SUPPLEMENTARY INFORMATION

Theorem 1 (Creating maximally entangled pairs)**.** *We can create an EPR pair between two nodes a and b of an arbitrary graph state using the X-protocol with at most as many measurements as with the repeater protocol.*

Before stating the proof of Theorem 1 we introduce some additional notation. For two subsets $A, B \subseteq V$ of vertices we denote by

$$E(A,B) := \{(a,b) : a \in A, b \in B, a \neq b\} \tag{1}$$

the set of all possible edges between the two sets. Note that $E(A,B)$ in general contains edges that are not contained in the edgeset of an arbitrary given graph $G = (V,E)$. For a vertex subset $W \subseteq V$ we denote by $E_{|W}$ the subset of E that contains every edge that connects to at least one vertex in W. We may subtract a set of edges F from E. That is, by $E \setminus F$ we denote the set of edges in E that are not contained in F. For such a second set of edges F we also define the *symmetric difference* of E and F as

$$E\Delta F := (E \cup F) \setminus (E \cap F). \tag{2}$$

If F happens to be a subset of E the symmetric difference $E\Delta F$ is identical to $E \setminus F$. Otherwise the common edges are removed from the union of the two sets.

If the qubit associated with vertex $v \in V$ of a graph state is X-measured, the transformation of the corresponding edge set E can be described in terms of symmetric differences (cf. Eq. (37) in Ref. [19]). Independent of a choice $w \in N_v$, the new edge set is given by

$$\left(E\Delta E_{vw}\Delta E_{v\cap w}\Delta E_{v\setminus w}\right) \setminus E_{|\{v\}}, \tag{3}$$

where $E_{vw} := E(N_w, N_v)$, $E_{v\cap w} := E(N_w \cap N_v, N_w \cap N_v)$ and $E_{v\setminus w} := E(\{w\}, N_v \setminus \{w\})$ are introduced as a shorthand notation. The subtraction of the set containing only the edges that connect to the vertex itself at the end of Eq. (3) represents the isolation of v due to the measurement. Given a distinct pair of vertices $a, b \in V$, a *path* of length k from a to b is an ordered list $(v_1, v_2, \ldots, v_k)$ such that $v_1 = a$, $v_k = b$, and for all $i \in [k-1]$, vertices v_i and v_{i+1} are adjacent. We denote by l the length of a shortest path from a to b within the graph at hand.

In the following we will describe how the neighborhoods N_{v_i} of vertices v_i change due to Pauli measurements on the graph state. To indicate that the graph and therefore some neighborhoods may have changed, we make use of an additional index t. By $N_{v_i}^{(t)}$ we denote the neighborhood of node v_i after the t^{th} Pauli measurement on the initially given graph state. We carry this notation over for symmetric differences. In the expression $E^{(t)}(\cdot,\cdot)$ the t indicates that all involved neigborhoods are regarded after the t^{th} Pauli measurement. From the context it will always be obvious which nodes are measured in which step. In particular

$$N_{v_1}^{(0)} \cup N_{v_2}^{(0)} \cup \ldots \cup N_{v_k}^{(0)} \tag{4}$$

is the joint neighborhood of a path $(v_1, v_2, \ldots, v_k)$ in the initially given graph before any measurements are made. In our proof we compare two measurement algorithms that both have the goal of establishing an EPR pair between the nodes a and b of a given graph state.

- The *repeater protocol* selects the shortest path connecting a to b that has the minimum combined neighborhood. Every node that lies in the combined neighborhood of this path but not on the path itself is then Z-measured. This isolates the path from the rest of the graph creating a repeater line. Finally, every intermediate vertex on the line is X-measured yielding the EPR pair between the two nodes.

- The *X-protocol* measures the intermediate vertices along the same shortest path in the X basis. Subsequently, the neighborhoods of the two nodes are Z-measured to create the desired EPR pair.

