Supplementary Information for "One-Shot Detection Limits of Quantum Illumination with Discrete Signals"

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SUPPLEMENTARY DISCUSSION 1: OPTIMIZATION OVER PURE STATES

The optimization of probe states for both quantum and conventional illumination is to maximize the trace norm of $\|\Omega_c(\rho_A)\|$ and $\|\Omega_q(\rho_{AB})\|$, where

$$\Omega_c(\rho_A) \equiv p_1\mathcal{E}_1(\rho_A) - p_0\mathcal{E}_0(\rho_A)\ , \tag{S-1}$$
$$\Omega_q(\rho_{AB}) \equiv p_1(\mathcal{E}_1\otimes\mathcal{I})(\rho_{AB}) - p_0(\mathcal{E}_0\otimes\mathcal{I})(\rho_{AB})\ . \tag{S-2}$$

For convenience, we can write these two quantity in the following form:

$$\Omega_c(\rho_A) \equiv p_1\,\eta\,\rho_A + \gamma\,\rho_E, \tag{S-3}$$
$$\Omega_q(\rho_{AB}) \equiv p_1\,\eta\,\rho_{AB} + \gamma\,\rho_E\otimes\rho_B\ , \tag{S-4}$$

where $\gamma := p_1(1-\eta) - p_0$, and $\rho_B := \mathrm{tr}[\rho_{AB}]$.

To begin with, we first show that the optimization over all pure state is sufficient.

Proposition 1. *In optimizing of the trace norm of the matrices $\Omega_c(\rho)$ or $\Omega_q(\rho)$ over all possible density matrices, we only need to maximize the set of pure states. i.e.,*

$$\max_{\rho_A\in\mathcal{H}} \|\,\Omega_c(\rho_A)\,\| = \max_{|\psi\rangle\in\mathcal{H}} \|\,\Omega_c\left(|\psi\rangle_A\langle\psi|\right)\|\ , \tag{S-5}$$

$$\max_{\rho_{AB}\in\mathcal{H}\otimes\mathcal{H}} \|\,\Omega_q(\rho_{AB})\,\| = \max_{|\psi_{AB}\rangle\in\mathcal{H}\otimes\mathcal{H}} \|\,\Omega_q\left(|\psi_{AB}\rangle\langle\psi_{AB}|\right)\|\ . \tag{S-6}$$

Proof. Let us denote,

$$\rho_* \equiv \sum\nolimits_i \lambda_i|\psi_i\rangle\langle\psi_i|\ , \tag{S-7}$$

in some diagonal basis $\{|\psi_i\rangle\}$ where all $\lambda_i \geqslant 0$ and $\sum_i\lambda_i = 1$, as the density matrix that maximizes the trace norm of Ω_c, i.e.,

$$\|\Omega_c(\rho_*)\| = \max_{\rho\in\mathcal{H}}\|\Omega_c(\rho)\|\ . \tag{S-8}$$

Ω_c is linear, which implies that

$$\Omega_c\left(\sum\nolimits_i\lambda_i|\psi_i\rangle\langle\psi_i|\right) = \sum\nolimits_i\lambda_i\Omega_c\left(|\psi_i\rangle\langle\psi_i|\right)\ , \tag{S-9}$$

and that the trace norm is convex, i.e.,

$$\left\|\sum\nolimits_i\lambda_i\Omega_c\left(|\psi_i\rangle\langle\psi_i|\right)\right\| \leqslant \sum\nolimits_i\lambda_i\left\|\Omega_c\left(|\psi_i\rangle\langle\psi_i|\right)\right\|\ . \tag{S-10}$$

We have $\|\Omega_c(\rho_*)\| \leqslant \sum_i\lambda_i\|\Omega_c(|\psi_i\rangle\langle\psi_i|)\|$, which implies the result advertised in Eq. (S-5). The case for Ω_q is similar. □

In the following, we discuss the conventional illumination and quantum illumination separately.

SUPPLEMENTARY DISCUSSION 2: RESULTS FOR CONVENTIONAL ILLUMINATION

Region I and II

Result 1: Suppose

(i) $p_0 \le 1/2$, and

(ii) $\eta \le \eta_* \equiv 1 - p_0/p_1$ (or equivalently, $\gamma \ge 0$),

then (a) the minimal errors for conventional illumination is equal to p_0, i.e.,

$$P_{\mathrm{err}} = p_0\ , \tag{S-11}$$

and (b) the bound can be achieved with any (pure or mixed) state.

