

Supplementary Information for “High-accuracy Ising machine using Kerr-nonlinear parametric oscillators with local four-body interactions”

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SUPPLEMENTARY NOTE 1. CONDITION ON C FOR SATISFYING THE FOUR-BODY CONSTRAINTS

We derive a formula that expresses the condition on C for satisfying the four-body constraints for $\{\tilde{s}_k\}$ minimizing E_{LHZ} . We then show that $C > 1$ is a sufficient condition under the normalization condition $\sum_{k=1}^L |J_k| = 1$.

First of all, it is notable that the second term in E_{LHZ} , $-C \sum_{\langle k,l,m,n \rangle} \tilde{s}_k \tilde{s}_l \tilde{s}_m \tilde{s}_n$, is simply expressed by $-C(L - N + 1 - 2b)$, where $(L - N + 1)$ is the number of all the constraints and b denotes the number of broken constraints. Thus in general, minimum E_{LHZ} with b broken constraints, which is denoted by $E_{\text{LHZ}}^{(b)}$, is expressed as

$$E_{\text{LHZ}}^{(b)} = - \sum_{k=1}^L J_k \tilde{s}_k^{(b)} - C(L - N + 1 - 2b), \quad (1)$$

where $\{\tilde{s}_k^{(b)}\}$ is the spin configuration minimizing E_{LHZ} with b broken constraints. By using these notations, the condition for satisfying the constraints is written as $E_{\text{LHZ}}^{(0)} < E_{\text{LHZ}}^{(b)}$ for all $b \geq 1$. Thus, using Eq. (1), we have the following inequality:

$$C > \frac{1}{2b} \left[- \sum_{k=1}^L J_k \tilde{s}_k^{(0)} + \sum_{k=1}^L J_k \tilde{s}_k^{(b)} \right]. \quad (2)$$

Note that the right-hand side in Eq. (2) offers a tight lower bound for C to satisfy the constraints for an arbitrary instance $\{J_k\}$. In the case of the uniform antiferromagnetic interaction in the main text ($L = 6$ and $J_k = -1/6$), the values of $-\sum_{k=1}^6 J_k \tilde{s}_k^{(b)}$ for $b = 0, 1, 2, 3$ are $-1/3, -2/3, -1$, and $-2/3$, respectively, and thus Eq. (2) gives $C > 1/6$.

Next, we roughly evaluate a condition on C valid for any instance $\{J_k\}$. The right-hand side of Eq. (2) can be estimated as

$$-\frac{1}{2b} \sum_{k=1}^L J_k \left[\tilde{s}_k^{(0)} - \tilde{s}_k^{(b)} \right] \leq \sum_{k=1}^L |J_k| = 1, \quad (3)$$

where we have used $b \geq 1$, $|\tilde{s}_k^{(0)} - \tilde{s}_k^{(b)}| \leq 2$, and the normalization condition $\sum_{k=1}^L |J_k| = 1$. Equation (3) shows that the right-hand side of Eq. (2) is at most 1, indicating that $C > 1$ is a sufficient condition for satisfying the four-body constraints.