Note that the X-protocol and the repeater protocol both allow to create EPR pairs due to the same inductive argument. Given nodes $a, b \in N_c$ that are not initially connected by an edge, the X-measurement of node c results in an edge between a and b. In particular, we find that (a,b) is not contained in any of the sets E, E_{ca}, $E_{c\cap a}$, $E_{|\{c\}}$. Equation 3 and $(a,b) \in E_{c\setminus a}$ then imply $(a,b) \in \left(E\Delta E_{ca}\Delta E_{c\cap a}\Delta E_{c\setminus a}\right) \setminus E_{|\{c\}}$. X-measuring along a shortest path therefore inductively establishes an edge between the first and the last node of the path.

We start our comparison by counting the number of measurements required in the repeater protocol. Along the minimal neighborhood path $(v_1, v_2, \ldots, v_l)$ connecting $v_1 = a$ to $v_l = b$, every neighboring vertex that does not lie on the path itself is measured in the Z basis. This requires $|N_{v_1}^{(0)} \cup N_{v_2}^{(0)} \cup \ldots \cup N_{v_l}^{(0)}| - l$ measurements. To obtain the desired EPR pair from this newly-created repeater line we then measure the vertices $v_2, v_3, \ldots, v_{l-1}$ in the X basis. These $l-2$ measurements remove the intermediate nodes of the path one by one. The repeater protocol thus requires

$$|N_{v_1}^{(0)} \cup N_{v_2}^{(0)} \cup \ldots \cup N_{v_l}^{(0)}| - 2 \tag{5}$$

Pauli measurements in total to establish the EPR pair.

In order to prove Theorem 1, we will now count the number of measurements required when using the X-protocol. The protocol starts by X-measuring along the path $(v_2, \ldots, v_{l-1})$. Here, t indicates the X-measurement of node v_{t+1} and $N_{v_i}^{(t)}$ is the neighborhood of v_i after the t^{th} X-measurement. We will need the following observation.

Observation (Minimizing measurements)**.** *To prove Theorem 1, it suffices to show*

$$N_a^{(l-2)} \cup N_b^{(l-2)} \subsetneq N_{v_1}^{(0)} \cup N_{v_2}^{(0)} \cup \ldots \cup N_{v_l}^{(0)} \tag{6}$$

and that we can find at least $l-2$ elements in the neighborhood of the initial (before any measurement) path between a and b that are not contained in the neighborhoods of a and b after the X-measurements of the X-protocol.

In total, the X-protocol requires $(l-2)$ X-measurements along the shortest path, and subsequently $|N_a^{(l-2)} \cup N_b^{(l-2)}| - 2$ Z-measurements on those vertices that have connecting edges to a or b (we need to subtract a and b from the count). From Eq. (5) it follows that Theorem 1 holds if

$$|N_a^{(l-2)} \cup N_b^{(l-2)}| + l - 2 \leq |N_{v_1}^{(0)} \cup N_{v_2}^{(0)} \cup \ldots \cup N_{v_l}^{(0)}|. \tag{7}$$

Therefore, in order to prove Theorem 1, it is sufficient to show that Eq. (6) is fulfilled.

Proof. In the following we will examine how the neighborhoods of $a = v_1$ and $b = v_l$ change with the sequence of X-measurements along the shortest path. The first such measurement is at vertex v_2. The measurement results in a new graph state with the same set of vertices and with an edge set that can be calculated via a series of symmetric differences, according to Eq. (3),

$$\left(E \Delta E_{v_1 v_2}^{(0)} \Delta E_{v_1 \cap v_2}^{(0)} \Delta E_{v_2 \setminus v_1}^{(0)}\right) \setminus E_{|\{v_2\},\cdot}. \tag{8}$$

By definition of $E(\cdot,\cdot)$ we find

$$E_{v_1 v_2}^{(0)} = \left\{(x_1, x_2) : x_i \in N_{v_i}^{(0)}, i = 1, 2; x_1 \neq x_2\right\}, \tag{9}$$

$$E_{v_1 \cap v_2}^{(0)} = \left\{(x, y) : x, y \in N_{v_1}^{(0)} \cap N_{v_2}^{(0)}, x \neq y\right\}, \tag{10}$$

$$E_{v_2 \setminus v_1}^{(0)} = \left\{(v_1, x_2) : x_2 \in N_{v_2}^{(0)} \setminus \{v_1\}\right\}. \tag{11}$$

