

Supplementary information: Synthesizing multi-phonon quantum superposition states using flux-mediated three-body interactions with superconducting qubits

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(Dated: October 3, 2019)

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S1. LAGRANGIAN-HAMILTONIAN DESCRIPTION OF THE CIRCUIT

The Lagrangian describing the electromechanical system in Fig.1(a) is

$$\mathcal{L} = \frac{m\dot{X}^2}{2} - \frac{m\omega_m^2 X^2}{2} + \sum_{i=1}^2 \left[\frac{1}{2} C_i \dot{\phi}_i^2 + E_{J_i} \cos \left(\frac{\phi_i}{\phi_0} \right) \right] + \frac{1}{2} C_c (\dot{\phi}_1 - \dot{\phi}_2)^2 + E_J^c \cos \left(\frac{\phi_1 - \phi_2}{\phi_0} \right), \quad (\text{S1})$$

where X , ϕ_i are variables representing the beam displacement and the flux on circuit node i [S1], respectively, and $\phi_0 = \hbar/2e$ is the reduced flux quantum. C_i , C_c and E_{J_i} , E_J^c denote the capacitances and Josephson energies of each transmon and the coupler, respectively, and m , ω_0 are the mass and frequency of the beam. Following a Legendre transformation $H = \sum_i \phi_i Q_i - \mathcal{L}$ we obtain the system Hamiltonian

$$H = \frac{P^2}{2m} + \frac{m\omega_m^2 X^2}{2} + \sum_{i=1}^2 \left[\frac{Q_i^2}{2\tilde{C}_i} - E_{J_i} \cos \left(\frac{\phi_i}{\phi_0} \right) \right] + \frac{C_c}{C_1 C_2} Q_1 Q_2 - E_J^c \cos \left(\frac{\phi_1 - \phi_2}{\phi_0} \right), \quad (\text{S2})$$

where $P = \frac{\partial \mathcal{L}}{\partial \dot{X}}$ is the mechanical conjugate momentum and $Q_i = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i}$ are the electrical conjugate momenta representing charges on each circuit node. $\tilde{C}_2 = C_2 + \frac{C_1 C_c}{C_1 + C_c}$ and $\tilde{C}_1 = C_1 + \frac{C_2 C_c}{C_2 + C_c}$ denote the modified transmon capacitances due to the coupling capacitance C_c .

A. Motion-dependent flux-tuneable Josephson energy

If the two junctions of the SQUID are identical, the total Josephson energy is $E_J^c(\Phi_b) = E_{J,\Sigma}^c |\cos(\pi\Phi_b/\Phi_0)|$, where $E_{J,\Sigma}^c$ is the sum of the Josephson energies of each junction. In the presence of an in-plane magnetic field, B , the SQUID loop can pick up an additional flux, $\beta_0 B l X$, due the beam displacement, where l is the length of the suspended arm and β_0 is a geometric constant associated with the mode shape of the beam (for the first mechanical mode $\beta_0 \sim 1$) [S2, S3]. Therefore, the flux- and motion-dependent Josephson energy reads

$$E_J^c = E_{J,\Sigma}^c |\cos(\pi\Phi_b/\Phi_0 + \alpha X)|, \quad (\text{S3})$$

where $\alpha = \pi\beta_0 B l / \Phi_0$ and $\Phi_0 = \hbar/2e$ is the magnetic flux quantum. Using basic trigonometry and expanding to lowest order in X , assuming $\pi\Phi_b/\Phi_0 \gg \alpha X$, we find

$$E_J^c \simeq E_{J,\Sigma}^c [\cos(\pi\Phi_b/\Phi_0)(1 - \alpha^2 X^2) - \sin(\pi\Phi_b/\Phi_0)\alpha X], \quad (\text{S4})$$

The above approximation holds for $\alpha X_{\text{ZPF}} \ll 1$, which is valid for the beams considered in similar experiments and small magnetic fields ($B < 1$ T) [S3].

In the above, the case of a symmetric SQUID $E_{J,1}^c = E_{J,2}^c$ is considered, however in realistic devices a finite asymmetry needs to be taken into account. The Josephson energy is therefore more accurately described by

$$E_J^c(\Phi_b, X) = E_{J,\Sigma}^c [\cos^2(\pi\Phi_b/\Phi_0 + \alpha X) + a_J^2 \sin^2(\pi\Phi_b/\Phi_0 + \alpha X)]^{1/2}, \quad (\text{S5})$$

where $a_J = |(E_{J,1}^c - E_{J,2}^c)/(E_{J,1}^c + E_{J,2}^c)|$ is the SQUID asymmetry [S4]. For $\pi\Phi_b/\Phi_0 \gg \alpha X$, we have

$$\begin{aligned} \cos^2(\pi\Phi_b/\Phi_0 + \alpha X) &\simeq \cos^2(\pi\Phi_b/\Phi_0) - 2\alpha X \cos(\pi\Phi_b/\Phi_0) \sin(\pi\Phi_b/\Phi_0), \\ \sin^2(\pi\Phi_b/\Phi_0 + \alpha X) &\simeq \sin^2(\pi\Phi_b/\Phi_0) + 2\alpha X \cos(\pi\Phi_b/\Phi_0) \sin(\pi\Phi_b/\Phi_0). \end{aligned} \quad (\text{S6})$$

Substituting into Eq. (S5), yields

$$E_J^c = E_{J,\Sigma}^c [1 + a_J^2 \tan^2(\pi\Phi_b/\Phi_0)]^{1/2} \cos(\pi\Phi_b/\Phi_0) \left[1 - \alpha X \frac{(1 - a_J^2) \tan(\pi\Phi_b/\Phi_0)}{1 + a_J^2 \tan^2(\pi\Phi_b/\Phi_0)} \right]^{1/2}, \quad (\text{S7})$$

which can further be simplified, assuming $\alpha X \frac{(1 - a_J^2) \tan(\pi\Phi_b/\Phi_0)}{1 + a_J^2 \tan^2(\pi\Phi_b/\Phi_0)} \ll 1$, into

$$E_J^c \simeq E_{J,\Sigma}^c [c_J \cos(\pi\Phi_b/\Phi_0) - s_J \sin(\pi\Phi_b/\Phi_0)\alpha X] + \mathcal{O}[X^2], \quad (\text{S8})$$

where $c_J = \sqrt{1 + a_J^2 \tan^2(\pi\Phi_b/\Phi_0)}$ and $s_J = (1 - a_J^2)/c_J$. The higher order terms

$$\mathcal{O}[X^2] = E_{J,\Sigma}^c \frac{s_J \sin^2(\pi\Phi_b/\Phi_0)}{2c_J \cos(\pi\Phi_b/\Phi_0)} \alpha^2 X^2, \quad (\text{S9})$$

are considered in the analysis below, however, as we will see later, they do not affect the system dynamics for the parameters used in this work.

