Quantum parameter estimation with general dynamics: Supplemental material

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BURES ANGLE BETWEEN EXTENDED CHANNELS

In this section we show that for any two given channels which maps from m_1- to m_2-dimensional Hilbert space, $K_1(\rho^S) = \sum_{i=1}^{d} F_{1i}\rho^S F_{1i}^{\dagger}$ and $K_2(\rho^S) = \sum_{i=1}^{d} F_{2i}\rho^S F_{2i}^{\dagger}$, the Bures angle between the extended channels $K_1 \otimes I_A$ and $K_2 \otimes I_A$ can be computed from the Kraus operators as following

$$\cos\Theta(K_1 \otimes I_A, K_2 \otimes I_A) = \max_{\|W\|\leq 1} \frac{1}{2}\lambda_{\min}(K_W + K_W^{\dagger}),$$

i.e.,

$$\begin{aligned}&\min_{\rho^{SA}} F[(K_1 \otimes I_A)(\rho^{SA}), (K_2 \otimes I_A)(\rho^{SA})]\\ &= \max_{\|W\|\leq 1} \frac{1}{2}\lambda_{\min}(K_W + K_W^{\dagger}),\end{aligned} \tag{1}$$

here $K_W = \sum_{ij} w_{ij} F_{1i}^{\dagger} F_{2j}$, w_{ij} is the ij-th entry of $d \times d$ matrix W and the optimization is over all $d \times d$ matrices W with operator norm $\|W\| \leq 1$. During the proof the analytical formulas that compute quantum Fisher information for arbitrary pure input states will be derived.

As the minimum fidelity can always be achieved with pure states, we can assume $\rho^{SA} = |\psi_{SA}\rangle\langle\psi_{SA}|$. Denote U_{ES1}, U_{ES2} as the unitary extension of K_1 and K_2 respectively, i.e.,

$$\begin{aligned}K_1(\rho^S) &= Tr_E(U_{ES1}(|0_E\rangle\langle 0_E| \otimes \rho^S)U_{ES1}^{\dagger}),\\ K_2(\rho^S) &= Tr_E(U_{ES2}(|0_E\rangle\langle 0_E| \otimes \rho^S)U_{ES2}^{\dagger}),\end{aligned} \tag{2}$$

where $|0_E\rangle$ denotes some standard state of the environment, U_{ES1} and U_{ES2} are unitary operators acting on both system and environment. The general form of U_{ES1} can be written as a pm_2-dimensional unitary, here $p \geq d$, $p-d$ zero Kraus operators can be added,

$$U_{ES1} = (W_1 \otimes I_{m_2}) \underbrace{\begin{bmatrix} F_{11} & * & * & \cdots & * \\ F_{12} & * & * & \cdots & * \\ \vdots & & \vdots & & \vdots \\ F_{1d} & * & * & \cdots & * \\ \mathbf{0} & * & * & \cdots & * \\ \vdots & & \vdots & & \vdots \\ \mathbf{0} & * & * & \cdots & * \end{bmatrix}}_{U_1} \tag{3}$$

where $W_1 \in U(p)$ and only the first m_1 columns of U_1 are fixed. Similarly

$$U_{ES2} = (W_2 \otimes I_{m_2}) \underbrace{\begin{bmatrix} F_{21} & * & * & \cdots & * \\ F_{22} & * & * & \cdots & * \\ \vdots & & \vdots & & \vdots \\ F_{2d} & * & * & \cdots & * \\ \mathbf{0} & * & * & \cdots & * \\ \vdots & & \vdots & & \vdots \\ \mathbf{0} & * & * & \cdots & * \end{bmatrix}}_{U_2} \tag{4}$$

where $W_2 \in U(p)$ and only the first m_1 columns of U_2 are fixed. Note that W_1 and W_2 only act on the environment and can be chosen arbitrarily.