In the following we analyse the consecutive symmetric differences in Eq. (8) step by step. In particular, we are interested in how the X-measurement on v_2 changes the neighborhoods of $a = v_1$ and $b = v_l$. Since we have $v_1 \in N_{v_2}^{(0)}$, the set $E_{v_1 v_2}^{(0)}$ contains all edges that where connected to v_1 before the measurement. From Eq. (9) we can thus infer that $N_{v_1}^{(1)}$ does not contain any of the elements that where previously contained in $N_{v_1}^{(0)}$. The second symmetric difference in Eq. (8) does not alter the neighborhood of the starting vertex v_1 by virtue of Eq. (10) and v_1 not being in the intersection of $N_{v_1}^{(0)}$ and $N_{v_2}^{(0)}$. If follows that the only contribution to $N_{v_1}^{(1)}$ comes from Eq. (11). We find

$$N_{v_1}^{(1)} = N_{v_2}^{(0)} \setminus \{v_1\} \tag{12}$$

and ascertain that the new graph after the X-measurement on v_2 has a path $(v_1, v_3, v_4, \ldots, v_l)$ of length $l-1$ connecting $a = v_1$ and $v_l = b$. We note that this new, shorter path is again a shortest path between a and b, since the X-measurement only alters the neighborhood of the measured node. The vertex v_2 is now isolated, that is, there are no edges that connect it to the remaining graph. The following measurements will remove the other intermediate vertices from the path one by one.

The next X-measurement on v_3 yields $N_{v_1}^{(2)} = N_{v_3}^{(1)} \setminus \{v_1\}$ and finally after the t^{th} measurement, it holds that

$$N_{v_1}^{(t)} = N_{v_{t+1}}^{(t-1)} \setminus \{v_1\}. \tag{13}$$

We now examine how the neighborhood of v_{t+2} is changed by the t^{th} measurement. Before we write down the general expression for $N^{(t)}_{v_{t+2}}$, we consider the special case $t = 1$, that is, the environment of vertex v_3 after the measurement on v_2. Again, Eq. (10) does not contribute to $N^{(1)}_{v_3}$, because $v_3 \in N^{(0)}_{v_1} \cap N^{(0)}_{v_2}$ would be a contradiction to $(v_1, v_2, \ldots, v_l)$ being a shortest path before the first measurement. Via Equation (9) we add those elements of $N^{(0)}_{v_1}$ that have not previously been connected to v_3 and remove those that were. Compared to $N^{(0)}_{v_3}$, the neighborhood $N^{(1)}_{v_3}$ also gains the element v_1 by virtue of Eq. (11), since v_3 is certainly an element of $N^{(0)}_{v_2} \setminus \{v_1\}$. To sum up this gives $N^{(1)}_{v_3} = \{v_1\} \cup \left(N^{(0)}_{v_3} \cup N^{(0)}_{v_1}\right) \setminus \left(N^{(0)}_{v_3} \cap N^{(0)}_{v_1}\right)$ and thus

$$N^{(t)}_{v_{t+2}} = \{v_1\} \cup \left(N^{(t-1)}_{v_{t+2}} \cup N^{(t-1)}_{v_1}\right) \setminus \left(N^{(t-1)}_{v_{t+2}} \cap N^{(t-1)}_{v_1}\right) \tag{14}$$

for the general case after the t^{th} measurement. For any $t = 2, 3, \ldots, l-2$ we can combine Eqs. (13) and (14) and obtain the expression

$$N^{(t)}_{v_1} = \left(N^{(t-2)}_{v_{t+1}} \cup N^{(t-2)}_{v_1}\right) \setminus \left(N^{(t-2)}_{v_{t+1}} \cap N^{(t-2)}_{v_1}\right) \tag{15}$$

In particular we can now write recursive expressions for $N^{(l-2)}_a$ and $N^{(l-2)}_b$. More specifically, we obtain

$$\begin{aligned} N^{(l-2)}_{v_1} &= \left(N^{(l-4)}_{v_{l-1}} \cup N^{(l-4)}_{v_1}\right) \setminus \left(N^{(l-4)}_{v_{l-1}} \cap N^{(l-4)}_{v_1}\right), \\ N^{(l-2)}_{v_l} &= \{v_1\} \cup \left(N^{(l-3)}_{v_l} \cup N^{(l-3)}_{v_1}\right) \setminus \left(N^{(l-3)}_{v_l} \cap N^{(l-3)}_{v_1}\right). \end{aligned} \tag{16}$$