B. Flux-mediated interactions

Expanding the last term in Eq. (S2) up to $\mathcal{O}[\phi^4 X^2]$, in combination with Eq. (S8), yields the following flux-mediated interaction terms

$$H_{\text{int}}^{(\text{flux})} = H_{3\text{-body}} + H_{\text{RP}} + H^{\{\phi^2\}} + H^{\{\phi^4\}} + H^{\{\phi^4 X\}} + H^{\{\phi^2 X^2\}} + H^{\{\phi^4 X^2\}}. \quad (\text{S10})$$

The first term describes a three-body interaction, between the two qubits and the beam

$$H_{3\text{-body}} = \alpha E_{\text{J},\Sigma}^c s_{\text{J}} \sin(\pi \Phi_{\text{b}}/\Phi_0) \frac{\phi_1 \phi_2}{\phi_0^2} X, \quad (\text{S11})$$

while the second term

$$H_{\text{RP}} = -\alpha E_{\text{J},\Sigma}^c s_{\text{J}} \sin(\pi \Phi_{\text{b}}/\Phi_0) \frac{(\phi_1^2 + \phi_2^2)}{2\phi_0^2} X, \quad (\text{S12})$$

describes a radiation-pressure type coupling of the mechanical mode with each qubit.

The next two terms describe flux-mediated qubit-qubit interactions, where

$$H^{\{\phi^2\}} = -E_{\text{J},\Sigma}^c c_{\text{J}} \cos(\pi \Phi_{\text{b}}/\Phi_0) \frac{\phi_1 \phi_2}{\phi_0^2}, \quad (\text{S13})$$

is a linear dipole coupling term, and

$$H^{\{\phi^4\}} = E_{\text{J},\Sigma}^c c_{\text{J}} \cos(\pi \Phi_{\text{b}}/\Phi_0) \left(-\frac{\phi_1^2 \phi_2^2}{4\phi_0^4} + \frac{\phi_1^3 \phi_2 + \phi_1 \phi_2^3}{6\phi_0^4} \right), \quad (\text{S14})$$

is a nonlinear interaction including cross-Kerr as well as pair- and correlated-hopping terms.

The last three terms

$$H^{\{\phi^4 X\}} = \alpha E_{\text{J},\Sigma}^c s_{\text{J}} \sin(\pi \Phi_{\text{b}}/\Phi_0) \frac{(\phi_1 - \phi_2)^4}{24\phi_0^4} X, \quad (\text{S15})$$

$$H^{\{\phi^2 X^2\}} = -\alpha^2 E_{\text{J},\Sigma}^c \cos(\pi \Phi_{\text{b}}/\Phi_0) \frac{(\phi_1 - \phi_2)^2}{2\phi_0^2} X^2, \quad (\text{S16})$$

and

$$H^{\{\phi^4 X^2\}} = \alpha^2 E_{\text{J},\Sigma}^c \cos(\pi \Phi_{\text{b}}/\Phi_0) \frac{(\phi_1 - \phi_2)^4}{24\phi_0^4} X^2, \quad (\text{S17})$$

describe nonlinear tripartite interactions, which are much weaker and do not contribute significantly to the dynamics compared to the leading order electromechanical terms of Eq. (S11) and (S12).

In addition, the last term in Eq. (S2) leads to corrections in the bare subsystem Hamiltonians. More specifically, the inductive energy term

$$E_{\text{J},\Sigma}^c c_{\text{J}} \cos(\pi \Phi_{\text{b}}/\Phi_0) \left(\frac{\phi_1^2 + \phi_2^2}{2\phi_0^2} - \frac{\phi_1^4 + \phi_2^4}{24\phi_0^4} \right), \quad (\text{S18})$$

results in an effective qubit Josephson energy

$$\tilde{E}_{\text{J}_i} = E_{\text{J}_i} + E_{\text{J},\Sigma}^c c_{\text{J}} \cos(\pi \Phi_{\text{b}}/\Phi_0), \quad (\text{S19})$$

while the potential energy term

$$E_{\text{J},\Sigma}^c \alpha X s_{\text{J}} \sin(\pi \Phi_{\text{b}}/\Phi_0) \quad (\text{S20})$$

leads to a displaced rest position

$$X_0 = \frac{\alpha E_{\text{J},\Sigma}^c s_{\text{J}} \sin(\pi \Phi_{\text{b}}/\Phi_0)}{2m\omega_{\text{m}}}. \quad (\text{S21})$$

The latter does not affect the dynamics and can be absorbed in a redefinition of $X \rightarrow (X - X_0)$.

S2. CIRCUIT QUANTISATION

We now switch to a quantum mechanical description of the circuit, promoting all canonical variables to quantum operators

$$\begin{aligned}\hat{X} &= X_{\text{ZPF}} (\hat{b} + \hat{b}^\dagger), \quad \hat{P} = P_{\text{ZPF}} i(\hat{b}^\dagger - \hat{b}), \\ \hat{\phi}_i &= \sqrt{\frac{\hbar Z_i}{2}} (\hat{c}_i + \hat{c}_i^\dagger), \quad \hat{Q}_i = \sqrt{\frac{\hbar}{2Z_i}} i(\hat{c}_i^\dagger - \hat{c}_i),\end{aligned}\tag{S22}$$

where $\hat{c}_i^{(\dagger)}$ and $\hat{b}^{(\dagger)}$ are ladder operators describing the annihilation (creation) of photons and phonons, respectively, and satisfy bosonic commutation relations $[\hat{c}_i, \hat{c}_j^\dagger] = \delta_{ij}$ and $[\hat{b}, \hat{b}^\dagger] = 1$. The zero-point fluctuations in the mechanical displacement and momentum are given by $X_{\text{ZPF}} = \sqrt{\hbar/(2m\omega_m)}$ and $P_{\text{ZPF}} = \sqrt{\hbar m\omega_m/2}$, respectively. $Z_i = \frac{\hbar}{e^2} \sqrt{E_{C_i}/2\tilde{E}_{J_i}}$ denotes the impedance of each transmon, where $E_{C_i} = \frac{e^2}{2C_i}$ is its charging energy. Since the qubits are in the transmon regime [S4], $\tilde{E}_{J_i} \gg E_{C_i}$, the uncoupled electromechanical system is well-described by

$$\hat{H}_0 = \hbar\omega_m \hat{b}^\dagger \hat{b} + \sum_{i=1}^2 \hbar\omega_i \hat{c}_i^\dagger \hat{c}_i - \frac{E_{C_i}}{2} \hat{c}_i^\dagger \hat{c}_i^\dagger \hat{c}_i \hat{c}_i,\tag{S23}$$

with qubit frequencies $\omega_i = \frac{1}{\hbar} \left(\sqrt{8\tilde{E}_{J_i} E_{C_i}} - E_{C_i} \right)$.

