Quantum coherence and state conversion: theory and experiment
Supplemental Material

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In this Supplemental Material, we provide the theoretical proofs missing in the main text and an extended discussion of the experiment. In the first section we gather useful theoretical results which we apply in the second section to prove our theoretical findings stated in the main text. The third section describes the experimental implementation in more detail.

I. USEFUL THEORETICAL RESULTS

In this section, we present results that we will use later in some proofs. If we can implement a stochastic transformation from a state ρ to a state σ with probability p, we will write $\rho \to p\sigma$. Using this notation, we present the following Proposition.

Proposition 1. *For two states ρ, σ and a probability p let there be a stochastic SIO achieving the transformation*

$$\rho \to p\sigma. \tag{1}$$

Then, for every incoherent state τ and every $0 \leq q \leq 1-p$, there exists a stochastic SIO achieving the transformation

$$\rho \to p\sigma + q\tau. \tag{2}$$

Proof. The key idea in this proof is that the set of strictly incoherent Kraus operators is closed under concatenation. Therefore, the overall map that describes the application of a SIO on post-selected output states of another SIO is still in SIO. From Prop. 1 follows that we can always complete a stochastic SIO for free. The part completing the map has, with probability $1-p$, a state μ as an output. Applying total dephasing to μ, we obtain an incoherent state μ', which we can transform into τ using SIO. In addition, we can do this only stochastically, which proves the Proposition. □

Another result we will need later is stated in the following Lemma.

Lemma 2. *For any correlated state $\rho^{AB} \neq \rho^A \otimes \rho^B$ there exists a two-element POVM $\{M^A, \mathbb{1}^A - M^A\}$ on Alice's side such that $\mathrm{Tr}\left[M^A\rho^{AB}\right] > 0$ and*

$$\frac{\mathrm{Tr}_A\left[M^A\rho^{AB}\right]}{\mathrm{Tr}\left[M^A\rho^{AB}\right]} \neq \rho^B := \mathrm{Tr}_A\left[\rho^{AB}\right]. \tag{3}$$

Proof. In the following, let $\{|i\rangle^B\}$ be an eigenbasis of Bob's state ρ^B. Consider now states of the form

$$\rho^{AB} = \sum_i p_i \rho_i^A \otimes |i\rangle\langle i|^B. \tag{4}$$

Following the notion of [1], we call these states quantum-incoherent with respect to an eigenbasis of Bob.

If the state ρ^{AB} shared by Alice and Bob is not of the form (4), there exists a POVM element M^A such that $\mathrm{Tr}\left[M^A\rho^{AB}\right] > 0$ and $\mathrm{Tr}_A\left[M^A\rho^{AB}\right]$ is not diagonal in the basis $\{|i\rangle^B\}$, see Thm. 2 in [1]. This proves the Lemma for states ρ^{AB} which are not quantum-incoherent with respect to an eigenbasis of Bob.

For completing the proof, let now ρ^{AB} be of the form (4). If the state ρ^{AB} is not a product state, the decomposition (4) must have at least two different states $\rho_i^A \neq \rho_j^A$. Because these states are different, there necessarily exists a POVM element M^A such that

$$\mathrm{Tr}\left[M^A\rho_i^A\right] \neq \mathrm{Tr}\left[M^A\rho_j^A\right]. \tag{5}$$

In the last step of the proof, note that Eq. (5) implies the following:

$$\frac{\langle i|\rho^B|i\rangle}{\langle j|\rho^B|j\rangle} \neq \frac{\langle i|\mathrm{Tr}_A\left[M^A\rho^{AB}\right]|i\rangle}{\langle j|\mathrm{Tr}_A\left[M^A\rho^{AB}\right]|j\rangle}. \tag{6}$$

Thus, also in this case Eq. (3) is fulfilled, and the proof of the Lemma is complete. □

II. TECHNICAL PROOFS

Here we give the missing proofs of the results in the main text, which we restate for readability.

