

Multi-mode ultra-strong coupling in circuit quantum electrodynamics: Supplementary material

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S1. MEASUREMENT SETUP

Supplementary Fig. S1 shows schematically the measurement setup used for the device characterization. The vector network analyser (VNA) outputs one or two continuous wave (CW) signals that are sent through a variable attenuator (0-120 dB) and combined with a directional coupler. From there the signal is sent into the dilution fridge, where it is attenuated (48 dB) before reaching the sample through a circulator. The reflected signal from the device is sent back to the VNA using two isolators and amplifiers.

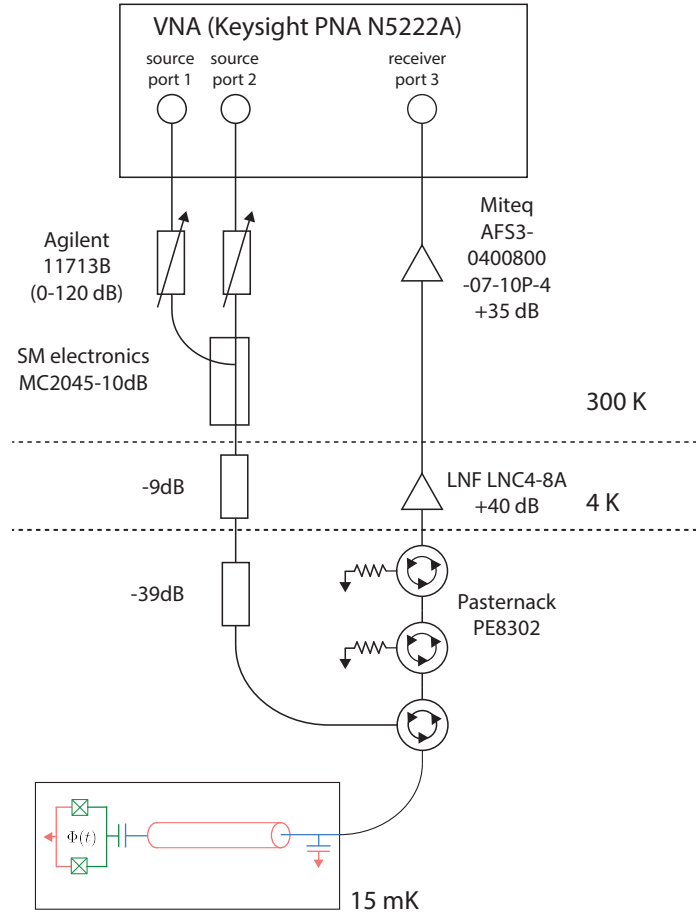


FIG. S1. Schematic of the measurement setup.

S2. SUPPLEMENTARY SPECTRA

Supplementary Fig. S2 shows the spectrum of device A (155 nm vacuum gap capacitor). The maximum frequency of the qubit is ~ 12 GHz, and the first harmonic of the cavity is at ~ 14 GHz, leading to only a single vacuum Rabi splitting in this device. The second energy level of the fundamental mode of the resonator is indicated in the blue dashed lines (labeled $2\omega_1/2\pi$), as the corresponding state hybridizes with the $|g\rangle \rightarrow |f\rangle$ transition we observed in Fig. 2(c) in the main text.

Supplementary Fig. S3 shows the response of device B (350 nm vacuum gap capacitor) in single-tone microwave reflectometry. From our model, we can attribute the faint lines bending downwards (opposite to the upper branch of the VRS mirrored in the line ω_1) to the dressed frequency of the resonator with the Cooper pair box in the excited state $|e\rangle$ (in this flux region the Josephson energy is too small to call the qubit a transmon). The response of the circuit also goes to the bare resonator frequency at half a flux quantum indicating that the junctions are also very symmetric in this device.