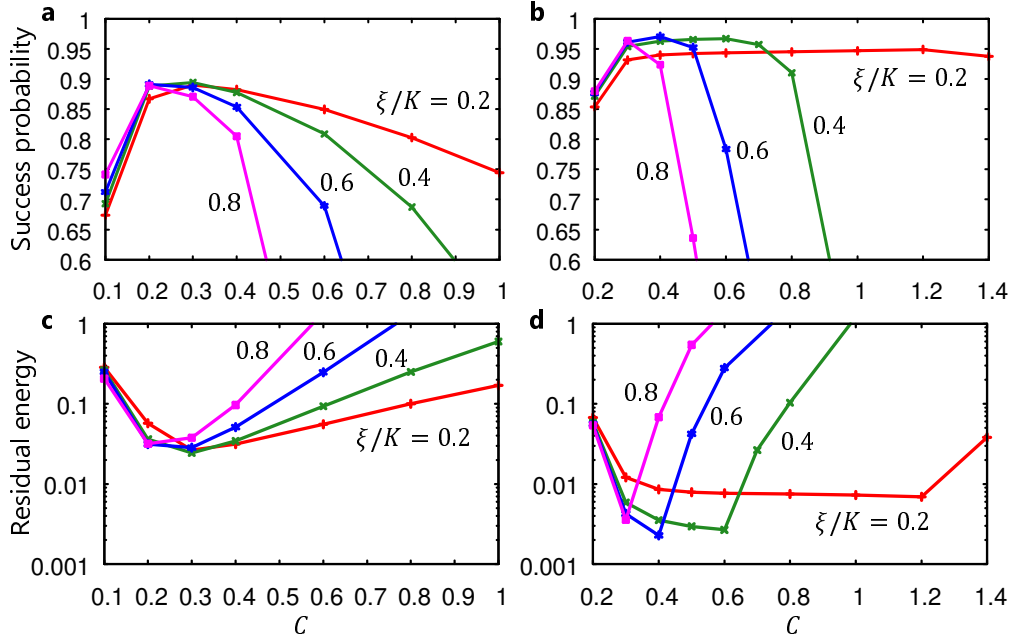
SUPPLEMENTARY NOTE 2. H_{LHZ} FOR 6 KPOS

Here we present H_{LHZ} more explicitly. The oscillators denoted by “Fixed” in the lowest row in Fig. 1 (in the main text) are assumed to be in a coherent state $|\beta\rangle$ with $\beta > 0$ [1]. As these oscillators do not bifurcate and β can be controlled by external fields, we replace corresponding annihilation and creation operators by β and regard β as a parameter. As a result, the LHZ part of the Hamiltonian is written as

$$H_{\text{LHZ}} = -\hbar\xi \left[A \sum_{k=1}^6 J_k a_k^\dagger + C \left(\beta a_1^\dagger a_2^\dagger a_4 + \beta a_2^\dagger a_3^\dagger a_5 + a_2^\dagger a_4^\dagger a_5 a_6 \right) + \text{h.c.} \right]. \quad (4)$$

SUPPLEMENTARY NOTE 3. AVERAGE SUCCESS PROBABILITIES AND RESIDUAL ENERGIES

Supplementary Figures 1a (without the correction) and 1b (with the correction) show the success probabilities averaged over the 100 random instances in the main text as functions of C and ξ . By the proposed correction, the



Supplementary Figure 1. **Average performance for the 100 random instances.** **a** Success probabilities averaged over the random instances discussed in the main text without the correction. **b** Corresponding results with the correction. **c** Average residual energies without the correction. **d** Corresponding results with the correction.

highest average success probability is increased from 89.5% at $(C, \xi/K) = (0.3, 0.3)$ to 97.1% at $(0.4, 0.6)$, which are used for Fig. 3 in the main text. The average residual energies are shown in Supplementary Figures 1c (without the correction) and 1d (with the correction), which are decreased by one order of magnitude by the correction.

The dependences of the average success probabilities on C are understood as follows. (Those of the residual energies can be understood in the same way.) We first explain the case without the correction shown in Supplementary Figure 1a. As a function of C , the average success probability shows a maximum. The low success probabilities for $C \ll 1$ are because the four-body constraints are frequently broken. With increasing C , the inhomogeneity in the photon numbers due to the four-body interactions is increased and degrades the success probabilities. Thus the average success probabilities are the highest at an intermediate C .

For the case with the correction shown in Supplementary Figure 1b, the average success probability also shows a maximum as a function of C . For small C , the success probabilities are low, again, owing to violated four-body constraints as mentioned above. For intermediate C , the correction can suppress the effects of the photon-number inhomogeneity, unlike the above case, leading to higher performance. For larger C , however, the success probabilities suddenly decrease, because the estimation given by Eq. (4) in the main text is no longer valid for these $(C, \xi/K)$, and the resultant large modulation of detunings according to Eq. (6) in the main text rather degrades the performance.