Eqs. (16) contain multiple expressions of the type $N^{(t)}_{v_{t+3}}$. These expressions can easily be simplified. For instance the case $t = 1$ entails $N^{(1)}_{v_4} = N^{(0)}_{v_4}$, since $v_4 \notin N_{v_i}$ for $i = 1, 2$. Otherwise this would be a contradiction to $(v_1, v_2, \ldots, v_l)$ being the shortest path before the first measurement. By the same argument we recursively get

$$N^{(t)}_{v_{t+3}} = N^{(t-1)}_{v_{t+3}} = \ldots = N^{(0)}_{v_{t+3}} \tag{17}$$

for all $t = 1, 2, \ldots, l-3$. Together with Eqs. (15) and (16), this recursively implies that we can write $N^{(l-2)}_a$ and $N^{(l-2)}_b$ as the union and intersection of sets of the type $N^{(0)}_{v_i}$ and $N^{(1)}_{v_i}$ where v_i is a vertex on the initial shortest path. For all v_i, except for $i = 1, 2, 3$, we have $N^{(0)}_{v_i} = N^{(1)}_{v_i}$, since the negation would imply $(v_1, v_2, \ldots, v_l)$ not being the shortest path before the first measurement. The neighborhood of v_2 is empty after the first measurement. Now, Eq. (12) and $N^{(1)}_{v_3} = \{v_1\} \cup \left(N^{(0)}_{v_3} \cup N^{(0)}_{v_1}\right) \setminus \left(N^{(0)}_{v_3} \cap N^{(0)}_{v_1}\right)$ imply

$$N^{(l-2)}_{v_1} \cup N^{(l-2)}_{v_l} \subsetneq N^{(0)}_{v_1} \cup N^{(0)}_{v_2} \cup \ldots \cup N^{(0)}_{v_l}. \tag{18}$$

The subset relation is proper, because the $l-2$ vertices $v_2, v_3, \ldots, v_{l-1}$ are contained in $N^{(0)}_{v_1} \cup N^{(0)}_{v_2} \cup \ldots \cup N^{(0)}_{v_l}$ but not in $N^{(l-2)}_{v_1} \cup N^{(l-2)}_{v_l}$. This implies

$$|N^{(l-2)}_{v_1} \cup N^{(l-2)}_{v_l}| + l - 4 \leq |N^{(0)}_{v_1} \cup N^{(0)}_{v_2} \cup \ldots \cup N^{(0)}_{v_l}| - 2, \tag{19}$$

which concludes the proof. □

Now we will prove Lemma 1 and thereby show the equivalence of successive X-measurements to Z-measurements on a graph in the LC orbit.

Lemma 1 (Equivalence of measurements)**.** *X-measurements along a shortest path between two nodes are equivalent to performing a series of local complementations on the path, followed by Z-measurements on the intermediate nodes.*

Proof. An X-measurement of a node is equivalent to locally complementing a neighbor, then locally complementing the actual node and Z-measuring, followed by a final local complementation of the same neighbor (cf. Eq. (37) in Ref. [19]). Suppose that the nodes v_i, $i = 1, \ldots, n$ constitute a shortest path. We denote by X_i and Z_i the X- and Z-measurements on node i respectively, and by LC_i the action of local complementation with respect to the node v_i. Then

$$X_2 = LC_1\ Z_2\ LC_2\ LC_1 \tag{20}$$

is a valid decomposition of X_2 in terms of local complementations and Z-measurements. If there is no shorter path connecting v_1 and v_3, this means that the X_2 measurement (and more specifically LC_2) creates a link between v_1 and v_3. Therefore, when we measure X_3, we can again choose v_1 as a neighbor and find

$$X_3 = LC_1 \; Z_3 \; LC_3 \; LC_1. \tag{21}$$

Continuing along the path, we finally find that

$$X_{n-1} \cdots X_2 = LC_1 \; Z_{n-1} \; LC_{n-1} \cdots Z_3 \; LC_3 \; Z_2 \; LC_2 \; LC_1, \tag{22}$$