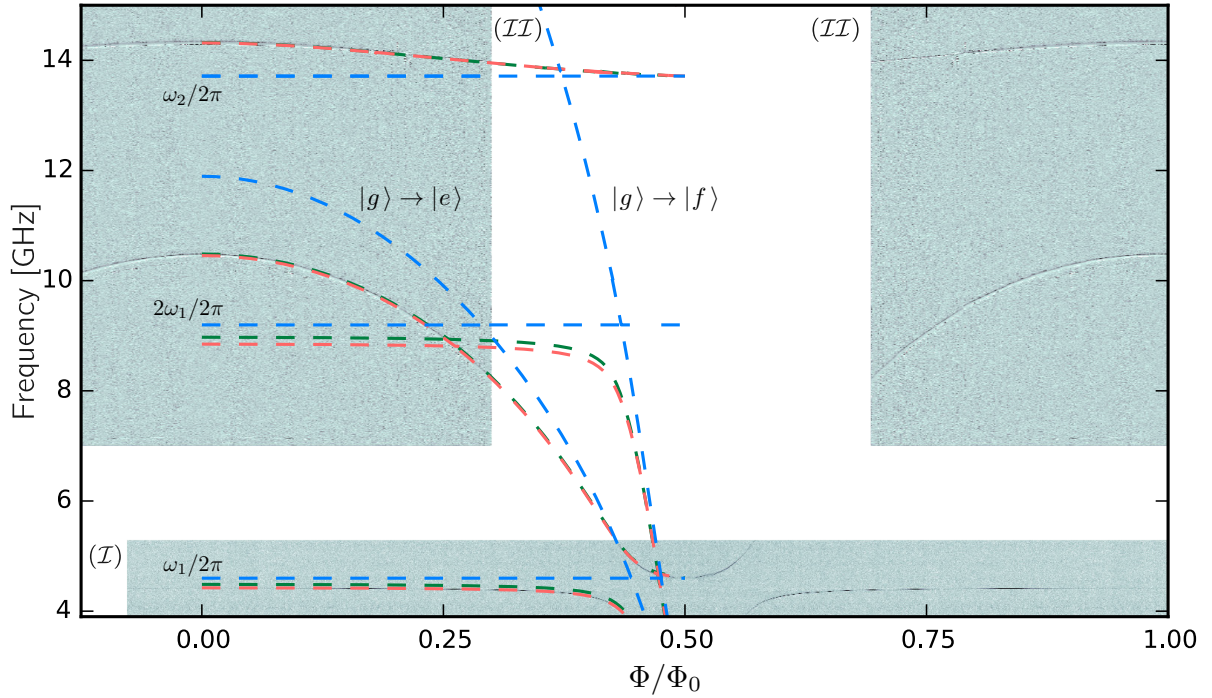


FIG. S2. Spectrum of device A, as represented as Fig. 3 in the main text for device B.

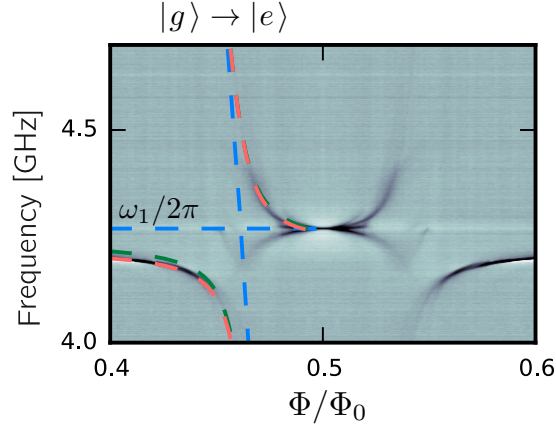


FIG. S3. Single-tone spectrum of device B, as represented as Fig. 2 in the main text for device A.

