (Fig. 4). In the DCE setup, by contrast, the Hilbert space is partitioned by photon-number parity (even or odd), because pair creation and annihilation couple only photonic Fock states of the same parity. (iii) Resonant condition and focus of prior work: The primary interest in the DCE community is the generation of photons from parametrically amplified vacuum fluctuations. In the DCE literature, the resonant condition $\omega_{\text{photon}} = \frac{1}{2}\omega_{\text{boundary}}$ is often assumed, which corresponds to $k = \frac{1}{2}, \Omega$ in our work. By contrast, the PTC community has focused on the transition point where photonic states move from the band into the momentum gap. This distinction arises because a PTC generally supports a continuum of momenta k, whereas a typical DCE setup involves a single resonant k.

- Lyapunov exponent as a probe of localization length: At the resonant condition, $\omega_{\text{photon}} = \frac{1}{2}, \omega_{\text{boundary}}$, Ref.[11] calculated the entanglement entropy generated by coherent photon-pair creation. That study demonstrated a clear connection between classical Lyapunov instability and its quantum manifestation in entanglement entropy: in one spatial dimension the entropy grows logarithmically with time, while in higher dimensions it grows linearly, with the slope set by the classical Lyapunov exponent. In our work, we instead employ the Lyapunov exponent to evaluate the localization length in the Floquet-photonic synthetic space outside the momentum gap, and we have cited Ref.[11] accordingly in the revised manuscript.
- Bridging DCE and PTC frameworks: To our knowledge, Ref. [12] is the study most closely related to the first part of our manuscript, namely the quantum theory of PTCs. Using the generalized Baker–Campbell–Hausdorff formulas [13], the authors derived an explicit expression for the exponential photon-growth rate as a function of the detuning parameter, which in our notation is $k - \frac{1}{2}\Omega$. We have duly acknowledged this important contribution in the revised manuscript.
- Novel atomic dissipation channel: In discussing the quantum dynamics of atomic states coupled to the DCE, we have acknowledged the contributions of Ref. [14, 15]. However, the physics addressed in those studies does not directly align with the findings of our manuscript. In particular, we report a novel phenomenon: the dissipation of atomic states into a half–half mixed state, $\rho_{\text{at}}(t \to \infty) = \frac{1}{2}|g\rangle\langle g| + \frac{1}{2}|e\rangle\langle e|$, which occurs irrespective of the initial condition $\rho_{\text{at}}(t = 0)$ once the eigenstates of the coupled quantum PTC become delocalized in the Floquet-photonic synthetic space. This irreversible decay mechanism is distinct from those considered in DCE, and we believe it constitutes an important new contribution to the understanding of atom–photon interactions in time-varying systems. Our formalism provides a precise description of this effect, establishing PTCs as an ideal platform for exploring quantum electrodynamics in driven media and for unveiling new avenues in the study of non-equilibrium dynamics in quantum photonics.

"Secondly, the central question of the manuscript involves relating the non-Hermitian Hamiltonian of classical models to a Hermitian quantum description. My understanding is it lies in the localization-delocalization transition of the wavefunction in the synthetic Floquet space. The discussion regarding this comparison is sparce and not well highlighted and should be expanded."

Our response: We sincerely thank the referee for pointing out the need for a more detailed discussion of the non-Hermitian physics of classical PTCs in the context of the quantum mechanical description. We appreciate this valuable feedback, as it allows us to further clarify the connection between the classical non-Hermitian description and its quantum counterpart. In the revised manuscript, we have expanded the relevant sections (the paragraph above and below Eq.(11) in the revised manuscript) to provide a more thorough explanation of how the localization-delocalization transition of the wavefunction in the synthetic Floquet photonic space manifests in both the classical and quantum descriptions. This expanded discussion highlights the relationship between the non-Hermitian behavior in classical models and the Hermitian quantum treatment, ensuring that the key insights are more clearly articulated. We trust that this revision will address the referee's concern and improve the overall clarity of the manuscript.