$$\begin{aligned} U_{ES1}^{\dagger} U_{ES2} &= \begin{bmatrix} F_{11}^{\dagger} & F_{12}^{\dagger} & * & \cdots & F_{1d}^{\dagger} & \mathbf{0} & * & \cdots & \mathbf{0} \\ * & * & * & \cdots & * & * & * & \cdots & * \\ \vdots & & & & \vdots & & & & \vdots \\ * & * & * & \cdots & * & * & * & \cdots & * \end{bmatrix} (W_1^{\dagger} \otimes I_{m_2})(W_2 \otimes I_{m_2}) \begin{bmatrix} F_{21} & * & * & \cdots & * \\ F_{22} & * & * & \cdots & * \\ \vdots & & & & \vdots \\ F_{2d} & * & * & \cdots & * \\ \mathbf{0} & * & * & \cdots & * \\ \vdots & & \vdots & & \vdots \\ \mathbf{0} & * & * & \cdots & * \end{bmatrix} \\ &= \begin{bmatrix} F_{11}^{\dagger} & F_{12}^{\dagger} & * & \cdots & F_{1d}^{\dagger} & \mathbf{0} & * & \cdots & \mathbf{0} \\ * & * & * & \cdots & * & * & * & \cdots & * \\ \vdots & & & & \vdots & & & & \vdots \\ * & * & * & \cdots & * & * & * & \cdots & * \end{bmatrix} (W_p \otimes I_{m_2}) \begin{bmatrix} F_{21} & * & * & \cdots & * \\ F_{22} & * & * & \cdots & * \\ \vdots & & & & \vdots \\ F_{2d} & * & * & \cdots & * \\ \mathbf{0} & * & * & \cdots & * \\ \vdots & & \vdots & & \vdots \\ \mathbf{0} & * & * & \cdots & * \end{bmatrix} \\ &= \begin{bmatrix} K_{W_d} & * & * & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & & & & \vdots \\ * & * & * & \cdots & * \end{bmatrix}, \end{aligned} \tag{5}$$

where $W_p = W_1^{\dagger} W_2 \in U(p)$, W_d is the first $d \times d$ submatrix of W_p, i.e.,

$$W_p = \begin{bmatrix} W_d & * \\ * & * \end{bmatrix} \tag{6}$$

and $K_{W_d} = \sum_{i=1}^{d} \sum_{j=1}^{d} w_{ij} F_{1i}^{\dagger} F_{2j}$ with w_{ij} as the ij-th entry of W_d. Then

$$\begin{aligned} & \min_{|\psi_{SA}\rangle} F[(K_1 \otimes I_A)(|\psi_{SA}\rangle\langle\psi_{SA}|), (K_2 \otimes I_A)(|\psi_{SA}\rangle\langle\psi_{SA}|)] \\ &= \min_{|\psi_{SA}\rangle} F[Tr_E(U_{ES1} \otimes I_A(|0_E\rangle\langle 0_E| \otimes |\psi_{SA}\rangle\langle\psi_{SA}|)U_{ES1}^{\dagger} \otimes I_A), Tr_E(U_{ES2} \otimes I_A(|0_E\rangle\langle 0_E| \otimes |\psi_{SA}\rangle\langle\psi_{SA}|)U_{ES2}^{\dagger} \otimes I_A)] \\ &= \min_{|\psi_{SA}\rangle} \max_{U_{ES2}} F(U_{ES1} \otimes I_A(|0_E\rangle\langle 0_E| \otimes |\psi_{SA}\rangle\langle\psi_{SA}|)U_{ES1}^{\dagger} \otimes I_A, U_{ES2} \otimes I_A(|0_E\rangle\langle 0_E| \otimes |\psi_{SA}\rangle\langle\psi_{SA}|)U_{ES2}^{\dagger} \otimes I_A), \end{aligned}$$

the second equality used Uhlmann's theorem which states that

$$F(\rho_1, \rho_2) = \max_{|\psi_2\rangle} F(|\psi_1\rangle\langle\psi_1|, |\psi_2\rangle\langle\psi_2|), \tag{7}$$

