The role of quantum measurement in stochastic thermodynamics
Supplementary material

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A microscopic interpretation of the quantum heat

Here we provide a simple interpretation of the quantum heat contribution in the context of cavity quantum electrodynamics. We consider a Qubit of transition frequency ω_0 prepared in the coherent superposition $|+_x\rangle = (|e\rangle + |g\rangle)/\sqrt{2}$, and then measured in its energy eigenbasis. Preparation and measurement fully define the thermodynamic transformation under study. For this elementary two-points quantum trajectory, the internal energy change equals $\Delta U = Q_{\mathrm{q}} = \pm\hbar\omega_0/2$.

It is possible to account for this energy change by introducing the classical source used to prepare the Qubit in a coherent superposition *before* the measurement. It is modeled by a large coherent field $|\alpha\rangle$ injected in the mode of a high quality factor cavity, resonantly coupled to the Qubit. Starting from the state $|g,\alpha\rangle$, the Qubit-field system evolves by coherently exchanging a quantum of energy (Rabi oscillation). After a time τ, the Qubit and the field share an entangled state $\mathcal{N}(\tau)(|e,\alpha_e(\tau)\rangle+|g,\alpha_g(\tau)\rangle)$, where $\mathcal{N}(\tau)$ is a normalisation constant. $|\alpha_e\rangle$ (resp. $|\alpha_g\rangle$) is the field's state correlated with the Qubit's excited (resp. ground) state, checking $\langle\alpha_e(\tau_{1/2})|N|\alpha_e(\tau_{1/2})\rangle = \langle\alpha_g(\tau_{1/2})|N|\alpha_g(\tau_{1/2})\rangle - 1$, where N is the photon number operator. In the classical limit where $|\alpha| \gg 1$, we have $\alpha_e(\tau) \sim \alpha_g(\tau)$ such that after tracing over the field, the Qubit remains in a pure state. This quantum-classical boundary has been theoretically studied by Gea-Banacloche [1], and experimentally checked by Auffèves *et al* [2].

An equally weighted coherent superposition of $|e\rangle$ and $|g\rangle$ is prepared if the Qubit-field interaction is frozen at time $\tau_{1/2}$, such that $P_e(\tau_{1/2}) = P_g(\tau_{1/2})$. P_e (resp. P_g) are the excited (resp. ground) state populations. Before the measurement, the quantum of energy is equally delocalized between the Qubit and the source, corresponding to the Qubit's energy change $\Delta U = \hbar\omega_0/2$. After the measurement, the Qubit is projected on the state $|e\rangle$ (resp. $|g\rangle$). Simultaneously, the field is projected on the state $|\alpha_e(\tau_{1/2})\rangle$ (resp. $|\alpha_g(\tau_{1/2})\rangle$). In the first case, the quantum of energy is localized in the Qubit, and in the source in the second case. The quantum heat accounts for this energy relocalization.

As mentioned in the main text, each physical situation can give rise to such analysis involving the remote sources used to prepare the superposition. However, these sources are most often not accessible; Moreover, in the classical limit of large numbers of photons in the field the degree of atom-field entanglement is negligible as the atomic purity approaches 1. This motivates the strategy proposed in the paper, i.e. not searching for further justifications of the quantum heat, and take it as a natural byproduct of the standard measurement postulate.

Quantum Jarzynski's equality

Just like in the classical situation, the entropy produced by some quantum measurement and its energetic imprint, e.g. the quantum heat, can be quantitatively related in the quantum Jarzynski's protocol. It involves a system initially thermalized in a bath of inverse temperature β, then decoupled from the bath. Its energy is measured with outcome $U(t_0)$, preparing the corresponding energy eigenstate. It is then driven by a Hamiltonian $H_{\mathrm{s}}(t)$, and its energy measured again with outcome $U(t_N)$, giving rise to a stochastic change of internal energy $\Delta U[\gamma] = U(t_N) - U(t_0)$. It is well known that