S3. MODEL

In this section we discuss the modeling of the device shown schematically in Fig. S4. In the GHz regime we are working in ($\omega/2\pi > 3.5$ GHz), the impedance of the $C_s = 30$ pF shunt capacitor is small ($|1/i\omega C_s| \simeq 2 \Omega \ll Z_0 = 50 \Omega$) such that it can effectively be considered as a short to ground [S1]. We can then use the model derived in the associated theoretical paper [S2], adding however two elements. First, the flux dependent Josephson term (see Ref. [S3])

$$-E_J \cos(\hat{\delta}) \rightarrow -E_J \cos(\hat{\delta}) \cos\left(\pi \frac{\Phi}{\Phi_0}\right) - dE_J \sin(\hat{\delta}) \sin\left(\pi \frac{\Phi}{\Phi_0}\right), \quad (\text{S1})$$

where $\hat{\delta}$ is the superconducting phase difference across the SQUID. We defined the total Josephson energy by $E_J = E_{J,1} + E_{J,2}$ and the asymmetry by $d = (E_{J,1} - E_{J,2})/(E_{J,1} + E_{J,2})$. Secondly, we offset the quantum number of Cooper pairs on the transmon island by a constant value n_{DC} to model an environmental offset charge present in any realistic system in the Cooper pair box regime [S3].

The Hamiltonian diagonalization is performed in two steps, first a diagonalization of the Cooper pair box Hamiltonian in the charge basis, secondly of an extended Rabi Hamiltonian. For more details, see the supplementary material of Ref.[S2]. In order to perform numerics, four simulation parameters should be fixed:

1. A maximum number of Cooper pairs on the transmons charge island N_{max}
2. The number of transmon levels N_q
3. The number of resonator modes N_{modes} ...
4. ...each with a certain number of photons $\{N_m\}_{m=0, \dots, N_{\text{modes}}-1}$

The size of the Hilbert space used for the diagonalization of the Cooper pair box Hamiltonian scales with $2N_{\text{max}} + 1$ and for the diagonalization of the extended Rabi Hamiltonian it scales with $N_q \prod_m N_m$. We should therefore have a small enough Hilbert space such that the diagonalizations

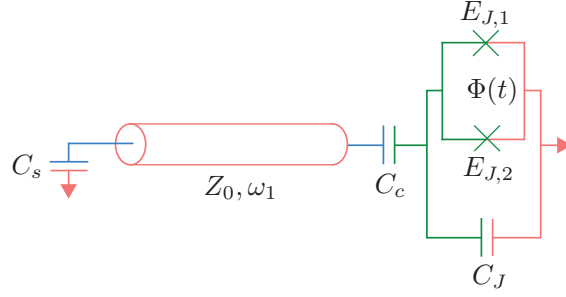


FIG. S4. Complete circuit model of our device. The transmon is a charge island connected to ground through a flux-biased SQUID and a capacitance C_J and connected to a $\lambda/4$ resonator through a capacitance C_c .

are feasible with the computer resources at our disposal, whilst ensuring that the neglected degrees of freedom do not significantly change the spectrum if we would have included them. We consider that a degree of freedom which changes the computed spectrum by less than a tenth of the measured linewidth can be neglected. This condition leads to the following simulation parameters $N_{\text{max}} = 20$, $N_{\text{modes}} = 4$, $N_q = 5$. The number of photon levels to include depends on the strength of the coupling and we have used 6, 4, 3 and 3 (5, 3, 2 and 2) photon levels in the fundamental, first, second and third modes of the resonator for device A (B). The considerable number of transmon levels needed illustrates how different the explored Hamiltonian is from that of the Rabi model. We show the impact of higher levels in Fig. S5, where the spectrum of Fig. 2(a) is plotted for different values of N_q .