Proposition 1. *Every stochastic quantum operation that can be decomposed into strictly incoherent Kraus operators is part of a deterministic SIO.*

Proof. Strictly incoherent Kraus operators K_n are of the form

$$K_n = \sum_i c_{i,n} |f_n(i)\rangle\langle i| \tag{7}$$

where $f_n(i)$ is a bijective function on $\{1, ..., d\}$. If they form a stochastic quantum operation, we have

$$\sum_n K_n^\dagger K_n = \sum_{n,i} |c_{i,n}|^2 |i\rangle\langle i| \leq \mathbb{1}. \tag{8}$$

Therefore $\sum_n |c_{i,n}|^2 \leq 1 \forall i$ and we can define

$$\tilde{c}_i = \sqrt{1 - \sum_n |c_{i,n}|^2} \tag{9}$$

and

$$\tilde{K} = \sum_i \tilde{c}_i |i\rangle\langle i|, \tag{10}$$

which is a strictly incoherent Kraus operator and has the property

$$\tilde{K}^\dagger \tilde{K} + \sum_n K_n^\dagger K_n = \mathbb{1}. \tag{11}$$

□

From Prop. 1 then follows Theorem 2.

Theorem 2. *Let ρ and σ be states of a single qubit. The following statements are equivalent:*
(1) There exists an IO transforming ρ with probability p to σ.
(2) There exists a SIO transforming ρ with probability p to σ.

Proof. An incoherent Kraus operator K is of the form

$$K = \sum_i c(i) |j(i)\rangle\langle i|, \tag{12}$$

and it is strictly incoherent if $j(i)$ is one-to-one [2]. Therefore all incoherent qubit Kraus operators are either also strictly incoherent or their output is, independent of the input, incoherent. Let us use the strictly incoherent ones to define a stochastic SIO. Then Prop. 1 finishes the proof. Note that this proof technique does not work in higher dimensions, since then, there exist $j(i)$ that have neither the same output for all i (and have thus incoherent output), nor are they one-to-one. □

This allows us to prove the main statement of our work.

Theorem 3. *A qubit state σ is reachable via a stochastic SIO or IO transformation from a fixed initial qubit state ρ with a given probability p iff*

$$r^2 s_z^2 + \left(1 - r_z^2\right) s^2 \leq r^2, \tag{13a}$$

$$p^2 s^2 \leq \frac{r^2}{1 + |r_z|} \left[2p - (1 - |r_z|)\right]. \tag{13b}$$

Proof. According to Thm. 2, we can focus on SIO transformations. In order to implement a stochastic qubit state transformation, we need a quantum instrument with two possible outcomes, success and failure, modelled by $\mathcal{E}_s^{\mathrm{SIO}}(\rho)$ and $\mathcal{E}_f^{\mathrm{SIO}}(\rho)$. In the case of SIO transformations, both $\mathcal{E}_s^{\mathrm{SIO}}(\rho)$ and $\mathcal{E}_f^{\mathrm{SIO}}(\rho)$ have to be decomposable into SIO Kraus operators. Due to Prop. 1, we can focus exclusively on $\mathcal{E}_s^{\mathrm{SIO}}$. According to [3], every $\mathcal{E}_s^{\mathrm{SIO}}$ can be represented by four SIO Kraus operators

$$K_1 = \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}, K_2 = \begin{pmatrix} 0 & b_2 \\ a_2 & 0 \end{pmatrix},$$
$$K_3 = \begin{pmatrix} a_3 & 0 \\ 0 & 0 \end{pmatrix}, K_4 = \begin{pmatrix} 0 & b_3 \\ 0 & 0 \end{pmatrix}. \tag{14}$$

Since overall phases of Kraus operators are physically irrelevant, we assume from here on $a_i, b_3 \geq 0$. Defining $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3)$, the condition that $\mathcal{E}_s^{\mathrm{SIO}}$ is trace non-increasing is equivalent to $l_a^2 := |\boldsymbol{a}|^2 \leq 1$ and $l_b^2 := |\boldsymbol{b}|^2 \leq 1$. Due to symmetries and as explained in [3], we can restrict our analysis to the case $s_y = r_y = 0$ and $s_x, r_x, s_z, r_z \geq 0$. More precisely, we assume $r_x > 0$ from here on, since otherwise we have the trivial case of incoherent initial states. From