A. Tripartite coupling

Following Eq. (S11) we can now write the Hamiltonian operator describing the three-body interaction

$$\begin{aligned}\hat{H}_{3\text{-body}} &= \hbar g (\hat{c}_1 + \hat{c}_1^\dagger)(\hat{c}_2 + \hat{c}_2^\dagger)(\hat{b} + \hat{b}^\dagger), \\ &= \hbar g (\hat{c}_1^\dagger \hat{c}_2 + \hat{c}_1 \hat{c}_2^\dagger)(\hat{b} + \hat{b}^\dagger),\end{aligned}\tag{S24}$$

where, in the second step, we have made a rotating-wave approximation (RWA) to neglect fast rotating terms $\hat{c}_1 \hat{c}_2 \hat{b}^{(\dagger)}$ and $\hat{c}_1^\dagger \hat{c}_2^\dagger \hat{b}^{(\dagger)}$, which do not contribute to the dynamics since $\omega_1, \omega_2 \gg \omega_m, g$. The tripartite coupling strength is given by

$$g = \frac{\alpha \sqrt{Z_1 Z_2}}{2\phi_0^2} E_{J,\Sigma}^c \sin(\pi \Phi_b / \Phi_0) X_{\text{ZPF}}.\tag{S25}$$

B. Radiation-pressure couplings

The radiation-pressure interaction between each qubit and the beam, in Eq. (S12), can be expressed in second quantisation form as

$$\hat{H}_{\text{RP}} = \hbar g_1 \hat{c}_1^\dagger \hat{c}_1 (\hat{b} + \hat{b}^\dagger) + \hbar g_2 \hat{c}_2^\dagger \hat{c}_2 (\hat{b} + \hat{b}^\dagger),\tag{S26}$$

following a RWA. The radiation-pressure coupling strengths are given by

$$g_i = \frac{\alpha Z_i}{2\phi_0^2} E_{J,\Sigma}^c \sin(\pi \Phi_b / \Phi_0) X_{\text{ZPF}}.\tag{S27}$$

C. Qubit-qubit couplings

Following circuit quantisation and a RWA, the linear qubit-qubit interaction Hamiltonian in Eq. (S13) becomes

$$H^{\{\phi^2\}} = -\hbar J_L (\hat{c}_1^\dagger \hat{c}_2 + \hat{c}_1 \hat{c}_2^\dagger) \quad (\text{S28})$$

where

$$J_L = \frac{\sqrt{Z_1 Z_2}}{2\phi_0^2} E_{J,\Sigma}^c \cos(\pi \Phi_b / \Phi_0), \quad (\text{S29})$$

The qubits also couple via their charge degrees of freedom ($Q_1 Q_2$ term in Eq. (S2)), which results in the same type of linear interaction with coupling strength

$$J_C = \frac{C_c}{2C_1 C_2} (Z_1 Z_2)^{-1/2}. \quad (\text{S30})$$

Combining these together leads to an overall exchange-type interaction

$$H_{12} = \hbar(J_C - J_L) (\hat{c}_1^\dagger \hat{c}_2 + \hat{c}_1 \hat{c}_2^\dagger), \quad (\text{S31})$$

which can be suppressed at the desired operating point (Φ_b) with the right choice of coupling capacitor C_c .

The nonlinear interaction between the two transmons in Eq. (S14) is given by

$$\begin{aligned} \hat{H}^{\{\phi^4\}} = & \frac{\hbar V}{4} (\hat{c}_1^\dagger + \hat{c}_1)^2 (\hat{c}_2^\dagger + \hat{c}_2)^2 \\ & + \frac{\hbar J_{n_1}}{3} (\hat{c}_1^\dagger + \hat{c}_1)^3 (\hat{c}_2^\dagger + \hat{c}_2) \\ & + \frac{\hbar J_{n_2}}{3} (\hat{c}_1^\dagger + \hat{c}_1) (\hat{c}_2^\dagger + \hat{c}_2)^3. \end{aligned} \quad (\text{S32})$$

The first term, following a RWA, results in a cross-Kerr interaction, $V \hat{c}_1^\dagger \hat{c}_1 \hat{c}_2^\dagger \hat{c}_2$, with coupling strength

$$V = -\frac{\hbar Z_1 Z_2}{4\phi_0^4} E_{J,\Sigma}^c \cos(\pi \Phi_b / \Phi_0). \quad (\text{S33})$$

Additionally the same term yields a pair-hopping interaction $\frac{V}{4} (\hat{c}_1^\dagger \hat{c}_1 \hat{c}_2^\dagger \hat{c}_2 + \text{H.c.})$, which however is not contributing to the dynamics for single-excitation levels. Finally, the other two terms in Eq. (S32) result in correlated hopping interactions $[\hat{c}_1^\dagger (J_{n_1} \hat{c}_1^\dagger \hat{c}_1 + J_{n_2} \hat{c}_2^\dagger \hat{c}_2) \hat{c}_2 + \text{H.c.}]$, with

$$J_{n_{1(2)}} = -\frac{\hbar \sqrt{Z_{1(2)}^3 Z_{2(1)}}}{24\phi_0^4} E_{J,\Sigma}^c \cos(\pi \Phi_b / \Phi_0), \quad (\text{S34})$$

which contribute to the linear coupling in Eq. (S31) as $J \rightarrow J + J_{n_1} + J_{n_2}$.