$$\langle \exp(-\beta(\Delta U[\gamma] - \Delta F))\rangle_\gamma = 1, \tag{1}$$

where $\langle\cdot\rangle_\gamma$ stands for the average over all trajectories and ΔF is the change in the system's free energy. With current definitions, the internal energy change is solely attributed to work exchanges: $\Delta U[\gamma] = W[\gamma]$. This gives rise to the quantum version of Jarzynski's equality, whose formal similarity with its classical counterpart has long been known. In particular, the entropy produced by the trajectory γ has the same expression as in the classical case $\Delta_{\mathrm{i}}s[\gamma] = \beta(W[\gamma] - \Delta F) = \beta Q_{\mathrm{irr}}[\gamma]$, where $Q_{\mathrm{irr}}[\gamma]$ is classically defined as the heat irreversibly dissipated in the bath. However, the physical situations addressed by each protocol are clearly different: While the classical protocol allows measuring irreversibility due to the departure from thermal equilibrium, the quantum protocol can involve purely quantum irreversibility, due to coherences' erasure induced by the final energy measurement [3, 4]. Such quantum irreversibility takes place if the system is driven in a non-adiabatic way, such that its state before the final measurement is not an energy eigenstate. Within the current framework, entropic contributions of measurement-induced irreversibility have remained hidden, while they clearly appear with our definitions: The change of internal energy now reads $\Delta U[\gamma] = W[\gamma] + Q_{\rm q}[\gamma]$, where $W[\gamma] = U(t_N^-) - U(t_0)$ is the work deterministically exchanged with the drive, and $Q_{\rm q}[\gamma] = U(t_N) - U(t_N^-)$ the stochastic heat exchange due to measurement back-action. Finally, the entropy production now splits into two components:

$$\begin{aligned} \Delta_{\rm i} s[\gamma] &= \beta(\Delta U[\gamma] - \Delta F) \\ &= \beta(W[\gamma] - \Delta F) + \beta Q_{\rm q}[\gamma]. \end{aligned} \tag{2}$$

The term $\beta Q_{\rm q}[\gamma]$ is a purely quantum contribution, that quantitatively relates the energetic and entropic signatures of quantum irreversibility. This term vanishes in the classical situation where the energy eigenbasis is fixed, or changes slowly (adiabatically). Note that if both quantities appear in similar situations of non-adiabatic driving, the quantum heat is not equivalent to the internal friction heat $Q_{\rm fric}$ [4], which is usually interpreted as the heat that would be dissipated in a fictitious heat bath and is strictly proportional to entropy production. Here the quantum heat can be negative, and corresponds to a real system's energy change that takes place during the measurement induced quantum jump.

We now show that the expression 2 for entropy production can be extended to the case of quantum open systems.

Jarzynski's equality for a quantum open system.

Here we consider a driven quantum system coupled to a thermal bath of inverse temperature β. This situation is routinely described by a Lindbladt master equation, that is unraveled within a QJ picture. Each jump operator induces a single transition between two states of a given basis $\mathcal{B} = \{|i\rangle\}$, described by Eq.(16) of main text. We compute the probability of a direct trajectory γ and the corresponding reverse trajectory $\gamma_{\rm r}$, applying formulæ17 and 18 of main text. $|\psi_0\rangle$ (resp. $|\psi_N\rangle$) is the initial (resp. final) energy eigenstate in which the system is just after the initial (resp. final) energy measurement. The boundary term in the entropy production is identical to the closed system case, verifying $\Delta_{\rm i}^{\rm b} s[\gamma] = -\log(p_{\rm d}(\psi_N)/p_{\rm d}(\psi_0)) = e^{-\beta(\Delta U[\gamma]-\Delta F)}$. On the other hand, the contribution of the conditional probabilities is now non-trivial. The equilibrium state $\pi_{\rm s}$ of the Lindbladian is a thermal state verifying