$$\mathcal{E}_s^{\mathrm{SIO}}(\rho) = p\sigma \tag{15}$$

then follow the Equations

$$\begin{aligned}
&p s_x = r_x \left(a_2 \,\mathbf{Re}(b_2) + a_1 \,\mathbf{Re}(b_1)\right), \\
&0 = a_2 \,\mathbf{Im}(b_2) - a_1 \,\mathbf{Im}(b_1), \\
&p(1 + s_z) = \left(a_1^2 + a_3^2\right)(1 + r_z) + \left(|b_2|^2 + b_3^2\right)(1 - r_z), \\
&p(1 - s_z) = a_2^2 (1 + r_z) + |b_1|^2 (1 - r_z)
\end{aligned} \tag{16}$$

or equivalently

$$\begin{aligned}
p s_x \; &= r_x \left(a_2 \,\mathbf{Re}(b_2) + a_1 \,\mathbf{Re}(b_1)\right), \\
0 \; &= a_2 \,\mathbf{Im}(b_2) - a_1 \,\mathbf{Im}(b_1), \\
2p \; &= l_a^2 (1 + r_z) + l_b^2 (1 - r_z), \\
2p s_z &= \left(a_1^2 + a_3^2 - a_2^2\right)(1 + r_z) \\
&\quad + \left(|b_2|^2 + b_3^2 - |b_1|^2\right)(1 - r_z).
\end{aligned} \tag{17}$$

The principal idea of our proof from here on is the following: For fixed r_x, r_z, p, we determine states (s_x, s_z) on the boundary of the region which is achievable with stochastic SIO, i.e., the region for which the Equations above have a solution for suitable $\boldsymbol{a}, \boldsymbol{b}$. Since the achievable region is convex and contains the free states (we can always mix incoherently with a free state), this will allow us to deduce the entire reachable region.

Now assume that (s_x, s_z) is on the boundary of the reachable region. Then one can choose $a_3 = 0$ and $b_3 = 0$, since K_3 and K_4 destroy all coherence. Formally, this can be shown considering

$$\begin{aligned} \boldsymbol{a}' &= (\sqrt{a_1^2 + a_3^2}, a_2, 0), \\ \boldsymbol{b}' &= (|b_1|, \sqrt{|b_2|^2 + b_3^2}, 0), \end{aligned} \tag{18}$$

which lead to

$$\begin{aligned} ps'_x &= r_x \left(a'_2\, \mathbf{Re}(b'_2) + a'_1\, \mathbf{Re}(b'_1)\right) \\ &= r_x \left(a_2 \sqrt{|b_2|^2 + b_3^2} + \sqrt{a_1^2 + a_3^2}\, |b_1|\right) \\ &\geq ps_x, \\ 2ps'_z &= 2ps_z, \\ l_{a'}^2 &= l_a^2, \\ l_{b'}^2 &= l_b^2, \\ 0 &= a'_2\, \mathbf{Im}(b'_2) - a'_1\, \mathbf{Im}(b'_1). \end{aligned} \tag{19}$$

Remember that we consider fixed r_x, r_z and $p > 0$. Thus $s'_x \geq s_x$ and $s'_z = s_z$. This mixing argument with the free states excludes boundaries of the achievable region parallel to the x-axis. Therefore $s'_x > s_x$ for $s'_z = s_z$ cannot happen if both (s_x, s_z) and (s'_x, s'_z) lie on the boundary and we will assume from here on $a_3 = b_3 = 0$ and $b_1, b_2 \geq 0$. This leads to the Equations

$$\begin{aligned} ps_x \;&= r_x \left(a_2 b_2 + a_1 b_1\right), \\ 2p \;&= l_a^2(1 + r_z) + l_b^2(1 - r_z), \\ 2ps_z &= \left(a_1^2 - a_2^2\right)(1 + r_z) + \left(b_2^2 - b_1^2\right)(1 - r_z). \end{aligned} \tag{20}$$

Next we notice that the second line in the above Equations defines an ellipse. Remembering that we excluded the trivial case of $r_z = 1$ by assuming $r_x > 0$, we can therefore use the parametrization

$$\begin{aligned} l_a &= \sqrt{\frac{2p}{1 + r_z}} \cos(t), \\ l_b &= \sqrt{\frac{2p}{1 - r_z}} \sin(t). \end{aligned} \tag{21}$$