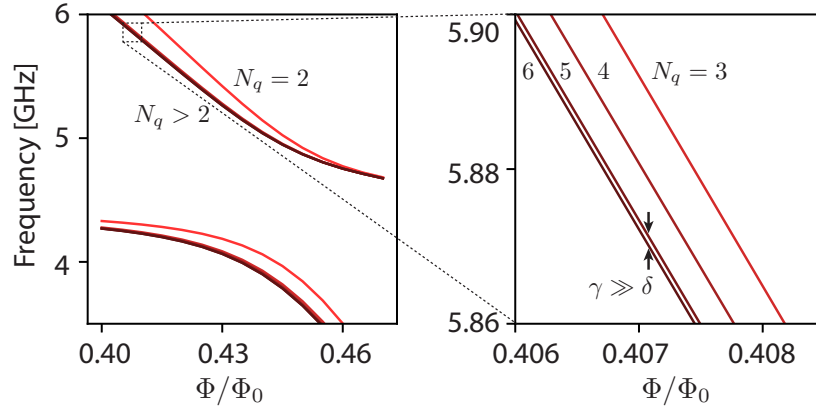


FIG. S5. Dressed spectrum shown in Fig. 2(a) of the main text illustrating the importance of higher transmon levels in obtaining the correct spectrum. The spectrum is plotted for different values of the number of included Transmon levels $N_q = 2, \dots, 6$. Including a 6-th level leads to a difference in the spectrum δ which is more than 10 times smaller than the measured linewidth. In the right panel, the relevant linewidth is that of the qubit $\gamma \sim 2\pi \times 20$ MHz and $\delta \sim 2\pi \times 1.2$ MHz.

S4. FITTING

S1. Fitting routine

We first experimentally perform a broad flux-dependent single tone measurement of the dressed cavity in order to extract the current periodicity and the point of maximum qubit frequency. This allows us to convert current (that we apply to a coil to bias the SQUID) to the flux through the SQUID (in units of flux quantum). Nine free parameters are to be determined for both devices: $\omega_1, C_c, C_J, n_{\text{DC}}, E_J, d$. Due to the effect of the shunt capacitor (neglected in the model) we also correct the frequency of the higher modes leading to three more parameters: α_n , such that $\omega_m = \alpha_m(2m - 1)\omega_1$, $m \geq 1$. We expect $\alpha_m \simeq 1$ and shall maintain this hypothesis until the last stage of the fit. Two other parameters can be fixed easily:

1. Through a high power single-tone measurement, we can drive the (dressed) fundamental mode of the resonator to its harmonic regime [S4] and determine the bare resonator frequency ω_1 .
2. For both devices, at a flux of $\Phi_0/2$, the dressed cavity frequency resumes its bare frequency which indicates that the coupling is negligible at that flux point. In simulation, this can only be achieved for very low asymmetry $d \simeq 0.01$.

We then perform a least-square minimization routine, fitting the extracted data – the dressed first and second cavity mode, and dressed first transmon transition frequency, as a function of flux – to a diagonalization of the Hamiltonian. The free parameters in this routine are $C_c, C_J, n_{\text{DC}}, E_J$. Due to the considerable time necessary to perform the diagonalization (approximately ten seconds per flux point on a commercial laptop computer), we parallelize the diagonalization over the different flux points using a high performance computing cluster. Finally we adjust the frequencies of the higher modes through the parameters α_m .

S2. Differentiating E_c and E_J

Since the frequency of the first transmon transition is approximated by $\hbar\omega_a \simeq \sqrt{8E_cE_J} - E_c$, simply fitting the dressed first transmon transition leads to an imprecision in the estimation of E_c and E_J . Namely different combinations of E_c and E_J lead to good fits to the data. In order to lift this indeterminacy, we measure the second level of the transmon (and thus the anharmonicity of the device) and insure that the fitted parameters accurately predict its dressed frequency. We used this approach for device B (see fig. S6) reproducing previously observed results [S5].

We do not have a measurement of the second transmon transition of device A (now unmeasurable). The two SQUIDs were however fabricated on the same wafer, simultaneously, and with the same design parameters. We will therefore consider that the Josephson energies of the two devices should be equal. Whilst this fact weakens the predictive power of our fit of device A, it does not impact our estimation of the coupling.