since two consecutive local complementations with respect to the same vertex cancel each other out. However, Z_i commutes with LC_j since measuring in Z removes the node and all adjacent edges. If $i \in N_j$, it does not matter whether a local complementation will connect v_i with any other node or not, since all connections will disappear after the measurement. We can therefore push all Z-measurements to the end and obtain

$$X_{n-1} \cdots X_2 = Z_{n-1} \cdots Z_2 \; LC_1 \; LC_{n-1} \cdots LC_1 \tag{23}$$

to conclude the proof. □

Now, building upon the X-protocol, we give a proof of Corollary 1 by a short case analysis.

Corollary 1 (Extraction of GHZ3 states)**.** *We can always distill a* 3*-partite GHZ state between arbitrary vertices of a connected graph state in polynomial time.*

Proof. Again we take $(a = v_1, v_2, \ldots, v_l = b)$ to be a shortest path in the initial graph. If $c \in N_{v_1}^{(0)} \cup N_{v_2}^{(0)} \cup \ldots \cup N_{v_l}^{(0)}$ lies in the neighborhood of the chosen path, there are two subcases.

- If $c = v_i$ for some $i \in \{1, 2, \ldots, l\}$, we measure the vertices $v_2, v_3, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{l-1}$ in the X-basis. After these $l-3$ measurements every vertex in $N_a^{(l-3)} \cup N_b^{(l-3)} \cup N_c^{(l-3)} \setminus \{a, b, c\}$ is measured in the Z-basis.
- If $c \neq v_i$ for all $i \in \{1, 2, \ldots, l\}$, we measure the vertices $v_2, v_3, \ldots, v_{l-1}$ in the X-basis. After these $l-2$ measurements vertex c is certainly contained in $N_a^{(l-2)} \cup N_b^{(l-2)}$. Every vertex but c from this set is then measured in the Z-basis.

If $c \notin N_{v_1}^{(0)} \cup N_{v_2}^{(0)} \cup \ldots \cup N_{v_l}^{(0)}$, we again measure the nodes $v_2, v_3, \ldots, v_{l-1}$ in the X-basis. Without loss of generality let the shortest path $b = w_1, w_2, \ldots, w_{l'} = c$ be shorter than all the paths from a to c. We continue by measuring all vertices in $N_a^{(l-2)}$ in the Z-basis followed by $w_2, w_3, \ldots, w_{l'-1}$ in the X-basis. Note that $w_i \in N_a^{(l-2)}$ for some $i \in \{2, 3, \ldots, l'\}$ would be a contradiction to the shortest path assumption. Finally, we measure every vertex but a that lies in the neighborhoods of b and c in the Z-basis. All of the above cases result in the desired 3-partite GHZ state between a, b and c independent of the choice of paths. □

Finally, we turn towards the generation of 4-partite GHZ states as stated in Proposition 3.

Proposition 3 (Extraction of GHZ4 states)**.** *We can always distill a* 4*-partite GHZ state from graph states when their underlying graph (i) is a repeater line, with at least one extra node between two pairs of the final GHZ4 nodes, or (ii) contains such a line as a vertex-minor.*

Proof. To be consistent with the figure in the main text, suppose that we want to have a GHZ4 state between nodes with labels $1, 2, 4$ and 5 in the original graph state. If the underlying graph has a repeater line as a vertex-minor, we may separate it from the remaining graph state via appropriate local complementations and measurements. By local complementation on the path and measurement on the nodes that are not part of the final GHZ4, we can always distill the required state, as seen in the figure in the main text. For a large subset of graph states there is however a more efficient way to generate the desired GHZ4 state in analogy to the X-protocol. Without isolating the repeater line first, we may perform local complementations with respect to nodes $2, 3, 4$ followed by Z-measurements on 3 and on every vertex that is connected to any of $1, 2, 4, 5$ and does not lie on the repeater-line itself. If, for example, there is no shorter path connecting 1 to 5 initially than the one given by the repeater line, the successive local complementations result in graphs that have a subgraph like the ones displayed in the figure in the main text.

□