Without loss of generality, we choose $0 \leq t \leq \pi/2$ and the condition $l_a, l_b \leq 1$ leads to

$$\begin{aligned} \cos(t) &\leq \sqrt{\frac{1 + r_z}{2p}}, \\ \sin(t) &\leq \sqrt{\frac{1 - r_z}{2p}}, \end{aligned} \tag{22}$$

which restricts the range of t further. Next we substitute

$$\begin{aligned} a_1 &= \sqrt{\frac{2p}{1 + r_z}} \cos(t) \cos\left(\frac{\theta - \phi}{2}\right), \\ a_2 &= \sqrt{\frac{2p}{1 + r_z}} \cos(t) \sin\left(\frac{\theta - \phi}{2}\right), \\ b_1 &= \sqrt{\frac{2p}{1 - r_z}} \sin(t) \sin\left(\frac{\theta + \phi}{2}\right), \\ b_2 &= \sqrt{\frac{2p}{1 - r_z}} \sin(t) \cos\left(\frac{\theta + \phi}{2}\right), \end{aligned} \tag{23}$$

which automatically satisfies the ellipse Equation. Since all left hand sides of these Equations are positive by assumptions, we can choose without loss of generality $0 \leq \theta \leq \pi/2$ and $-\theta \leq \phi \leq \theta (\Leftrightarrow 0 \leq \frac{\theta-\phi}{2}, \frac{\theta+\phi}{2} \leq \frac{\pi}{2})$. The remaining two Equations are then (since $p > 0$)

$$\begin{aligned} s_x &= \frac{r_x \sin(2t) \sin(\theta)}{\sqrt{1 - r_z^2}}, \\ s_z &= \cos(2t) \sin(\theta) \sin(\phi) + \cos(\theta) \cos(\phi). \end{aligned} \tag{24}$$

When we know for every reachable s_x the largest possible s_z, we achieved our goal of determining the boundary of the reachable region. Therefore we fix s_x and and maximize s_z. For fixed s_x, we obtain from the first Equation a relation between t and θ,

$$\sin(\theta(t)) = \frac{\sqrt{1 - r_z^2} s_x}{r_x \sin(2t)}. \tag{25}$$

Using $0 \leq \theta \leq \pi/2$, we can rewrite the second Equation as

$$s_z(t, \phi) = \cos(2t) \sin(\theta(t)) \sin(\phi) + \sqrt{1 - \sin^2(\theta(t))} \cos(\phi),$$

which is maximal either on the boundary or for

$$\begin{aligned} 0 &= \frac{\partial s_z(t, \phi)}{\partial \phi} \\ &= \sin(\theta(t)) \cos(2t) \cos(\phi) - \sqrt{1 - \sin^2(\theta(t))} \sin(\phi). \end{aligned}$$

Since we have $-\pi/2 \leq -\theta \leq \phi \leq \theta \leq \pi/2$, this is equivalent to

$$\phi = \arctan\left(\frac{\sin(\theta(t)) \cos(2t)}{\sqrt{(1 - \sin^2(\theta(t)))}}\right). \tag{26}$$

Using that $\arctan(x)$ is monotonically increasing in x, we find

$$\begin{aligned} \phi &\geq \arctan\left(\frac{-\sin(\theta)}{\sqrt{1 - \sin(\theta)^2}}\right) = -\theta, \\ \phi &\leq \arctan\left(\frac{\sin(\theta)}{\sqrt{1 - \sin(\theta)^2}}\right) = \theta \end{aligned} \tag{27}$$

and therefore ϕ inside the allowed region. Then the $s_z(t)$, the s_z optimized over ϕ, is independent of t and given by

$$s_z(t) = \sqrt{1 - \frac{\left(1 - r_z^2\right) s_x^2}{r_x^2}}. \tag{28}$$

Note that the expression under the square root is, due to Eq. (24), never negative.