Proof. If $p_0 \le 1/2$ and $\eta \le \eta_*$, we have $\gamma \geqslant 0$. This also implies that the Hermitian matrix, Ω_c, is a positive sum of two density matrices (with positive eigenvalues).

Consequently, all eigenvalues $\lambda_i = \langle i|\,\Omega_\mathrm{c}\,|i\rangle$, with an eigenvector $|i\rangle$, are positive, i.e., $\lambda_i \geqslant 0$. In this case the trace norm of Ω_c can be obtained directly by taking the trace, i.e.,

$$\|\Omega_\mathrm{c}\| = |p_1\eta + \gamma| = p_1 - p_0\ , \tag{S-12}$$

which implies the result stated in Eq. (S-11). Note that the argument above is applicable to any pure state. □

Result 2: Suppose

(i) $p_0 \geqslant 1/2$, and

(ii) $\eta \leqslant -\eta_{*}(\frac{\lambda_{\min}}{1-\lambda_{\min}})$,

with $\lambda_{\min} = \lambda_{\min}(\rho_E) \geqslant 0$, and the minimum eigenvalue of the environment signal ρ_E, then (a) the minimum error for conventional illumination is equal to p_1, i.e.,

$$P^c_{\text{err}} = p_1 \ , \tag{S-13}$$

and (b) the bound can be achieved with any (pure or mixed) state.

Proof. Let's consider the conventional illumination first. We now express Ω_{c} as follows:

$$\Omega_{\text{c}} = (p_1\eta + \gamma\lambda_{\min})\,\rho + \gamma\,(\rho_E - \lambda_{\min}\rho) \ . \tag{S-14}$$

Note that the matrix, $\rho_E - \lambda_{\min}\rho$, contains non-negative eigenvalues. Now suppose the following conditions are satisfied, (i) $p_0 \geqslant 1/2$ (or equivalently $p_0 \geqslant p_1$), which implies that $\gamma \equiv p_1(1-\eta) - p_0 \leqslant p_0(1-\eta) - p_0 = -p_0\eta \leqslant 0$, and (ii) $\eta \leqslant (\frac{p_0}{p_1}-1)(\frac{\lambda_{\min}}{1-\lambda_{\min}})$, which further implies that

$$p_1\eta + \gamma\lambda_{\min} \leqslant 0 \ , \tag{S-15}$$

then the trace norm of Ω_c is given by the trace of Ω_c, i.e.,

$$\|\Omega_c(\rho)\| = |p_1\eta + \gamma| = p_0 - p_1 \ , \tag{S-16}$$

which implies the result in Eq. (S-13). The proof for quantum case is similar. □

Region III

For conventional illumination and quantum illumination in region III, we have $\gamma < 0$. Therefore, we can always write

$$\Omega_c(|\psi\rangle\langle\psi|) \propto \rho_E - \alpha\,|\psi\rangle\langle\psi| \ , \tag{S-17}$$

$$\Omega_q(|\psi\rangle\langle\psi|) \propto \rho_E \otimes \rho_B - \alpha\,|\psi\rangle\langle\psi| \ , \tag{S-18}$$

for $\alpha := p_1\eta/|\gamma| > 0$. And thus P_{err} can be written as

$$P^c_{\text{err}} = \frac{1}{2}\left(1 - |\gamma|\cdot\|\rho_E - \alpha\,|\psi\rangle\langle\psi|\|\right) \ , \tag{S-19}$$

$$P^q_{\text{err}} = \frac{1}{2}(1 - |\gamma|\cdot\|\rho_E\otimes\rho_B - \alpha\,|\psi\rangle\langle\psi|\|) \ . \tag{S-20}$$

We shall see that (i) the trace norm of Ω_c is determined by the minimum eigenvalue of the matrix

$$\rho_E - \alpha\,|\psi\rangle\langle\psi| \ , \tag{S-21}$$

and (ii) the smallest eigenvalue is minimized by choosing the signal state as the eigenstate $|e_k\rangle$ with the minimum eigenvalue. These results come from the following lemmas.