D. Higher-order tripartite interactions

The next-to-leading order electromechanical interactions are given by

$$\begin{aligned} \hat{H}^{\{\phi^4 X\}} = & \hbar g_{\phi_1^2 \phi_2^2 x} (\hat{c}_1^\dagger + \hat{c}_1)^2 (\hat{c}_2^\dagger + \hat{c}_2)^2 (\hat{b} + \hat{b}^\dagger) \\ & + \hbar g_{\phi_1^3 \phi_2 x} (\hat{c}_1^\dagger + \hat{c}_1)^3 (\hat{c}_2^\dagger + \hat{c}_2) (\hat{b} + \hat{b}^\dagger) \\ & + \hbar g_{\phi_1 \phi_2^3 x} (\hat{c}_1^\dagger + \hat{c}_1) (\hat{c}_2^\dagger + \hat{c}_2)^3 (\hat{b} + \hat{b}^\dagger), \end{aligned} \quad (\text{S35})$$

with nonlinear coupling strengths

$$g_{\phi_1^2 \phi_2^2 x} = \frac{\hbar \alpha Z_1 Z_2}{16\phi_0^4} E_{J,\Sigma} \sin(\pi \Phi_b / \Phi_0) X_{\text{ZPF}}, \quad (\text{S36})$$

$$g_{\phi_1^3\phi_2x} = \frac{\hbar\alpha Z_1^{3/2}Z_2^{1/2}}{24\phi_0^4}E_{J,\Sigma}\sin(\pi\Phi_b/\Phi_0)X_{\text{ZPF}}, \quad (\text{S37})$$

and

$$g_{\phi_1\phi_2^3x} = \frac{\hbar\alpha Z_1^{1/2}Z_2^{3/2}}{24\phi_0^4}E_{J,\Sigma}\sin(\pi\Phi_b/\Phi_0)X_{\text{ZPF}}. \quad (\text{S38})$$

The last two terms also lead to a small correction of the tripartite coupling strength $g \rightarrow g + 3g_{\phi_1^3\phi_2x} + 3g_{\phi_1\phi_2^3x}$. These terms, although weaker, are not negligible (since they are of the same order or larger than dissipation rates) and they are include in simulating the system dynamics.

The $\mathcal{O}[\phi^2X^2]$ terms in Eq. (S16) can be written as,

$$\hat{H}\{\phi^2X^2\} = \sum_{i,j}\hbar g_{\phi_i\phi_jx^2}\hat{c}_i^\dagger\hat{c}_j(\hat{b} + \hat{b}^\dagger)^2 \quad (\text{S39})$$

where

$$g_{\phi_i\phi_jx^2} = \frac{\alpha^2\sqrt{Z_iZ_j}s_J\tan(\pi\Phi_b/\Phi_0)}{2\phi_0^2c_J}E_{J,\Sigma}\sin(\pi\Phi_b/\Phi_0)X_{\text{ZPF}}^2. \quad (\text{S40})$$

For the parameters used in this work ($\alpha X_{\text{ZPF}} \sim 10^{-6}$) the coupling is negligible and does not affect the system dynamics (the same holds for the even weaker $\mathcal{O}[\phi^4X^2]$ terms). Interestingly, however, for $i = j$ we find a dispersive (cross-Kerr) interaction $\hat{c}_i^\dagger\hat{c}_i\hat{b}^\dagger\hat{b}$ of each qubit with the resonator, which is active even when the qubits are far detuned and could potentially be employed for phonon-sensitive measurements of the mechanical state [S5].

S3. PROTOCOLS FOR ARBITRARY QUANTUM STATE GENERATION

A. States with arbitrary complex coefficients

In the main text we demonstrated a protocol for creating multi-phonon quantum superposition states by alternating the qubit frequencies such that different parts of the tripartite interaction become resonant. By controlling the interaction times and post-selecting on the qubit state at the end of the protocol, we showed the possibility of creating interesting classes of superposition states with high fidelity. However, this protocol alone is not sufficient for generating mechanical superposition states $|\psi\rangle = \sum_{n=0}^N c_n|n_m\rangle$ with arbitrary complex coefficients c_n , since there is no phase degree of freedom to control in the protocol. This is due to the fact that the coupling constant g is not an adjustable complex number and in practical implementations the interaction time is the only parameter that can be varied.

However, there is another tuning knob that can be employed by taking advantage of the qubit-qubit exchange-type interaction $J(\hat{c}_1^\dagger\hat{c}_2 + \hat{c}_1\hat{c}_2^\dagger)$. Detuning the qubits by $\Delta = |\omega_1 - \omega_2| \gg \omega_m$, while tuning the coupling SQUID such that there is a finite exchange-type coupling strength $J \geq \Delta$, results in a resonant qubit-qubit interaction that couples $|0_11_2\rangle$ and $|1_10_2\rangle$. For example, starting from $|0_11_2\rangle$ one would end up with state $[\cos(Jt)|0_11_2\rangle - i\sin(Jt)|1_10_2\rangle]$ after time t (at $t = \pi/(2J)$ this realises a SWAP gate). Furthermore, by detuning the qubit frequencies such that $\Delta > J$ it is also possible to introduce a relative phase between the qubit states $|0_11_2\rangle$ and $|1_10_2\rangle$ (C-Phase gate), as in Ref. [S6]. Combining this gate with the resonant qubit-qubit interaction, acting on $|0_11_2\rangle$ would result in

$$U_J(t, \theta)|0_11_2\rangle = \cos(Jt)|0_11_2\rangle - ie^{i\theta}\sin(Jt)|1_10_2\rangle. \quad (\text{S41})$$

We note that our system, is analogous to the one studied theoretically by Law and Eberly [S7] and experimentally by Hofheinz et al. [S8]. The system studied in these references considers a resonator that is controllably coupled to a qubit (with local qubit driving) via a resonant exchange-type interaction (Jaynes-Cummings). It is shown that an arbitrary resonator state can be generated by interleaving the Jaynes-Cummings evolution with qubit driving. The two-qubit states $|0_11_2\rangle$ and $|1_10_2\rangle$ in our system can be mapped to the single qubit basis $|0\rangle$ and $|1\rangle$ considered in Law and Eberly. Furthermore, the tripartite interaction can be mapped to the qubit-resonator Jaynes-Cummings interaction, where in our system we additionally have the possibility of realising the equivalent of a counter-rotating Jaynes-Cummings interaction by exchanging the qubit frequencies. The equivalent of qubit driving can then be performed by controlling the evolution under the exchange qubit-qubit interaction U_J described in Eq. (S41).