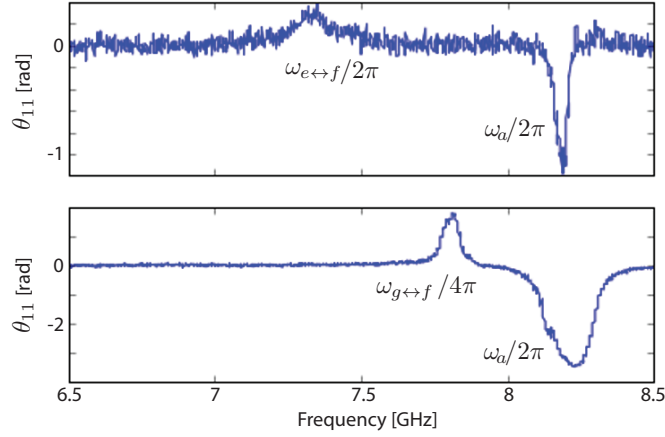


FIG. S6. In the top panel we measure the excited to second excited transition frequency of the transmon by performing two-tone spectroscopy whilst exciting the qubit with a third drive tone. In the lower panel we perform a similar experiment however substituting the third tone with high probe tone powers. On the y -axis we plot $\theta_{11}(\omega)$ the phase of the cavity response at ω_1 , denoted by $\tilde{S}_{11}(\omega)$ in the main text. This allows us to measure the two-photon process which excites the second excited transmon state from the ground state. The transition frequencies are shifted and broadened for strong probe powers (observed previously, for example in [S6] Fig. 8.7), but the measured anharmonicity is constant. The two measurements were performed at the same flux point (0.3752 flux quantum). They both provide a measurement of the anharmonicity and the low power three-tone measurement also provides an exact measurement of the ground to excited and second excited state at that particular flux point.

S3. Fitted Device Parameters

In supplementary tables S1, S2 and S3 we tabulate all the device characteristics, either fixed or extracted from the fit of our model as described above.

Quantity	Symbol	Device A	Device B	unit
resonator parameters				
bare fundamental frequency	$\omega_1/2\pi$	4.603	4.268	GHz
internal quality factor	Q_{int}	3800	9600	
external quality factor	Q_{ext}	2600	4100	
resonator impedance	Z_0	50	50	Ω
harmonics deviation	$[\alpha_m]_m$	[1.0, 0.994, 1.0, 1.0]	[1.0, 0.983, 0.983, 1.0]	

TABLE S1. Parameters of the $\lambda/4$ microwave resonators of the device.

Quantity	Symbol	Device A	Device B	unit
transmon parameters				
charging energy	E_c/h	426	700	MHz
Josephson energy at 0 flux	$E_J(0)/h$	36.3	37.4	GHz
transmon parameter at 0 flux	$E_J(0)/E_c$	78.6	53.4	
maximum qubit frequency at 0 flux	$\omega_a/2\pi$	10.67	13.74	GHz
transmon capacitance to ground	C_J	5.13	8.73	fF
vacuum-gap capacitor	C_c	40.3	18.9	fF
total capacitance	C_Σ	45.4	27.7	fF
capacitance participation ratio	β	0.89	0.68	
distance of vacuum-gap capacitor	d	155	330	nm

TABLE S2. Transmon qubit parameters.

Quantity	Symbol	Device A	Device B	unit
circuit QED parameters				
Rabi splitting with ω_1	$\Delta_1/2\pi$	1.19	0.63	GHz
Coupling ratio with ω_1	$\Delta_1/2\omega_1$	0.13	0.07	
Rabi splitting with ω_2	$\Delta_2/2\pi$	none	1.82	GHz
Coupling ratio with ω_2	$\Delta_2/2\omega_2$	none	0.07	
Coupling at 0 flux	$g(0)/2\pi$	897	568	MHz
Coupling ratio at 0 flux with ω_1	g/ω_1	0.19	0.13	

TABLE S3. Coupling parameters.

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