Lemma 1 (Positivity of eigenvalues II). *Let ρ be a d-dimentional density matrix, let $|\psi\rangle$ be a pure quantum state, and $\alpha > 0$ a real number. Then there is at most one negative eigenvalue of the operator $\rho - \alpha\,|\psi\rangle\langle\psi|$.*

Proof. Suppose we can find two distinct negative eigenvalues such that $E_d < E_{d-1} < 0$. Consider the subspace spanned by the corresponding eigenvectors, $V = \text{Span}\{|e_d\rangle, |e_{d-1}\rangle\}$, of the matrix $\rho - \alpha\,|\psi\rangle\langle\psi|$. Clearly, there exists a linear combination, denoted by

$$|\psi^{\perp}\rangle \equiv \beta|e_d\rangle + \theta|e_{d-1}\rangle \neq 0 \ , \tag{S-22}$$

which is orthogonal to $|\psi\rangle$, i.e., $\langle\psi\,|\psi^{\perp}\rangle = 0$. Then, $\rho - \alpha|\psi\rangle\langle\psi|$ maps $|\psi^{\perp}\rangle$ to $\rho|\psi^{\perp}\rangle$, i.e.,

$$(\rho - \alpha|\psi\rangle\langle\psi|)\,|\psi^{\perp}\rangle = \rho\,|\psi^{\perp}\rangle \ . \tag{S-23}$$

On the other hand, when restricted to the subspace V, the operator $\rho - \alpha|\psi\rangle\langle\psi|$ is negative definite, because it has its all eigenvalues negative. Explicitly, $\langle\psi^{\perp}|\,(\rho - \alpha|\psi\rangle\langle\psi|)\,|\psi^{\perp}\rangle = |\beta|^2 E_d + |\theta|^2 E_{d-1} < 0$, which is equivalent to

$$\langle\psi^{\perp}|\rho|\psi^{\perp}\rangle < 0 \ . \tag{S-24}$$

This conclusion contradicts the fact that ρ is a density matrix, which must be positive semidefinite. There exists at most one negative eigenvalue. □

Lemma 2 (Problem of eigenvalue minimization). *Following the previous lemma, if the minimum eigenvalue $E_d \leq 0$, then the minimum error P_{err} is minimized by minimizing the negative eigenvalue E_d of the matrix $\rho_E - \alpha\,|\psi\rangle\langle\psi|$ with fixed ρ_E and α.*

Proof. Let us now express the minimum error as

$$P_{\text{err}} = (1 - |\gamma|\cdot||\rho_E - \alpha\,|\psi\rangle\langle\psi|\,||)\,/2 \ , \tag{S-25}$$

where $\alpha = p_1\eta/|\gamma|$. Denote E_k's as the eigenvalues of the matrix $\rho_E - \alpha\,|\psi\rangle\langle\psi|$. Then, we have

$$P_{\text{err}} = (1 - |\gamma|\textstyle\sum_k |E_k|)/2 = [1 - |\gamma|\,(\sum_{k\neq d} E_k - E_d)]/2 \ , \tag{S-26}$$

where we have applied the result of the previous lemma. Now, we have $\sum_{k\neq d} E_k = 1 - \alpha - E_d$ and hence

$$P_{\text{err}} = \frac{1}{2}[1 - |\gamma|\,(1 - \alpha - 2E_d)] \ , \tag{S-27}$$

which depends linearly with the smallest eigenvalue E_d; the more negative E_d is, the smaller P_{err} becomes. □

Lemma 3 (Eigenvector for minimization). *The smallest eigenvalue of the matrix $\rho_E - \alpha\,|\psi\rangle\langle\psi|$, with fixed ρ_E and α, can be minimized by choosing the pure state as the eigenvector associated with the smallest eigenvalue $\lambda_{\min}$ of ρ_E, i.e., $|\psi\rangle = |\theta_d\rangle$ where $(\rho_E - \lambda_{\min})\,|\theta_d\rangle = 0$.*

Proof. First, the minimum eigenvalue $\lambda_{\min}(A+B)$ of the sum of two Hermitian matrices A and B is bounded by the sum of the minimum eigenvalues of the individual matrix, i.e.,

$$\lambda_{\min}(A) + \lambda_{\min}(B) \leqslant \lambda_{\min}(A+B) \ . \tag{S-28}$$

As a result, the minimum eigenvalue E_d of the matrix $\rho_E - \alpha\,|\psi\rangle\langle\psi|$ is bounded by $\lambda_{\min} - \alpha \leqslant E_d$. Furthermore, this bound can be saturated by choosing $|\psi\rangle = |\theta_d\rangle$. □

Therefore, for region III, we only need to consider the minimum error resulted from sending the eigenstates of ρ_E with the minimum eigenvalue. The results of region III are summarized as follow:

Result 3: The minimum error $P_{\rm err}$ over all possible conventional input states is given by

$$P_{\rm err}^{c} = p_0 + \gamma\,(1-\lambda_d)\,, \tag{S-29}$$

which is obtained by choosing $|\psi\rangle = |\theta_d\rangle$ to be the eigenvector of ρ_E associated with the smallest eigenvalue.