We will consider the case where an arbitrary state $|\psi\rangle_m = \sum_{n=0}^N a_n|n_m\rangle$ is generated following post-selection from the arbitrary entangled state $|\psi\rangle = \sum_{n=0}^{N-1}(a_n|0_1n_m1_2\rangle + b_n|1_1n_m0_2\rangle) + a_N|0_1N_m1_2\rangle$. That is, if we can create the

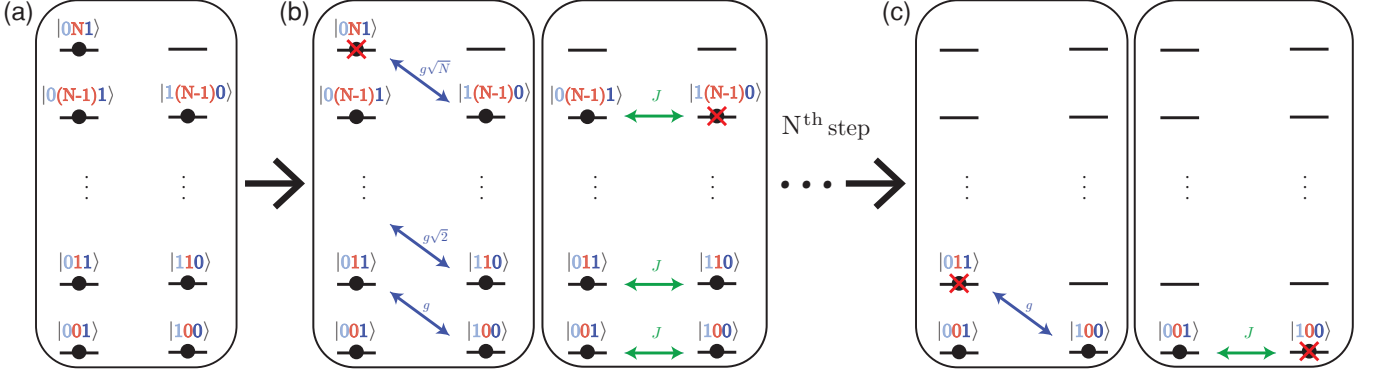


FIG. S1. **Protocol for arbitrary state generation.** (a) The protocol is calculated in reverse by emptying an arbitrary entangled state $|\psi\rangle = \sum_{n=0}^{N-1} (a_n|0_1 n_m 1_2\rangle + b_n|1_1 n_m 0_2\rangle) + a_N|0_1 N_m 1_2\rangle$ (from this state it is always possible to obtain an arbitrary mechanical state by post-selecting on $|0_1 1_2\rangle$). (b) The first step of the protocol is shown, which consists of two substeps: At first we turn on the (inverse of the) C-Phase gate and the tripartite interaction $\hat{U}(t) = e^{i(\hat{c}_1^\dagger \hat{b} \hat{c}_2 + \text{H.c.})t}$ such that all population from $|0_1 N_m 1_2\rangle$ is emptied to $|1_1(N-1)_m 0_2\rangle$. Then, by applying another C-Phase gate in combination with resonant qubit-qubit interaction for a variable time it is possible to empty $|1_1(N-1)_m 0_2\rangle$ to $|0_1(N-1)_m 1_2\rangle$. (c) Following this procedure N times it is possible to empty the original state in (a) and end up in $|0_1 0_m 1_2\rangle$.

above state $|\psi\rangle$ with arbitrary complex coefficients, it is then straightforward to collapse it to $|\psi\rangle_m$ following post-selection on the $|0_1 1_2\rangle$ qubit state. Inspired by Refs. [S7, S8], let us now consider the problem of generating the arbitrary state $|\psi\rangle$ from the inverse point of view, i.e. by proving that it is always possible to empty it. Suppose we start from $|\psi\rangle = \sum_{n=0}^{N-1} (a_n|0_1 n_m 1_2\rangle + b_n|1_1 n_m 0_2\rangle) + a_N|0_1 N_m 1_2\rangle$ (Fig. S1a). Then by applying the (inverse of the) C-Phase gate and the tripartite interaction $\hat{U}^\dagger(t) = e^{i(\hat{c}_1^\dagger \hat{b} \hat{c}_2 + \text{H.c.})t}$ for phase θ_A and time t_A , the probability amplitude of state $|0_1 N_m 1_2\rangle$ becomes

$$a_N \rightarrow a_N \cos(g\sqrt{N}t_A) + ie^{\theta_A} b_{N-1} \sin(g\sqrt{N}t_A). \quad (\text{S42})$$

By appropriately choosing θ_A and t_A , it is possible to make the above probability amplitude zero, such that all phonons occupying the state $|0_1 N_m 1_2\rangle$ are transferred to $|1_1(N-1)_m 0_2\rangle$ (see Fig. S1b).

The step above also results in incomplete transfer of phonons between states $|0_1 n_m 1_2\rangle$ and $|1_1(n-1)_m 0_2\rangle$ (for $n < N$), after which we are left with the state $|\psi'\rangle = \sum_{n=0}^{N-1} (a'_n|0_1 n_m 1_2\rangle + b'_n|1_1 n_m 0_2\rangle)$. Now by combining the (inverse of the) C-Phase gate and resonant qubit-qubit interaction $U_J^\dagger(t_J, \theta_J)|\psi'\rangle$ the probability amplitude of state $|1_1(N-1)_m 0_2\rangle$ becomes

$$b_{N-1} \rightarrow b_{N-1} \cos(Jt_J) + ie^{\theta_J} a_{N-1} \sin(Jt_J), \quad (\text{S43})$$

which can be made zero by appropriately choosing θ_J and t_J such that all phonons occupying $|1_1(N-1)_m 0_2\rangle$ are transferred to $|0_1(N-1)_m 1_2\rangle$ (see Fig. S1b). By applying the above two steps N times it is possible to completely empty the initial state, leading the system to $|0_1 0_m 1_2\rangle$ (Fig. S1c). Therefore, reversing the problem, it is possible to generate any arbitrary mechanical state $|\psi\rangle_m = \sum_{n=0}^N a_n|n_m\rangle$ by creating the arbitrary entangled state

$$\prod_{j=N}^1 \hat{U}_{\theta_{A_j}} \hat{U}(t_{A_j}) \hat{U}_J(t_{J_j}, \theta_{J_j}) |0_1 0_m 1_2\rangle = \sum_{n=0}^{N-1} (a_n|0_1 n_m 1_2\rangle + b_n|1_1 n_m 0_2\rangle) + a_N|0_1 N_m 1_2\rangle, \quad (\text{S44})$$

and post-selecting on $|0_1 1_2\rangle$.