Now we need to check the boundaries. To do this, we express t in terms of θ and define $X = \sin^2(\theta)$ (therefore $(1 - r_z^2)s_x^2/r_x^2 \leq X \leq 1$, again from Eq. (24)). For the moment, we assume $\cos(2t(\theta)) \geq 0$. This leads to

$$\begin{aligned} s_z^+(\phi = \theta, \theta) &= \cos(2t(\theta))\sin^2(\theta) + \cos^2(\theta) \\ &= \sqrt{1 - \frac{(1 - r_z^2)s_x^2}{r_x^2 \sin^2(\theta)}} \sin^2(\theta) + \cos^2(\theta) \\ &= 1 - X + \sqrt{1 - \frac{(1 - r_z^2)s_x^2}{r_x^2 X}} X \\ &= s_z(X). \end{aligned} \tag{29}$$

Since

$$0 = \frac{\partial}{\partial X}\left(1 - X + \sqrt{1 - y/X}\, X\right) \tag{30}$$

has for $y \neq 0$ no solutions, $s_z(X)$ attains its extrema on the boundaries. The exact maximum on the boundary depends on t, but it is lower than the maximum of

$$\begin{aligned} s_z^+(X = (1 - r_z^2)s_x^2/r_x^2) &= 1 - \frac{(1 - r_z^2)s_x^2}{r_x^2}, \\ s_z^+(X = 1) &= \sqrt{1 - \frac{(1 - r_z^2)s_x^2}{r_x^2}}. \end{aligned} \tag{31}$$

and thus smaller than the extrema inside the allowed region. In the case of $\cos(2t(\theta)) \leq 0$, we have

$$\begin{aligned} s_z^-(\phi = \theta, \theta) &= \cos(2t(\theta))\sin^2(\theta) + \cos^2(\theta) \\ &= -\sqrt{1 - \frac{(1 - r_z^2)s_x^2}{r_x^2 \sin^2(\theta)}} \sin^2(\theta) + \cos^2(\theta) \\ &= 1 - X - \sqrt{1 - \frac{(1 - r_z^2)s_x^2}{r_x^2 X}} X \\ &\leq s_z^+(\phi = \theta, \theta). \end{aligned} \tag{32}$$

For the boundary with $\phi = -\theta$, the above considerations are the same, with the roles of $\cos(2t(\theta)) \geq 0$ and $\cos(2t(\theta)) \leq 0$ inverted. We thus confirmed that the maximal s_z for given s_x is indeed given by Eq. (28) and independent of θ and t.

In order to finish the proof, we need to determine the reachable range of s_x which depends according to Eq. (24) on t and therefore through Eqs. (22) on r_z and p. By the convexity of the reachable region, it is again sufficient to find the maximal reachable s_x. This corresponds to finding the allowed t closest to $\pi/4$ (see again Eq. (24)), for which we will consider different cases. The first case is that neither of the conditions in Eq. (22) restricts t, which is equivalent to

$$p \leq \frac{1 - r_z}{2} \tag{33}$$

and therefore

$$s_x \leq \frac{r_x}{\sqrt{1 - r_z^2}}. \tag{34}$$

If

$$p \leq \frac{1 + r_z}{2}, \tag{35}$$

the constraints are

$$0 \leq t \leq \arcsin\left(\sqrt{\frac{1 - r_z}{2p}}\right). \tag{36}$$

For $p < 1 - r_z$, the upper bound on t is larger than $\pi/4$, and we find the same bounds on s_x as in the first case. Using

$$\sin\left(2 \arcsin x\right) = 2x\sqrt{1 - x^2}, \tag{37}$$

we find

$$s_x \leq \frac{r_x}{\sqrt{1 + r_z}} \frac{1}{p} \sqrt{2p - (1 - r_z)} \tag{38}$$

otherwise. In the last case, for

$$p \geq \frac{1 + r_z}{2}, \tag{39}$$

we have a lower and an upper bound on t,

$$\arccos\left(\sqrt{\frac{1 + r_z}{2p}}\right) \leq t \leq \arcsin\left(\sqrt{\frac{1 - r_z}{2p}}\right). \tag{40}$$

From Eq. (20), we see that the lower bound is always smaller than the upper. In addition,

$$\arccos\left(\sqrt{\frac{1 + r_z}{2p}}\right) \leq \arccos\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}. \tag{41}$$

Therefore, we end up with the same conclusions as in the second case.