Proof. Let us consider the matrix,

$$\Omega_{\rm c}(|\theta_d\rangle\langle\theta_d|) = p_1\eta\,|\theta_d\rangle\langle\theta_d| + \gamma\rho_E\,. \tag{S-30}$$

Here we consider the range where

$$\gamma \equiv p_1(1-\eta) - p_0 < 0 \tag{S-31}$$

but $p_1\eta + \gamma\lambda_d > 0$. Note that $\mathrm{tr}(\Omega_{\rm c}) = p_1\eta + \gamma$ is a sum of the $d-1$ negative eigenvalues and one positive eigenvalue, $p_1\eta + \gamma\lambda_d$. Therefore, the trace norm is obtained by $\|\Omega_c\| = -\mathrm{tr}(\Omega_c) + 2\,(p_1\eta + \gamma\lambda_d)$, or

$$\|\Omega_c\| = p_1\eta - \gamma + 2\lambda_d\gamma\,. \tag{S-32}$$

Finally, as $P_{\rm err}^{c} = (1 - ||\Omega_c||)/2$, we have $P_{\rm err}^{c} = p_0 + \gamma\,(1-\lambda_d)$. □

To check the consistency of the result above, when $\lambda_d = 0$, we have

$$P_{\rm err}^{c} = p_0 + \gamma = p_1\,(1-\eta)\ . \tag{S-33}$$

Note that when $\gamma = 0$, we have $P_{\rm err}^{c} = p_0$. Moreover, when

$$\eta = (\tfrac{p_0}{p_1} - 1)(\tfrac{\lambda_d}{1-\lambda_d})\,, \tag{S-34}$$

we have $p_1\eta = -\gamma\lambda_d$, which implies that $P_{\rm err}^{c} = p_0 + \gamma - \gamma\lambda_d = p_0 + p_1\eta + (p_1\,(1-\eta) - p_0) = p_1$. In the special case where $p_0 = p_1 = 1/2$ and $d = 2$ we have $\gamma = -\eta/2$. Therefore,

$$P_{\rm err}^{c} = 1/2 - \eta/4\,, \tag{S-35}$$

agrees with the example.

SUPPLEMENTARY DISCUSSION 3: RESULTS FOR QUANTUM ILLUMINATION

For region I of quantum illumination, the result and the proof is similar to the conventional illumination case. Before discussing region II and III, we first introduce some useful lemmas.

Lemma 4 (Lower bound of eigenvalue value). *Given a density matrix $\rho_E = \sum_{i=1}^{d}\lambda_i\,|\theta_i\rangle\langle\theta_i|$, where $|\theta_i\rangle$'s are orthonormal, and a pure state $|\psi\rangle = \sum_{i=1}^{d}|\theta_i\rangle\,|u_i\rangle$, the minimum eigenvalue (associated with the eigenvector $|g_\psi\rangle = \sum_{i=1}^{d}|\theta_i\rangle\,|v_i\rangle$), $E_g \equiv \lambda_{\min}(\rho_E\otimes\rho_B - \alpha\,|\psi\rangle\langle\psi|)$, is bounded below by*

$$E_g \geqslant \sum_{i=1}^{d}\lambda_i x_i^2 - \alpha(\sum_{i=1}^{d}x_i)^2\,, \tag{S-36}$$

where $x_i \equiv |\langle u_i\,|v_i\rangle|$. Here $|u_i\rangle$'s are not assumed to be normalized nor mutually orthogonal. However, to achieve the lower bound, we have $|u_i\rangle$'s mutually orthogonal to each other and $|g_\psi\rangle = |\psi\rangle$.

Proof. Let us consider the explicit form of the smallest eigenvalue :

$$E_g = \langle g_\psi|\,\rho_E\otimes\rho_B\,|g_\psi\rangle - \alpha\langle g_\psi\,|\psi\rangle\,\langle\psi|\,g_\psi\rangle\,, \tag{S-37}$$

where $\langle\psi\,|g_\psi\rangle = \sum_{i=1}^{d}\langle u_i\,|v_i\rangle$ and $\langle g_\psi|\,\rho_E\otimes\rho_B\,|g_\psi\rangle = \sum_{i=1}^{d}\lambda_i\,\langle v_i|\,\rho_B\,|v_i\rangle$. Now, we can obtain an upper bound of $|\langle\psi\,|g_\psi\rangle|$ by (i) taking absolute values for each term, i.e.,

$$|\langle\psi\,|g_\psi\rangle| \leqslant \sum_{i=1}^{d}|\langle u_i\,|v_i\rangle| \tag{S-38}$$