B. States with arbitrary phonon number probability distributions

The protocol presented in the previous section can enable the creation of any mechanical quantum state with arbitrary coefficients. Although this protocol is experimentally feasible, it can become complex as it requires a lot of additional tuning to realise the C-Phase and exchange-type gates between the qubits. Here we describe an alternative protocol that only employs the tripartite interactions, therefore requiring only alternating between the

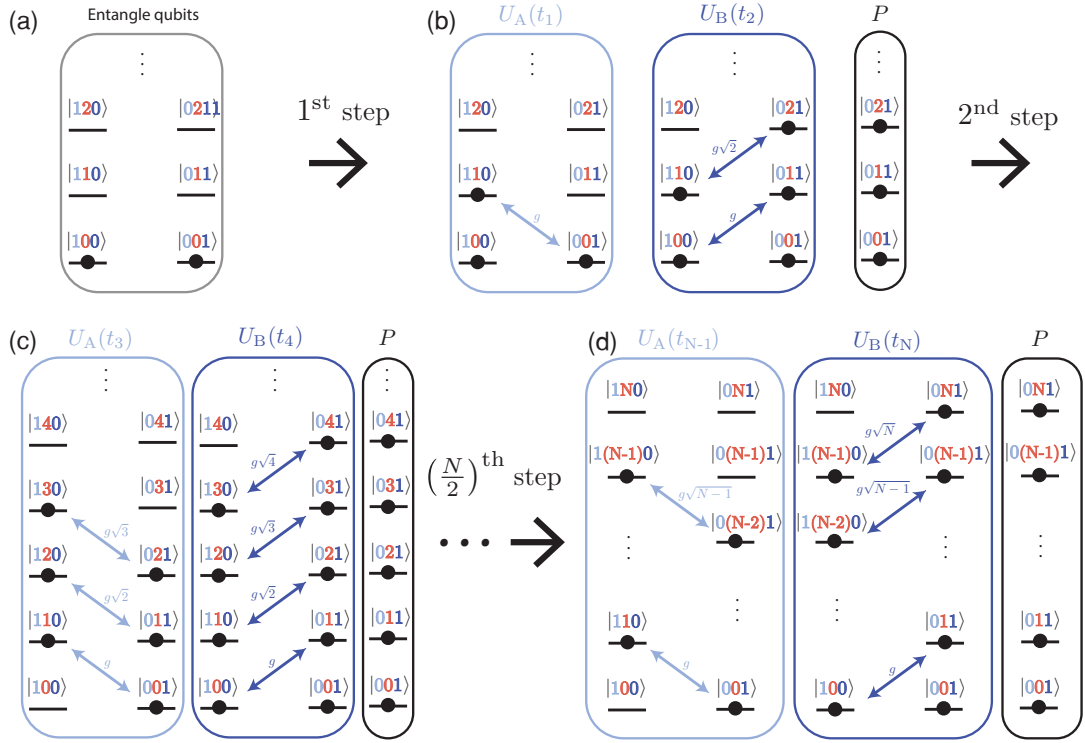


FIG. S2. **Protocol for generating states with arbitrary phonon number probability distributions using flux pulsing and projective measurements.** (a) Assuming the mechanical resonator is in the ground state, we can start the protocol with the qubits in an entangled state. This can be achieved by letting the qubits interact for a variable time without interacting with the resonator, e.g. by tuning the system such that the condition $J \geq |\omega_1 - \omega_2| \gg \omega_m$ is satisfied. (b) The first step of the protocol is shown, which consists of three substeps: At first we turn on the interaction $\hat{H}_A = \hat{c}_1^\dagger \hat{b}^\dagger \hat{c}_2 + \text{H.c.}$, by bringing the system in the resonance condition $\omega_2^+ = \omega_1 + \omega_m$, for variable time t_1 (light blue frame). Then, by tuning to $\omega_2^- = \omega_1 - \omega_m$, the system evolves according to $\hat{H}_B = \hat{c}_1 \hat{b}^\dagger \hat{c}_2^\dagger + \text{H.c.}$ for a different time t_2 (dark blue frame). Finally, we measure either qubit 1 or 2 and post-select on the outcome $|1_1 0_2\rangle$ (black frame). The freedom in choosing the coefficients of the initial two-qubit state together with the choice of interaction times t_1 and t_2 give enough degrees of freedom for generating a superposition state with arbitrary phonon number probability distribution up to the third level. (c) Second step of the protocol leading to a superposition state up to the fifth level. (d) General case for creating states with arbitrary phonon number probability distribution up to N , following $N/2$ steps.

qubit frequencies, and post-selective measurements at each step. Reducing the complexity of tuning pulses comes at the cost of not being able to create states with arbitrary complex coefficients, although it is possible to generate states with arbitrary phonon number probability distributions as we see below.

The protocol relies on initially preparing the two qubits in an arbitrary entangled state (Fig. S2a). Assuming the resonator is in the ground state, the tripartite system is initially described by the following wavefunction

$$|\psi\rangle_0 = (\alpha|0_1 1_2\rangle + \beta|1_1 0_2\rangle)|0_m\rangle, \quad (\text{S45})$$

where α, β are complex numbers that can be chosen arbitrarily and are related by $|\alpha|^2 + |\beta|^2 = 1$. As described in the previous section, one can prepare this state by activating the exchange-type qubit-qubit interaction $U_J(t, \theta)$. This can be done by detuning the qubits sufficiently such that their frequency difference is much greater than the mechanical frequency, and at the same time smaller than the direct qubit-qubit coupling, i.e. $J \geq |\omega_1 - \omega_2| \gg \omega_m$, which can be adjusted by changing the flux Φ_b on the coupling SQUID.

The next step, schematically depicted in Fig. S2b, consists of the following three substeps:

1. Flux pulse into resonance condition $\omega_2^+ = \omega_1 + \omega_m$ for variable time t_1
2. Flux pulse into resonance condition $\omega_2^- = \omega_1 - \omega_m$ for variable time t_2 ,
3. Projective measurement on one of the qubits and post-selection on $|0_1 1_2\rangle$.

The initial wavefunction is transformed into

$$|\psi\rangle_1 = \hat{P} \hat{U}_B(t_2) \hat{U}_A(t_1) |\psi\rangle_0, \quad (\text{S46})$$

where $\hat{P} = |0_1 1_2\rangle\langle 0_1 1_2|$ and $\hat{U}_{A,B}(t_{1,2}) = e^{-i\hat{H}_{A,B}t_{1,2}}$, with interaction Hamiltonians $\hat{H}_A = \hat{c}_1^\dagger \hat{b}^\dagger \hat{c}_2 + \text{H.c.}$ and $\hat{H}_B = \hat{c}_1 \hat{b}^\dagger \hat{c}_2^\dagger + \text{H.c.}$, respectively.