Finally, using the symmetry and mixing arguments, the reachable region is defined by the inequalities

$$\begin{aligned} & s_z^2 \leq 1 - \frac{1 - r_z^2}{r_x^2 + r_y^2}\left(s_x^2 + s_y^2\right) \\ & \begin{cases} p < 1 - |r_z|: & s_x^2 + s_y^2 \leq \frac{r_x^2 + r_y^2}{1 - r_z^2} \\ p \geq 1 - |r_z|: & s_x^2 + s_y^2 \leq \frac{r_x^2 + r_y^2}{1 + |r_z|} \frac{1}{p^2}\left(2p - (1 - |r_z|)\right) \end{cases} \end{aligned} \tag{42}$$

Rearranging the terms in the above Equations and using the short hand notations leads to

$$r^2 s_z^2 + \left(1 - r_z^2\right) s^2 \le r^2, \tag{43}$$

$$\begin{cases} p < 1 - |r_z| : & \left(1 - r_z^2\right) s^2 \le r^2, \\ p \ge 1 - |r_z| : & p^2 s^2 \le \frac{r^2}{1+|r_z|} \left(2p - (1 - |r_z|)\right), \end{cases} \tag{44}$$

formally also including the trivial cases of $r_x = r_y = 0$. Now one can easily see that the condition for $p \le 1 - |r_z|$ is always satisfied if condition (43) is satisfied. If we insert $p = 1 - |r_z|$ into the condition for $p \ge 1 - |r_z|$, we obtain after simplifications

$$(1 - r_z^2) s^2 \le r^2, \tag{45}$$

which is also always satisfied if condition (43) is satisfied. Therefore the condition

$$p^2 s^2 \le \frac{r^2}{1 + |r_z|} \left(2p - (1 - |r_z|)\right) \tag{46}$$

is for $p \le 1 - |r_z|$ automatically satisfied, if condition (43) holds. This leads us to the Theorem. □

As stated in the main text, these results allow us to evaluate the optimal conversion probability $P(\rho \to \sigma)$ via IO and SIO for any two states ρ and σ of a single qubit.

Corollary 4. *The maximal probability $P(\rho \to \sigma)$ for a successful transformation from a coherent qubit state ρ to a coherent qubit state σ using IO or SIO is zero iff*

$$r^2 s_z^2 + \left(1 - r_z^2\right) s^2 > r^2 \tag{47}$$

and

$$\min\left\{ \frac{r^2}{(1 + |r_z|) s^2} \left(1 + \sqrt{1 - \frac{s^2 \left(1 - r_z^2\right)}{r^2}} \right), 1 \right\} \tag{48}$$

otherwise.

Proof. From Thm. 3 and its discussion in the main text, we get that a transformation from ρ to σ (with ρ coherent, i.e., $r > 0$ and therefore $r_z^2 < 1$) is possible with probability $p > 0$ iff

$$r^2 s_z^2 + \left(1 - r_z^2\right) s^2 \le r^2. \tag{49}$$

As soon as we are inside this ellipsoid, the maximal probability of success is bounded by Eq. (13b). Now we want to maximize p such that this inequality is still satisfied. This is the case if we choose the larger p for which

$$p^2 s^2 = \frac{r^2}{1 + |r_z|} \left(2p - (1 - |r_z|)\right). \tag{50}$$

Together with the assumptions that $p_{\max}$ is a probability, this finishes the proof. □

Next we prove the partial generalization of these results to higher dimensions.

Theorem 5. *A necessary condition for the existence of a stochastic SIO transformation from ρ to σ is*

$$C_{\Delta,R}(\sigma) \le C_{\Delta,R}(\rho). \tag{51}$$

Proof. Assume that there exists a SIO transformation which maps ρ to σ with probability $p \neq 0$. Due to Prop. 1, we can restrict ourselves to the case of successful transformations which we write as,

$$\Lambda[\rho] = p\sigma. \tag{52}$$

From here on, we will suppress unnecessary brackets for readability and write for example $\Lambda\rho$ instead of $\Lambda[\rho]$. Since Λ can be decomposed into strictly incoherent Kraus operators, we have $\Delta\Lambda = \Lambda\Delta$ and therefore

$$\Lambda\Delta\rho = \Delta\Lambda\rho = p\Delta\sigma. \tag{53}$$

Defining

$$s_\tau = \max\{s | \Delta\tau + s(\Delta\tau - \tau) \ge 0\}, \tag{54}$$

ensures that

$$\tilde{\rho} = \Delta\rho + s_\rho(\Delta\rho - \rho) \tag{55}$$

is a valid density operator with the property

$$\Lambda\tilde{\rho} = p(\Delta\sigma + s_\rho(\Delta\sigma - \sigma)) =: p\tilde{\sigma}, \tag{56}$$