(can be achieved by choosing the vectors, $|u_i\rangle$'s and $|v_i\rangle$'s, to be proportional to each other), and (ii) dropping all the cross terms in an expansion of $\langle v_i|\,\rho_B\,|v_i\rangle$, i.e.,

$$\langle v_i|\,\rho_B\,|v_i\rangle = \sum_{j=1}^{d}|\langle u_j\,|v_i\rangle|^2 \geqslant |\langle u_i\,|v_i\rangle|^2\,. \tag{S-39}$$

By defining $x_i \equiv |\langle u_i\,|v_i\rangle|$, we obtain a lower bound for the smallest eigenvalue E_g, $E_g \geqslant \sum_{i=1}^{d}\lambda_i x_i^2 - \alpha(\sum_{i=1}^{d}x_i)^2$. The equality holds if and only if $|u_i\rangle$'s proportional to $|v_i\rangle$'s, and $|v_i\rangle$'s are mutually orthogonal to each other. □

In the next step, one need to minimize the lower bound of E_g, subject to a constraint,

$$\sum_{i=1}^{d}x_i \equiv C\,, \tag{S-40}$$

where $C \leqslant 1$ is not greater than unity as $|\langle\psi\,|g_\psi\rangle| \leqslant 1$. We found that E_g is minimized when $\alpha > \lambda_h$ where we have $C = 1$. Here $\lambda_h \equiv (\sum_{i=1}^{d}\lambda_i^{-1})^{-1}$ is proportional to the harmonic mean of the set of eigenvalues $\{\lambda_i\}$. When there are eigenvalues of ρ_E that are equal to 0, λ_h should be 0. Thus we have $\lambda_h \leqslant \lambda_{\min}(\rho_E)$, where the equality holds only for $\lambda_{\min}(\rho_E) = 0$. On the other hand, when $\alpha \leqslant \lambda_h$, the smallest eigenvalue of H_Q is positive, i.e., $E_g \geqslant 0$. In this case, the trace norm is equal to the trace, i.e., $\|H_q\| = \mathrm{tr}H_q$.

Lemma 5 (**Minimization with Lagrange multiplier**). *The minimum eigenvalue* $E_g = \lambda_{\min}(\rho_E \otimes \rho_B - \alpha |\psi\rangle\langle\psi|)$ *is bounded below by a value depending on* α. *(i) When* $\alpha \leqslant \lambda_h \equiv (\sum_{i=1}^d \lambda_i^{-1})^{-1}$, $E_g \geqslant 0$. *(ii) When* $\alpha > \lambda_h$, $E_g \geqslant \lambda_h - \alpha$.

Proof. Let us introduce a Lagrange multiplier l_m. The the lower bound of E_g, labeled by

$$f \equiv \sum_{i=1}^{d} \lambda_i x_i^2 - \alpha(\sum_{i=1}^{d} x_i)^2 , \tag{S-41}$$

is minimized when the condition,

$$\partial f / \partial x_i = l_m \partial g / \partial x_i , \tag{S-42}$$

holds, where $g \equiv \sum_{i=1}^d x_i - C$. Explicitly, we have

$$\frac{\partial f}{\partial x_i} = 2(\lambda_i x_i - \alpha C), \quad \frac{\partial g}{\partial x_i} = 1 , \tag{S-43}$$

When $\lambda_{\min}(\rho) \neq 0$, this gives $x_i = (\alpha C + l_m/2)/\lambda_i$, and hence $C = \sum_{i=1}^d x_i = (\alpha C + l_m/2)(\sum_{i=1}^d \lambda_i^{-1})$. As a result, the minimum value of f is given by,

$$f \geqslant C^2((\sum_{i=1}^{d} \lambda_i^{-1})^{-1} - \alpha) . \tag{S-44}$$

If there exist i_0 such that $\lambda_{i_0} = 0$, we have $x_i = 0$ for all $i \neq i_0$, and the value for x_{i_0} can be chosen freely. As a result, the minimum value of f is given by,

$$f \geq -\alpha C^2 . \tag{S-45}$$

Therefore, when $\alpha \leqslant \lambda_h \equiv (\sum_{i=1}^d \lambda_i^{-1})^{-1}$, f is minimized by choosing $C = 0$, which implies that all eigenvalues of H_q are positive, i.e., $\lambda_i(H_q) \geqslant 0$. However, when $\alpha > \lambda_h$, the lower bound f is minimized by setting $C^2 = 1$, which implies $E_g \geqslant \lambda_h - \alpha$. □