where $|\psi_1\rangle$ is an arbitrary purification of ρ_1 and the maximization runs over all purifications $|\psi_2\rangle$ of ρ_2[1]. Thus

$$\begin{aligned}
&\min_{|\psi_{SA}\rangle} F[(K_1\otimes I_A)(|\psi_{SA}\rangle\langle\psi_{SA}|),(K_2\otimes I_A)(|\psi_{SA}\rangle\langle\psi_{SA}|)]\\
&=\min_{|\psi_{SA}\rangle}\max_{U_{ES2}} F[U_{ES1}\otimes I_A(|0_E\rangle\langle 0_E|\otimes|\psi_{SA}\rangle\langle\psi_{SA}|)U_{ES1}^\dagger\otimes I_A, U_{ES2}\otimes I_A(|0_E\rangle\langle 0_E|\otimes|\psi_{SA}\rangle\langle\psi_{SA}|)U_{ES2}^\dagger\otimes I_A]\\
&=\min_{|\psi_{SA}\rangle}\max_{U_{ES2}} F[|0_E\rangle\langle 0_E|\otimes|\psi_{SA}\rangle\langle\psi_{SA}|, U_{ES1}^\dagger U_{ES2}\otimes I_A(|0_E\rangle\langle 0_E|\otimes|\psi_{SA}\rangle\langle\psi_{SA}|)U_{ES2}^\dagger U_{ES1}\otimes I_A]\\
&=\min_{|\psi_{SA}\rangle}\max_{U_{ES2}} |\langle\psi_{SA}|\langle 0_E|U_{ES1}^\dagger U_{ES2}\otimes I_A|0_E\rangle|\psi_{SA}\rangle|\\
&=\min_{|\psi_{SA}\rangle}\max_{U_{ES2}} |\langle\psi_{SA}|\langle 0_E|\begin{bmatrix} K_{W_d} & * & * & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & & & & \vdots \\ * & * & * & \cdots & * \end{bmatrix}\otimes I_A|0_E\rangle|\psi_{SA}\rangle|\\
&=\min_{|\psi_{SA}\rangle}\max_{W_d} |\langle\psi_{SA}|K_{W_d}\otimes I_A|\psi_{SA}\rangle|\\
&=\min_{\rho^S}\max_{W_d} |Tr(\rho^S K_{W_d})|\\
&=\min_{\rho^S}\max_{W_d} |Tr(\sum_{ij} w_{ij}\rho^S F_{1i}^\dagger F_{2j})|\\
&=\min_{\rho^S}\max_{W_d} |\sum_{ij} w_{ij} Tr(\rho^S F_{1i}^\dagger F_{2j})|
\end{aligned} \tag{8}$$

where $\rho^S = Tr_A(\rho^{SA})$. As W_d is the first $d\times d$ submatrix of $W_p \in U(p)$, $\|W_d\| \leq 1$, here $\|\dot{\|}$ denotes the operator norm which equals to the maximum singular value. Conversely any W_d satisfies $\|W_d\| \leq 1$ can be imbedded as a submatrix of a unitary[3], thus

$$\min_{|\psi_{SA}\rangle} F[(K_1\otimes I_A)(|\psi_{SA}\rangle\langle\psi_{SA}|),(K_2\otimes I_A)(|\psi_{SA}\rangle\langle\psi_{SA}|)] = \min_{\rho^S}\max_{\|W_d\|\leq 1} |\sum_{ij} w_{ij} Tr(\rho^S F_{1i}^\dagger F_{2j})|. \tag{9}$$