where $\tilde{\sigma}$ is a valid density operator too. This implies that

$$s_\sigma \ge s_\rho \tag{57}$$

is a necessary condition for the existence of a stochastic transformation from ρ to σ using SIO. From the definition of s_τ follows directly that this quantity can be calculated using a semidefinite program. In addition, we will show now how Eq. (57) can be reformulated in terms of the Δ robustness of coherence $C_{\Delta,R}$ [4, 5]

$$C_{\Delta,R}(\rho) = \min\left\{ t \ge 0 \middle| \frac{\rho + t\tau}{1 + t} \in \mathcal{I}, \tau \ge 0, \Delta\rho = \Delta\tau \right\}. \tag{58}$$

Because $\Delta\rho = \Delta\tau$, this is equivalent to

$$C_{\Delta,R}(\rho) = \min\left\{ t \ge 0 \middle| \rho = (1 + t)\Delta\rho - t\tau, \tau \ge 0, \Delta\rho = \Delta\tau \right\}.$$

Using the same technique as in [6], we can further simplify this expression to

$$C_{\Delta,R}(\rho) = \min\left\{ t \ge 0 \middle| \rho \le (1 + t)\Delta\rho \right\}. \tag{59}$$

To prove this, we first note that for $t, \tau \ge 0$,

$$\rho = (1 + t)\Delta\rho - t\tau \tag{60}$$

implies $\rho \leq (1+t)\Delta\rho$. To show the converse, assume that $\rho \leq (1+t)\Delta\rho$. Then we can define $\tau := [(1+t)\Delta\rho - \rho]/t$ and it is easy to check that $\Delta\tau = \Delta\rho$, $\rho = (1+t)\Delta\rho - t\tau$ and $\tau \geq 0$. Substituting s by $1/t$, we find

$$\begin{aligned} s_\tau &= \max\{s|\Delta\tau + s(\Delta\tau - \tau) \geq 0\} \\ &= \max\{s \geq 0|\Delta\tau + s(\Delta\tau - \tau) \geq 0\} \\ &= \max\{1/t|\Delta\tau + (\Delta\tau - \tau)/t \geq 0, t \geq 0\} \\ &= \max\{1/t|\tau \leq (1+t)\Delta\tau, t \geq 0\}. \end{aligned} \tag{61}$$

A comparison with Eq. (59) shows that $C_{\Delta,R}(\tau) = 1/s_\tau$, and therefore Eq. (57) is equivalent to

$$C_{\Delta,R}(\sigma) \leq C_{\Delta,R}(\rho). \tag{62}$$

□

Now we turn to the proofs of our results concerning optimal conversion with assistance. We begin with the proof of Thm. 6 from the main text.

Theorem 6. *Let Alice and Bob share a pure two-qubit state $|\psi\rangle^{AB}$ and denote Bob's local state by ρ^B. The maximal probability $P_{\mathrm{a}}(|\psi\rangle^{AB} \to \sigma^B)$ to prepare the qubit state σ^B on Bob's side via one-way LQICC is given by*

$$P_{\mathrm{a}}\left(|\psi\rangle^{AB} \to \sigma^B\right) = \min\left\{1, (1-|r_z|)\,\frac{1+\sqrt{1-s^2}}{s^2}\right\}, \tag{63}$$

where $\boldsymbol{r}$ and $\boldsymbol{s}$ are the Bloch vectors of ρ^B and σ^B, respectively.

Proof. In the following, we will prove the more general case in which Alice holds an arbitrary purification of Bob's qubit state. Recall that by performing local measurements on Alice's side and using classical communication, Bob can obtain any decomposition $\{q_i, \rho_i^B\}$ of his local state $\rho^B = \sum_i q_i \rho_i^B$ [7]. After Alice's measurement has been performed, Bob applies incoherent operations to stochastically transform his post-measurement states ρ_i^B into the desired state σ^B. First we will show that there always exists an optimal decomposition containing only pure states.