Region II of quantum illumination

Note that the condition of $\alpha \leqslant \lambda_h$ is equivalent to

$$\eta \leqslant \tfrac{\lambda_h}{1-\lambda_h}(\tfrac{p_0}{p_1} - 1) . \tag{S-46}$$

This defines the region II of quantum illumination. There, all the eigenvalues of Ω_q has the same sign. Similar to the conventional illumination, the minimum error is given by

$$P_{\text{err}}^q = p_1 , \tag{S-47}$$

which can be achieved by any states. So Result. 2 is also applied for quantum illumination, with different boundary. Recall that the boundary of region II for conventional illumination is given by

$$\eta \leqslant (\tfrac{p_0}{p_1} - 1)\tfrac{\lambda_{\min}(\rho_E)}{1-\lambda_{\min}(\rho_E)} , \tag{S-48}$$

and the error probability is the same as the trivial strategy. Quantum illumination is capable of shrinking the boundary to

$$\eta \leqslant \tfrac{\lambda_h}{1-\lambda_h}(\tfrac{p_0}{p_1} - 1) , \tag{S-49}$$

as $\lambda_h \leq \lambda_{\min}$, but the error probability remains the same as the trivial strategy.

Region III of quantum illumination

The Region III of quantum illumination is given by $\alpha > \lambda_h$ and $\eta > 1 - p_0/p_1$, which is equivalent to

$$\eta > \max\left\{1 - \frac{p_0}{p_1}, \frac{\lambda_h}{1-\lambda_h}(\frac{p_0}{p_1} - 1)\right\} . \tag{S-50}$$

In this region, the reflected signal is useful for the one-shot detection. Upon the Helstrom POVM measurement, the minimum error probability of the quantum illumination is lower bounded by

$$P_{\text{err}} \geq p_0 + \gamma(1 - \lambda_h) . \tag{S-51}$$

This lower bound can be saturated by picking appropriate probe states, which is summarized as follows.

Result 4: Optimal state for quantum illumination The lower bound, $\lambda_h - \alpha$, of E_g can be achieved by the input states,

$$|\psi\rangle = I_A \otimes U_B \sum_{i=1}^{d} \mu_i |\theta_i\rangle |\theta_i\rangle , \tag{S-52}$$

where $\mu_i = \sqrt{\lambda_h/\lambda_i}$, and U_B can be any local unitaries on the ancillary system B, when $\lambda_{\min}(\rho_E) \neq 0$.

In particular, one usually works under the frequency basis in practice, so the following probe state can be used,

$$|\psi\rangle = \sum_{i=1}^{d} \mu_i |\theta_i\rangle |\theta_i\rangle . \tag{S-53}$$

Proof. Let us consider the following ansatz, $|\psi\rangle = I_A \otimes U_B \sum_{i=1}^d \mu_i |\theta_i\rangle |\theta_i\rangle$, which has been proved in Lemma.4 to be the only type of states that can saturate the lower bound. Without loss of generality, we can assume the amplitudes μ_i's to be non-negative ($\mu_i \geqslant 0$) and normalized ($\sum_{i=1}^d \mu_i^2 = 1$). The expectation value, $\langle H_q \rangle = \langle\psi| H_q |\psi\rangle = \langle\psi| \rho_E \otimes \rho_B |\psi\rangle - \alpha$, is given by

$$\langle H_q \rangle = \sum_{i=1}^{d} \lambda_i \mu_i^2 - \alpha . \tag{S-54}$$

We can achieve the lower bound, $\langle H_q \rangle = \lambda_h - \alpha$, by setting $\mu_i^2 = \lambda_h/\lambda_i$. (One can readily check that the normalization condition is satisfied as $\sum_{i=1}^d \mu_i^2 = \lambda_h \sum_{i=1}^d 1/\lambda_i = 1$.) □

Remark: When the environmental background has noiseless modes, i.e. $\lambda_{\min}(\rho_E) = 0$, $\lambda_h = 0$, the minimum error probability is given by $P_{\text{err}} = p_1(1-\eta)$, which can be saturated with probe state $\rho_E^{\perp} \otimes \rho_B$, where $\rho_E^{\perp}$ is any quantum state that is orthogonal to the environmental noise. In this special case, the performance of quantum illumination coincides with conventional illumination.