Below we derive the resulting wavefunction after each substep:

$$1. \hat{U}_A(t_1)|\psi\rangle_0 = \alpha(\cos(gt_1)|0_1 0_m 1_2\rangle - i\beta \sin(gt_1)|1_1 1_m 0_2\rangle) + \beta|1_1 0_m 0_2\rangle, \quad (\text{S47})$$

$$2. \hat{U}_B(t_2)\hat{U}_A(t_1)|\psi\rangle_0 = \alpha \cos(gt_1)|0_1 0_m 1_2\rangle + \beta (\cos(gt_2)|1_1 0_m 0_2\rangle + \sin(gt_2)|0_1 1_m 1_2\rangle) \\ + \alpha \sin(gt_1) \left(\cos(g\sqrt{2}t_2)|1_1 1_m 0_2\rangle + \sin(g\sqrt{2}t_2)|0_1 2_m 1_2\rangle \right) \quad (\text{S48})$$

$$3. \hat{P}\hat{U}_B(t_2)\hat{U}_A(t_1)|\psi\rangle_0 = \frac{1}{P} \left(\alpha \cos(gt_1)|0_m\rangle + \beta \sin(gt_2)|1_m\rangle + \alpha \sin(gt_1) \sin(g\sqrt{2}t_2)|2_m\rangle \right) |0_1 1_2\rangle, \quad (\text{S49})$$

where P is a normalisation factor determined by the Born rule.

Evidently, following this first step one has enough degrees of freedom (α, t_1, t_2) to create states with arbitrary phonon number probability distribution up to three levels, $|\psi\rangle_1 = \sum_{n=0}^2 c_n |n_m\rangle |0_1 1_2\rangle$. Following a second step (Fig. S2c), we find

$$\hat{P}\hat{U}_B(t_4)\hat{U}_A(t_3)|\psi\rangle_1 = |0_1 1_2\rangle \{ c_0 \cos(gt_3)|0_m\rangle + c_1 \cos(g\sqrt{2}t_3) \cos(gt_4)|1_m\rangle \\ + (c_0 \sin(gt_3) \sin(g\sqrt{2}t_4) + c_2 \cos(g\sqrt{3}t_3) \cos(g\sqrt{2}t_4)) |2_m\rangle \\ + c_1 \sin(g\sqrt{2}t_3) \sin(g\sqrt{3}t_4)|3_m\rangle + c_2 \cos(g\sqrt{4}t_4)|4_m\rangle \}. \quad (\text{S50})$$

The addition of two more variables t_3, t_4 enables the creation of states with arbitrary phonon number probability distribution up to $|4\rangle$.

Suppose we start from an arbitrary mechanical state $|\psi\rangle = \sum_{n=0}^{N-2} c_n |n_m\rangle |0_1 1_2\rangle$, which can be created by applying the above protocol $(N/2 - 1)$ times. Then, following another step we have

$$\prod_{j=N/2}^1 \hat{P}\hat{U}_B(t_{2j})\hat{U}_A(t_{2j-1})|\psi\rangle_0 = \hat{P}\hat{U}_B(t_{N-1})\hat{U}_A(t_N) \left(\sum_{n=0}^{N-2} c_n |n\rangle |0_1 1_2\rangle \right) = \sum_{n=0}^N c'_n |n\rangle |0_1 1_2\rangle, \quad (\text{S51})$$

where the new coefficients are determined by the previous ones according to the following relations:

$$c'_0 = c_0 \cos(gt_{N-1}) \quad (\text{S52})$$

$$c'_1 = c_1 \cos(g\sqrt{2}t_{N-1}) \cos(gt_N) \quad (\text{S53})$$

$$c'_2 = c_0 \sin(gt_{N-1}) \sin(g\sqrt{2}t_N) + c_2 \cos(g\sqrt{3}t_{N-1}) \cos(g\sqrt{2}t_N) \quad (\text{S54})$$

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$$c'_n = c_{n-2} \sin(g\sqrt{n-1}t_{N-1}) \sin(g\sqrt{n}t_N) + c_n \cos(g\sqrt{n+1}t_{N-1}) \cos(g\sqrt{n}t_N) \quad (\text{S55})$$

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$$c'_{N-1} = c_{N-3} \sin(g\sqrt{N-2}t_{N-1}) \sin(g\sqrt{N-1}t_N) \quad (\text{S56})$$

$$c'_N = c_{N-2} \sin(g\sqrt{N-1}t_{N-1}) \sin(g\sqrt{N}t_N). \quad (\text{S57})$$

Therefore, applying this protocol $N/2$ times can lead to the creation of states with arbitrary phonon number probability distribution up to $|N\rangle$.

S4. VALIDITY FOR NON-IDEAL SYSTEM PARAMETERS

Although in the simulations we have considered realistic parameters taken from recent experiments, we realise that in an experimental scenario system parameters such as the qubit coherence may significantly deviate from the ones

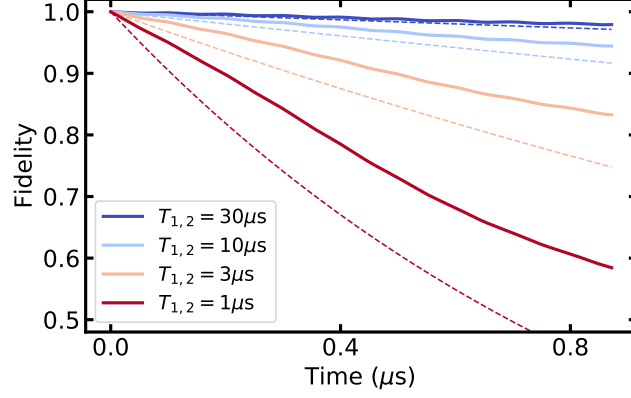


FIG. S3. **Effect of qubit coherence on quantum state preparation.** Evolution of the fidelity of the prepared state to the ideal state, $\cos(gt)|0_1 0_m 1_2\rangle - i \sin(gt)|1_1 1_m 0_2\rangle$, using the protocol presented in Fig. 3, for different values of relaxation and pure dephasing times (assuming $T_1 = T_2$). The dashed curves correspond to e^{-t/T_1} for each case.

considered in Table 1. Although bad qubit coherence may not affect the success of the cooling protocol, it can pose limits on the quantum state preparation protocols. We have therefore examined the dependence of the fidelity of the prepared quantum state in Fig. 3 to the ideal evolution $|\psi_{\text{ideal}}\rangle = U(t)|0_1 0_m 1_2\rangle = \cos(gt)|0_1 0_m 1_2\rangle - i \sin(gt)|1_1 1_m 0_2\rangle$ on the qubit coherence. In Fig. S3 we plot the evolution of the fidelity of the prepared density matrix ρ to the ideal one $\rho_{\text{ideal}} = |\psi_{\text{ideal}}\rangle\langle\psi_{\text{ideal}}|$, for different values of T_1 (assuming $T_2 = T_1$). The dashed curves correspond to e^{-t/T_1} . We find that high-fidelity state preparation ($> 90\%$) can be achieved with qubit coherence times of $T_1, T_2 \sim 10 \mu\text{s}$ which are standard in the superconducting qubit community and achievable in the presence of 10 mT magnetic fields [S9].