"Finally, for the delocalized state, the Lyapunov exponent expressed as the acosh(Z). Therefore, the delocalization should increase with Z. Since Z is proportional to the inverse of the modulation amplitude α_c, this suggests that the delocalization occurs more for zero modulation. That is counter to the classical description where the amplification/imaginary eigenvalue grows with increasing permittivity modulation."

Our response: We sincerely thank the referee for raising this insightful comment. The distinction between the localized and delocalized regimes is an important point, and we appreciate the opportunity to clarify the role of the Lyapunov exponent in our model. We reproduce Eq. (9) below:

$$\xi^{-1} = \text{arcosh}\left(\frac{1-\frac{\Omega}{2k}}{1-\frac{\Omega}{2k_{c-}}}\right) = \text{arcosh}\left(\frac{1-\frac{\Omega}{2k}}{-\alpha_c/2}\right) \tag{1}$$

where in the last equality $k_{c-}=\frac{\Omega}{2+\alpha_c}$ is used. As the referee mentioned, 'Z' increases by lowering α_c. At the same time, when 'Z' increases, the localization length gets smaller. It implies that eigenstates are more localized when α_c is lowered. The confusion might have arisen here because we employ the Lyapunov exponent γ in the *localized* regime when $k \in \mathcal{K}_{\text{band}}$ to compute the localization length $\xi = 1/\gamma$ according to the transfer matrix method, $\psi_n \sim e^{-\gamma n} = e^{-n/\xi}$: The larger value of Lyapunov exponent leads the smaller value of the localization length.

In the *delocalized* regime, we did not compute the localization length or Lyapunov exponent, as the states are delocalized. Instead, we focus on the exponential growth rate of photonic energy, $\gamma_{\text{qu}} = \langle\hat{N}\rangle^{-1}\frac{d\langle\hat{N}\rangle}{dt}$, which captures the energy dynamics of the system. This approach is distinct from the use of the Lyapunov exponent in Ref. [11], where it characterizes the rate of linear entanglement growth in a delocalized system. In contrast, in our work, the Lyapunov exponent is specifically used to quantify the degree of localization in the synthetic lattice in the localized regime.

We hope this clarification addresses the referee's concern and provides a clear understanding of how the Lyapunov exponent is applied in different regimes in our manuscript. We trust that this distinction further strengthens the logic of our analysis and highlights the significance of the localization-delocalization transition in the PTC.

"A few minor points for clarity: In the text description of Figure 1d, a blue dotted line is referenced. Perhaps the authors mean the purple band? The pop-out figures in 4c and 4b do not have labels on the axes of color scales."

Our response: We sincerely thank the referee for pointing out these minor issues. Regarding the description of Figure 1d, we agree that there was a discrepancy in the reference to a "blue dotted line," and we have corrected this to refer to the "purple band" in the revised manuscript. Additionally, we appreciate the comment about the pop-out figures in panels 4c and 4b. We have now added the necessary labels on the axes of the color scales to ensure that these figures are fully clear and self-contained.

We trust these updates will enhance the clarity of the manuscript and appreciate the referee's careful attention to these details.

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In response to the Referee

Comment: "The revision provided by the Authors significantly clarifies the results presented. I especially appreciate the review and contrast of this manuscript with prior work. After reviewing the revisions, I do recommend this for publication in Nature Communications."

Response: We thank the reviewer for the positive evaluation and for recommending our manuscript for publication.

Comment: "I will note, that there are a few minor grammatical errors in the added text and encourage the Authors to review carefully each sentence for clarity and meaning."

Response: We carefully proofread the revised manuscript and corrected minor grammatical and stylistic issues to improve clarity. All changes have been incorporated into the main text and the Supplementary Information.

We appreciate the reviewer's constructive feedback and are grateful for the recommendation for publication.