SUPPLEMENTARY DISCUSSION 4: COMPLIMENTARY RESULTS

Result 5: Upper bound of minimal error The error probability P_{err} is upper bounded by either p_0 or p_1, i.e.,

$$P_{\text{err}} \leqslant \frac{1}{2}\left(1 - |p_0 - p_1|\right) = \min\left\{p_0, p_1\right\} \ . \tag{S-55}$$

Proof. Mathematically, this result comes from the definition of the trace norm. Let us consider the case of $p_0 \geq p_1$ first. Denote e_k's as the eigenvalues of the operator $p_0\rho_0 - p_1\rho_1$. Furthermore, we label the non-negative eigenvalues as $e_k^+ \geq 0$ and the negative eigenvalues as $e_k^- < 0$. In this way, we can write the trace norm as the difference between these two sets of eigenvalues, i.e.,

$$\|p_0\rho_0 - p_1\rho_1\| = \sum\nolimits_k e_k^+ - \sum\nolimits_k e_k^- . \tag{S-56}$$

In fact, using $\mathrm{tr}(p_0\rho_0 - p_1\rho_1) = p_0 - p_1 = \sum_k e_k^+ + \sum_k e_k^-$, we have $\|p_0\rho_0 - p_1\rho_1\| = p_0 - p_1 - 2\sum_k e_k^- \geqslant p_0 - p_1$, which yields

$$P_{\text{err}} \leqslant \tfrac{1}{2}\left(1 - (p_0 - p_1)\right) \leqslant p_1 \ . \tag{S-57}$$

The result for $p_0 < p_1$ can be obtained by the same argument. □

Result 6: Monotonicity of minimum error For a given reflectivity η, and density matrix ρ, we denote the minimum error as $P_{\text{err}}(\eta) = \frac{1}{2}\left(1 - \|p_1\rho_1(\eta) - p_0\rho_0\|\right)$, where $\rho_1(\eta) = \eta\rho + (1-\eta)\rho_0$. The minimum error is a non-increasing function of the reflectivity, i.e., if $\eta \geqslant \eta'$, then

$$P_{\text{err}}(\eta) \leqslant P_{\text{err}}(\eta') \ . \tag{S-58}$$

Proof. First of all, we define a trace-preserving quantum operation that represents the action of the reflection,

$$\varepsilon_\eta(\rho) = \eta\rho + (1-\eta)\rho_0 \ , \tag{S-59}$$

and hence $\varepsilon_\eta(\rho_0) = \rho_0$. Obviously, there exist a quantum operation,

$$\varepsilon_\Delta(\rho) = \Delta\rho + (1-\Delta)\rho_0 \ , \tag{S-60}$$

where $\Delta\eta = \eta'$, connecting the two, i.e.,

$$\varepsilon_\Delta\left(\varepsilon_\eta(\rho)\right) = \varepsilon_{\eta'}(\rho) \ . \tag{S-61}$$

In this way, all we need to show is that

$$\|p_1\varepsilon_{\eta'}(\rho) - p_0\varepsilon_{\eta'}(\rho_0)\| \leqslant \|p_1\varepsilon_\eta(\rho) - p_0\varepsilon_\eta(\rho_0)\| \ , \tag{S-62}$$

or equivalently,

$$\|p_1\varepsilon_\Delta(\rho) - p_0\varepsilon_\Delta(\rho_0)\| \leqslant \|p_1\rho - p_0\rho_0\| \ , \tag{S-63}$$

for any density matrix ρ. The proof Eq. (S-63) is essentially the same as for the well-known result that a trace-preserving operation is contractive under the measure of trace distance.

Alternatively, one can prove it by contradiction as follows: first, the Helstrom bound states that the minimum error for distinguishing between density matrices ρ and ρ_0 (occurring with probabilities p_0 and p_1 respectively) is determined by the trace norm $\|p_1\rho - p_0\rho_0\|$, over all possible POVM measurements. Since any channel can be regarded as a type of POVM, if the opposite, i.e.,

$$\|p_1\varepsilon_\Delta(\rho) - p_0\varepsilon_\Delta(\rho_0)\| \geqslant \|p_1\rho - p_0\rho_0\| \ , \tag{S-64}$$

were true, then one can find a POVM that yields a smaller error than the Helstrom bound, which causes a contradiction. □

Mathematical origin of the performance advantage

Mathematically, this quantum advantage originates from the difference between the *induced trace norm* [S1] and the *diamond norm* [S2], which can be directly observed from the definitions of these two norms. Defining $\Omega \equiv p_1\mathcal{E}_1 - p_0\mathcal{E}_0$, then $\|\Omega_c\|_1$ is the induced trace norm of Ω, and $\|\Omega_q\|_\diamond$ is the diamond norm of Ω. The quantum advantage, in terms of difference in detection error probability, is just the difference of these two norms.