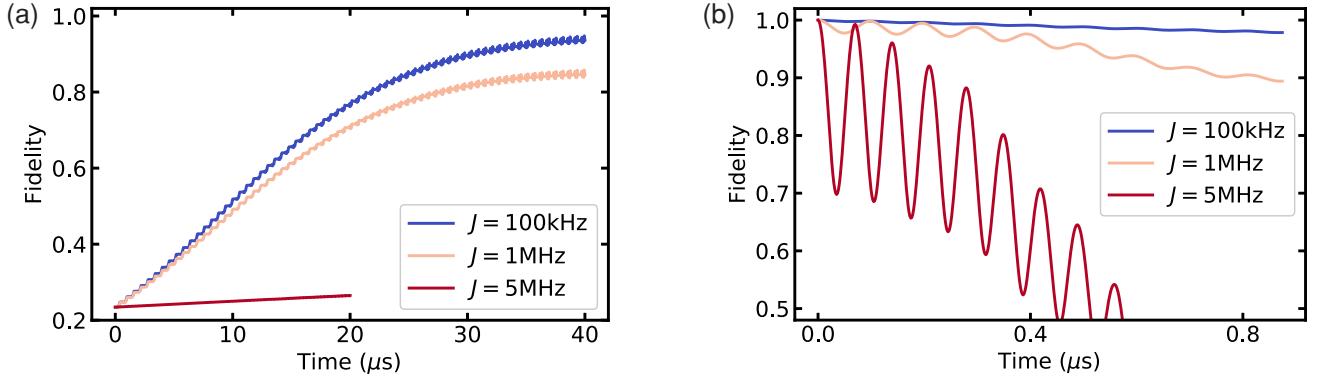


FIG. S4. **Effect of flux fluctuations and stray qubit-qubit coupling on ground-state cooling and quantum state preparation.** (a) Evolution of the fidelity of the prepared mechanical state to the vacuum state using the cooling protocol of Fig. 2, for different values of qubit-qubit coupling $J = 100 \text{ kHz}$, 1 MHz and 5 MHz , corresponding to flux bias fluctuations of $\delta\Phi_b/\Phi_0 = 10^{-6}$, 10^{-5} and 5×10^{-5} , respectively. The same number of cooling steps were used in all simulations, where for $J = 5 \text{ MHz}$ a cooling step of 100 ns instead of 200 ns was chosen as an optimal cooling time. (b) Evolution of the fidelity of the prepared state to the ideal state, $\cos(gt)|0_1 0_m 1_2\rangle - i \sin(gt)|1_1 1_m 0_2\rangle$, following the protocol used in Fig. 3, for the same variations on J and $\delta\Phi_b/\Phi_0$ as in (a).

One effect that is not considered in the main text is that of fluctuations on the flux bias channel $\delta\Phi_b$. These can occur as a result of environmental magnetic field noise or noise of the current source used for biasing the flux line. Given the stability of our current sources and assuming flux noise levels reported in similar devices [S10, S11], we estimate such fluctuations to be around $\delta\Phi_b/\Phi_0 \sim 10^{-6}$ during the course of the preparation protocols. The most important effect of flux fluctuations would be to introduce a stray qubit-qubit exchange-type coupling as $J_L \propto E_{J,\Sigma}^c \cos(\pi(\Phi_b + \delta\Phi_b)/\Phi_0)$. More specifically, adding a fluctuation of $\delta\Phi_b/\Phi_0 \sim 10^{-6}$ would result in $J_L \rightarrow J_L + 100 \text{ kHz}$, while $\delta\Phi_b/\Phi_0 \sim 10^{-5}$ and $\delta\Phi_b/\Phi_0 \sim 10^{-4}$ translate to additional coupling of 1 and 10 MHz, respectively. The latter would be detrimental for the state preparation protocols as it is of the order of the mechanical frequency. Note that the effect of flux noise on the qubit-qubit coupling is also amplified by our choice of a large coupling Josephson energy amplitude

$E_{J,\Sigma}^c/h = 200$ GHz. In Fig. S4 we plot the effect of a finite qubit-qubit coupling on the cooling and quantum state preparation protocols. We find that for $J < 1$ MHz (therefore $\delta\Phi_b < 10 \mu\Phi_0$) it is possible to maintain high-fidelity quantum state preparation. Note that despite the sharp dependence of the tripartite coupling on the flux bias (Fig. 1), this change is less than 0.1 % in the case of $\delta\Phi_b < 10 \mu\Phi_0$.

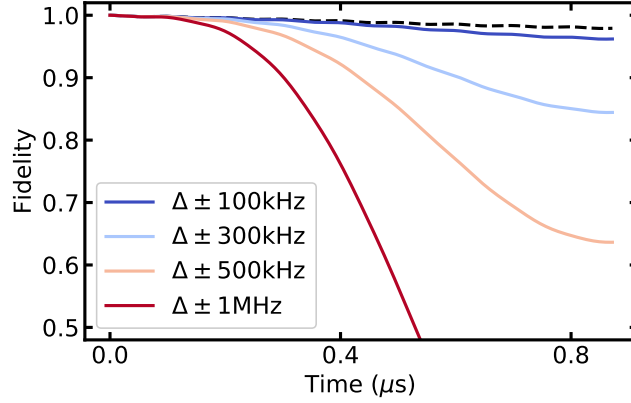


FIG. S5. **Effect of imperfect flux pulsing on quantum state preparation.** Evolution of the fidelity of the prepared state to the ideal state using the protocol presented in Fig.3, for different variations of the qubit-qubit detuning Δ . The black dashed curve corresponds to no variation, i.e. $\Delta = \omega_m$.

Additionally, in Fig. S5 we examine the effect of imperfect qubit flux pulsing on the fidelity of the prepared quantum state. We find that the fidelity of the protocol is sensitive to this parameter and that targeting the qubit frequencies within ~ 100 kHz is required for high-fidelity quantum state preparation.

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