SUPPLEMENTARY NOTE 4. THE GROUND STATE OF THE INITIAL HAMILTONIAN

We consider a condition for the vacuum to be the ground state of H at initial time for $L = 6$. As we set $p/K = A = \beta = 0$ and $\Delta_k = \Delta$ at $t = 0$, H is written as

$$H = \hbar \sum_{k=1}^6 \left(\frac{K}{2} a_k^\dagger a_k^\dagger a_k^2 + \Delta a_k^\dagger a_k \right) - \hbar \xi C \left(a_2^\dagger a_4^\dagger a_5 a_6 + \text{h.c.} \right). \quad (5)$$

We evaluate the ground state by the variational method, using the coherent state $|\alpha_1\rangle \cdots |\alpha_6\rangle$ as a trial wave function [2, 3]. $\partial \langle H \rangle / \partial \alpha_k^* = 0$ gives, for $k = 1, 3$,

$$K \alpha_k^* \alpha_k^2 + \Delta \alpha_k = 0, \quad (6)$$

and for the others,

$$K\alpha_2^*\alpha_2^2 + \Delta\alpha_2 - \xi C\alpha_4^*\alpha_5\alpha_6 = 0, \quad (7)$$

$$K\alpha_4^*\alpha_4^2 + \Delta\alpha_4 - \xi C\alpha_2^*\alpha_5\alpha_6 = 0, \quad (8)$$

$$K\alpha_5^*\alpha_5^2 + \Delta\alpha_5 - \xi C\alpha_2\alpha_4\alpha_6^* = 0, \quad (9)$$

$$K\alpha_6^*\alpha_6^2 + \Delta\alpha_6 - \xi C\alpha_2\alpha_4\alpha_5^* = 0. \quad (10)$$

KPOs 1 and 3 are decoupled and their ground states are the vacuums since $K > 0$ and $\Delta > 0$. Equations (7)–(10) can be transformed to

$$\left[\left(K |\alpha_2|^2 + \Delta \right) \left(K |\alpha_4|^2 + \Delta \right) \left(K |\alpha_5|^2 + \Delta \right) \left(K |\alpha_6|^2 + \Delta \right) - (\xi C)^4 |\alpha_2\alpha_4\alpha_5\alpha_6|^2 \right] \alpha_2\alpha_4\alpha_5\alpha_6 = 0. \quad (11)$$

If $K \geq \xi C$, the quantity in the square bracket is always positive and thus $\alpha_2\alpha_4\alpha_5\alpha_6$ must be zero. $\alpha_2\alpha_4\alpha_5\alpha_6 = 0$ and Eqs. (7)–(10) result in $\alpha_2 = \alpha_4 = \alpha_5 = \alpha_6 = 0$. Around $\alpha_k = 0$, $\langle H \rangle$ is approximated by $\langle H \rangle \simeq \hbar \sum_{k=1}^6 \Delta |\alpha_k|^2$, which shows that $\alpha_k = 0$ corresponds to the minimum. This result means that if $\xi C/K \leq 1$ then the initial ground state is the vacuum within the estimation by the variational method.

SUPPLEMENTARY NOTE 5. TIME DEPENDENCE OF PARAMETERS

The explicit forms of the functions in Fig. 4 (in the main text) are as follows:

$$\frac{p}{K} = 3 \sin \frac{\pi t}{2T}, \quad (12)$$

$$\frac{\Delta}{K} = \cos \frac{\pi t}{2T}, \quad (13)$$

$$A = \left(\frac{p}{K} - \frac{\Delta}{K} \tanh \frac{p}{\Delta} \right)^{3/2}, \quad (14)$$

$$\beta = \left(\frac{p}{K} - \frac{\Delta}{K} \tanh \frac{p}{\Delta} \right)^{1/2}. \quad (15)$$

SUPPLEMENTARY REFERENCES

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- [1] Puri, S., Andersen, C. K., Grimsmo, A. L. & Blais, A. Quantum annealing with all-to-all connected nonlinear oscillators. *Nat. Commun.* **8**, 15785 (2017).
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