Note that we can always adjust the phase of w_{ij} so that $\sum_{ij} w_{ij} Tr(\rho^S F_{1i}^\dagger F_{2j})$ is a positive real number, thus

$$\begin{aligned}
&\max_{\|W_d\|\leq 1} |\sum_{ij} w_{ij} Tr(\rho^S F_{1i}^\dagger F_{2j})|\\
&=\max_{\|W_d\|\leq 1} Re\{\sum_{ij} w_{ij} Tr(\rho^S F_{1i}^\dagger F_{2j})\} \quad =\max_{\|W_d\|\leq 1} Re[Tr(\rho^S K_{W_d})] = \max_{\|W_d\|\leq 1}\frac{1}{2}Tr\{\rho^S K_{W_d} + [\rho^S K_{W_d}]^\dagger\}\\
&=\max_{\|W_d\|\leq 1}\frac{1}{2}Tr[\rho^S K_{W_d} + K_{W_d}^\dagger\rho^S]\\
&=\max_{\|W_d\|\leq 1}\frac{1}{2}Tr[\rho^S(K_{W_d} + K_{W_d}^\dagger)],
\end{aligned} \tag{10}$$

then $\min_{|\psi_{SA}\rangle} F[(K_1\otimes I_A)(|\psi_{SA}\rangle\langle\psi_{SA}|),(K_2\otimes I_A)(|\psi_{SA}\rangle\langle\psi_{SA}|)] = \min_{\rho^S}\max_{\|W_d\|\leq 1}\frac{1}{2}Tr[\rho^S(K_{W_d}+K_{W_d}^\dagger)]$, where $\rho^S = Tr_A(|\psi_{SA}\rangle\langle\psi_{SA}|)$ and $K_{W_d} = \sum_{i=1}^d\sum_{j=1}^d w_{ij}F_{1i}^\dagger F_{2j}$ with w_{ij} as the ij-th entry of W_d. For the extended channel since all the reduced states ρ^S forms a convex set, $\{W_d | \|W_d\| \leq 1\}$ is convex and compact and $\frac{1}{2}Tr[\rho^S(K_{W_d}+K_{W_d}^\dagger)]$ is a bilinear function of entries of ρ^S and W_d, thus from Sion's theorem[2], the sequence of min-max can be exchanged, so

$$\begin{aligned}
&\min_{\rho^S}\max_{\|W_d\|\leq 1}\frac{1}{2}Tr[\rho^S(K_{W_d}+K_{W_d}^\dagger)]\\
&=\max_{\|W\|\leq 1}\min_{\rho^S}\frac{1}{2}Tr[\rho^S(K_W+K_W^\dagger)]\\
&=\max_{\|W\|\leq 1}\frac{1}{2}\lambda_{\min}(K_W+K_W^\dagger),
\end{aligned} \tag{11}$$

here $\lambda_{\min}(K_W+K_W^\dagger)$ denotes the minimum eigenvalue of $K_W+K_W^\dagger$. We used a different symbol W after the exchange of min-max to emphasis that although the optimal value is not affected by the exchange of min-max, the optimal point that achieves the optimal value can change. If we substitute K_1 with K_x and K_2 with K_{x+dx}, then we get the formula to compute the Bures angle between the extended channels

$$\begin{aligned} &\cos\Theta(K_x\otimes I_A, K_{x+dx}\otimes I_A) \\ =&\min_{\rho^{SA}} F[(K_1\otimes I_A)(\rho^{SA}),(K_2\otimes I_A)(\rho^{SA})] \\ =&\max_{\|W\|\le 1}\frac{1}{2}\lambda_{\min}(K_W+K_W^\dagger), \end{aligned} \tag{12}$$

which then gives the maximal quantum Fisher information for the extended channel $K_x\otimes I_A$

$$\begin{aligned} &\max_{\rho^{SA}} J(K_x\otimes I_A(\rho^{SA})) \\ =&\lim_{dx\to 0} 8\frac{1-\cos\Theta(K_x\otimes I_A, K_{x+dx}\otimes I_A)}{dx^2} \\ =&\lim_{dx\to 0}\frac{8(1-\max_{\|W\|\le 1}\frac{1}{2}\lambda_{\min}(K_W+K_W^\dagger)}{dx^2}, \end{aligned} \tag{13}$$

where $K_W=\sum_{ij} w_{ij}F_i^\dagger(x)F_j(x+dx)$.