$$\pi_{\rm s} = \sum_{i=1}^{\mathcal{N}_s} \pi_{\rm s}(i)|i\rangle\langle i|, \tag{3}$$

with $\pi_{\rm s}(i) = \exp(-\beta(U_i(t_n) - F(t_n)))$. $U_i(t_n)$ is the energy of state $|i\rangle$ and $F(t_n)$ the equilibrium free energy. Therefore, the reversed measurement operators $M^{\rm r}_{\mathcal{K}}$ verify Eq.20 of main text with $Q_{\rm cl}(\mathcal{K}) = U_{j(\mathcal{K})} - U_{i(\mathcal{K})}$. Finally, the conditional probability of the reversed trajectory reads:

$$\begin{aligned} P_{\rm r}[\gamma_{\rm r}|\psi_N] &= |\langle\psi_0| \left(\prod_{n=1}^{N} \exp\left[\beta Q_{\rm cl}(\mathcal{K}_\gamma(t_{N+1-n}))\right] M^\dagger_{\mathcal{K}_\gamma(t_{N+1-n})} \right) |\psi_N\rangle|^2 \\ &= \exp\left[\beta \sum_{n=1}^{N} Q_{\rm cl}(\mathcal{K}_\gamma(t_{N+1-n}))\right] |\langle\psi_0| \left(\prod_{n=1}^{N} M^\dagger_{\mathcal{K}_\gamma(t_{N+1-n})} \right) |\psi_N\rangle|^2 \\ &= \exp\left[\beta Q_{\rm cl}(\gamma)\right] |\langle\psi_N| \left(\prod_{n=1}^{N} M_{\mathcal{K}_\gamma(t_n)} \right) |\psi_0\rangle|^2. \end{aligned} \tag{4}$$

We have used the equalities $\sum_{n=1}^{N} Q_{\rm cl}(\mathcal{K}_\gamma(t_n)) = Q_{\rm cl}[\gamma]$, and $\left(\prod_{n=1}^{N} M^\dagger_{\mathcal{K}_\gamma(t_{N+1-n})}\right)^\dagger = \prod_{n=1}^{N} M_{\mathcal{K}_\gamma(t_n)}$. It is then straigthforward to derive $P_{\rm r}[\gamma_{\rm r}|\psi_N] = e^{\beta Q_{\rm cl}[\gamma]} P_{\rm d}[\gamma|\psi_0]$. We eventually get:

$$\Delta_i s[\gamma] = \beta\left(\Delta U - Q_{\rm cl}[\gamma] - \Delta F\right). \tag{5}$$

Using the first law at the single trajectory level:

$$\Delta U = W[\gamma] + Q_{\rm q}[\gamma] + Q_{\rm cl}[\gamma], \tag{6}$$

we finally find

$$\Delta_i s[\gamma] = \beta\left(W[\gamma] - \Delta F\right) + \beta Q_{\rm q}[\gamma], \tag{7}$$

which extends the validity of Eq.2 (See previous section). Again, the second term involving the quantum heat contribution quantifies the irreversibility induced by the quantum measurements performed on the system. Such measurements involve the continuous monitoring of the environment and the final energy measurement performed by the operator.

In particular, this formalism captures genuinely quantum situations for which the time-dependent Hamiltonian $H_\mathrm{s}(t_n)$ does not commute with the stationary state defined by the Lindbladian $\pi_\mathrm{s}(t_n)$ at the same time [6, 7]. Here βQ_q is non-zero because after each jump, the system is projected on one of the states of $\mathcal{B}$, i.e. an eigenstate of π_s, which is not an eigenstate of the Hamiltonian H_s: It therefore develops coherences during the deterministic evolution, before a new jump takes place. Again, this expression for entropy production is fully compatible with current derivations of FTs: The main difference of our approach is that the heat increment now has a classical and a quantum component, and solely the classical component is involved in the expression of the time-reversed operators.

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