Discord cannot fully explain the quantum advantage

Here, we give an example showing that quantum discord cannot fully explain the noise resilience of quantum illumination when the noise spectrum is nontrivial, i.e. not white noise.

As demonstrated by Eq.(3) in Ref [S3], the quantum advantage over conventional illumination, quantified by difference in accessible information, coincides exactly to the discord of encoding: $\Delta I = \delta_{enc}(x)$ for completely mixed noise. (Here we follow the same definitions and notations used in Ref [S3]). In below, we show that if the noise is not completely mixed, there is example when Eq.(3) does not hold.

Let's consider the qubits case. Suppose the noise for a two mode detection is modeled by the following quantum state:

$$\begin{aligned} \rho_E &= \lambda_1|1\rangle\langle 1| + \lambda_2|2\rangle\langle 2| \\ &= \frac{2}{3}|1\rangle\langle 1| + \frac{1}{3}|2\rangle\langle 2| \ , \end{aligned}$$

According to our result, one can find that the optimal quantum state for conventional illumination is

$$|\phi_A\rangle = |2\rangle,$$

and the optimal quantum state for quantum illumination is

$$|\psi_{\rm AB}\rangle = \sqrt{\frac{1}{3}}|1\rangle|1\rangle + \sqrt{\frac{2}{3}}|2\rangle|2\rangle\ . \tag{S-65}$$

Following the definitions in Ref [S3], one can calculate that for e.g. reflectivity $\eta = 0.287, p_0 = 0.7$, the quantum advantage is $\Delta I = 0.0077$, which does not equal the discord of encoding.

The critical fact that prevents such equivalence from extending to more general noise spectrum is that $I(\phi)$ is no longer the same for all pure states (a comment made between Eq.A3 to Eq.A4 on page 8 of Ref [S3]) due to the breaking of the local unitary invariance symmetry.

SUPPLEMENTARY DISCUSSION 5: CONTRIBUTION OF LOCAL DISTINGUISHABILITY AND QUANTUM CORRELATION TO THE PERFORMANCE

Based on our results with general noise model, we can identify two factors that contribute to the performance of illumination. The first factor is quantum correlation, and the second factor is local distinguishability.

Generally speaking, these two factors compete with each other. To increase the local distinguishability from the noise state, the reduced density state of the probe state becomes more close to a pure state, and thus the quantum correlation in the probe state between the signal system and probe system decreases. On the other hand, with more quantum correlation, the probe state of the signal and the ancilla can be more easily differentiated from the noise and the reduced state of the ancilla. However, with more quantum correlation in the probe state, its reduced density state becomes more mixed and thus the local distinguishability is reduced. These two factors therefore compete with each other. The optimal probe state is a subtle balance between these two factors, depending on the noise state. To see this, one can consider two extreme cases.

The local distinguishability dominates when the environmental noise is a pure state, or it has at least one mode that is noiseless. In this case, the optimal probe state for quantum illumination can be chosen as $\rho_{AB} = |\theta_d\rangle\langle\theta_d| \otimes \rho_B$ for any quantum state ρ_B, where $|\theta_d\rangle$ is the eigenstate of the noise with smallest eigenvalue. The performance of the quantum illumination reduces to the conventional illumination, where the error probability is $P_{\rm err} = \min\{p_0, p_1(1-\eta)\}$. The quantum advantage vanishes, and the performance of the illumination comes from local distinguishability. In other words, the quantum correlation between the signal and the ancilla does not contribute to the performance. And it may even decrease the performance, since the optimal state is a product state.

On the other hand, the correlation factor dominates the performance gain of quantum illumination when the environmental noise is white noise. In this case, the state describing the environment becomes isotropic. And the quantum advantage ΔI has been proved to be equal to the discord of encoding in Ref. [S3]. In this case, the reduced density state of the optimal probe state is completely mixed, which minimize the local distinguishability with the noise state. In other words, the correlation factor dominates the performance of quantum illumination, because the optimal probe state minimize the local distinguishability.

When the environmental noise is at finite temperature, the best probe state for quantum illumination is a subtle balance between the quantum correlation and the local distinguishability. It cannot be characterized by one factor alone.

SUPPLEMENTARY REFERENCES

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