OPTIMAL W CAN BE NON-UNITARY

We provide an example showing that the optimal W in Eq.(13) can be non-unitary.

Consider two 8-dimensional channels with two Kraus operators each. The first channel $K_1(dx)$ has the following operators:

$$F_{11}=\mathrm{diag}(\alpha,\alpha,\alpha,\alpha,\alpha,\alpha,\alpha,\alpha) \tag{14}$$

$$F_{12}=\mathrm{diag}(\beta,-\beta,\beta,-\beta,i\beta,-i\beta,i\beta,-i\beta) \tag{15}$$

The second channel $K_2(dx)$ has the following:

$$F_{21}=F_{11} \tag{16}$$

$$F_{22}=\mathrm{diag}(\beta,\beta,-\beta,-\beta,\beta,\beta,-\beta,-\beta) \tag{17}$$

where $\alpha=\sqrt{1-dx^2}$ and $\beta=|dx|$ are real. Note that these two channels can be regarded as two neighbouring channels parameterized continuously in x, $\phi(x)=\begin{cases} K_1(x) & x\le 0 \\ K_2(x) & x>0 \end{cases}$.

We let $W=\begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22}\end{bmatrix}$, and compute $\frac{1}{2}\lambda_{\min}(K_W+K_W^\dagger)$, where $K_W=w_{11}F_{11}^\dagger F_{21}+w_{12}F_{11}^\dagger F_{22}+w_{21}F_{12}^\dagger F_{21}+w_{22}F_{12}^\dagger F_{22}$. First, note that if $W=\begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix}$, then $\frac{1}{2}\lambda_{\min}(K_W+K_W^\dagger)=\alpha^2$.

Now consider general W, since all matrices $F_{1i}^\dagger F_{2j}$ are diagonal in this case, we will only need to look at the diagonal elements. The diagonal elements of K_W are

$$\alpha(w_{11}\alpha+w_{12}\beta)\ +\beta(w_{21}\alpha+w_{22}\beta) \tag{18}$$

$$+\qquad -\qquad + \tag{19}$$

$$-\qquad +\qquad - \tag{20}$$

$$-\qquad -\qquad - \tag{21}$$

$$+\qquad -i\qquad + \tag{22}$$

$$+\qquad +i\qquad + \tag{23}$$

$$-\qquad -i\qquad - \tag{24}$$

$$-\qquad +i\qquad - \tag{25}$$

Thus, $Tr(K_W) = 8\alpha^2 w_{11}$, thus $Tr(\frac{K_W+K_W^\dagger}{2}) = 8\alpha^2 Re(w_{11})$. If $Re(w_{11}) < 1$, the average eigenvalue of $\frac{K_W+K_W^\dagger}{2}$ will be smaller than α^2 thus the minimum eigenvalue will also be smaller than α^2, which is worse than the case of $W = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, thus not optimal. So for optimal W, w_{11} must be 1. With $w_{11} = 1$, the most general W with $\|W\| \leq 1$ is $W = \begin{bmatrix} 1 & 0 \\ 0 & re^{i\theta} \end{bmatrix}$ with $0 \leq r \leq 1$. The diagonal elements of K_W are then $\alpha^2 \pm re^{i\theta}\beta^2$, $\alpha^2 \pm ire^{i\theta}\beta^2$. And the diagonal elements of $\frac{K_W+K_W^\dagger}{2}$ are

$$\alpha^2 \pm \beta^2 r\cos(\theta), \alpha^2 \pm \beta^2 r\sin(\theta).$$

If $r > 0$, one of them must be less than α^2. So the optimal choice of W in this case is $W = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, which is not unitary. And it is easy to check that the minimum value α^2 for $F[(K_1(dx) \otimes I_A)(\rho^{SA}), (K_2(dx) \otimes I_A)(\rho^{SA})]$ in this case can be achieved by preparing the input state as the maximally entangled state $\frac{1}{\sqrt{8}}\sum_{i=1}^{8}|ii\rangle$.

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