To do this, we use a decomposition of every mixed qubit state ρ into two pure states, which we will also use later in this proof. The Bloch vector $\boldsymbol{r}$ corresponding to ρ can be written as a convex combination of two points $\boldsymbol{t}$ and $\boldsymbol{u}$ on the surface of the Bloch sphere having the same z-coordinate as $\boldsymbol{r}$, i.e.,

$$\boldsymbol{r} = q\boldsymbol{t} + (1-q)\boldsymbol{u}, \tag{64a}$$

$$r_z = t_z = u_z, \tag{64b}$$

$$|\boldsymbol{t}| = |\boldsymbol{u}| = 1. \tag{64c}$$

Now assume that an optimal decomposition of Bob's local state contains a mixed state ρ_x which occurs with probability q_x. Since every decomposition can be obtained, Bob can also obtain a decomposition in which the pair $\{q_x, \rho_x\}$ is replaced by the two corresponding pure states from Eqs. (64) and probabilities $q_x q, q_x(1-q)$. From Cor. 4 follows that the transformation probability from both of these states to any target state is at least as high as the probability from ρ_x: In case the transformation from ρ_x to a target state is forbidden by Eq. (47), there is nothing to prove. If not, the transformation probability from ρ_x and the two pure states to the target is determined by Eq. (48), since a pure initial state can never satisfy Eq. (47). Remember that by choice, we have $r_z = t_z = u_z$ and $t, u \geq r$. Now note that for a fixed target state σ (fixed $\boldsymbol{s}$) and fixed r_z, the quantity in Eq. (48) increases if r increases. From this follows the claim, which implies that the new decomposition is also optimal. Eliminating all mixed states using this procedure, we end up with an optimal decomposition $\{q_i, |\psi_i\rangle^B\}$ which only contains pure states.

If we denote the corresponding maximal conversion probability by $P_{\mathrm{a}}\left(|\psi_i\rangle^B \to \sigma^B\right)$, our figure of merit can be written as

$$P_{\mathrm{a}}\left(|\psi\rangle^{AB} \to \sigma^B\right) = \max \sum_i q_i P\left(|\psi_i\rangle^B \to \sigma^B\right), \tag{65}$$

where the maximization is performed over all pure state decompositions $\{q_i, |\psi_i\rangle^B\}$ of Bob's local state.

In the next step, we deduce from Eq. (48) that the maximal probability for stochastic conversion between two single-qubit states $|\psi_i\rangle^B$ and σ^B can be expressed as

$$P\left(|\psi_i\rangle^B \to \sigma^B\right) = \min\left\{1, \left(1-|r_z^i|\right)\frac{1+\sqrt{1-s^2}}{s^2}\right\}. \tag{66}$$

This result allows us to bound the average conversion probability for a decomposition $\{q_i, |\psi_i\rangle^B\}$ of the state ρ^B as follows:

$$\begin{aligned} &\sum_i q_i P\left(|\psi_i\rangle^B \to \sigma^B\right) \\ &\qquad = \sum_i q_i \min\left\{1, \left(1-|r_z^i|\right)\frac{1+\sqrt{1-s^2}}{s^2}\right\} \\ &\qquad \leq \sum_i q_i \times \left(1-|r_z^i|\right)\frac{1+\sqrt{1-s^2}}{s^2} \\ &\qquad = \left(1-\sum_i q_i |r_z^i|\right)\frac{1+\sqrt{1-s^2}}{s^2} \\ &\qquad \leq (1-|r_z|)\,\frac{1+\sqrt{1-s^2}}{s^2}, \end{aligned} \tag{67}$$

where in the last inequality we used the fact that $\sum_i q_i |r_z^i| = \sum_i q_i |\langle\psi_i|\sigma_z|\psi_i\rangle| \geq \left|\mathrm{Tr}[\rho^B\sigma_z]\right| = |r_z|$. Since Eq. (67) holds for any decomposition $\{q_i, |\psi_i\rangle^B\}$ of the state ρ^B, it implies that the probability for assisted conversion $P_{\mathrm{a}}\left(|\psi\rangle^{AB} \to \sigma^B\right)$ is bounded as

$$P_{\mathrm{a}}\left(|\psi\rangle^{AB} \to \sigma^B\right) \leq \min\left\{1, (1-|r_z|)\,\frac{1+\sqrt{1-s^2}}{s^2}\right\}. \quad (68)$$

To complete the proof of the Theorem, let Alice perform a two-outcome measurement such that Bob's post-measurement states $|\psi_i\rangle^B$ fulfill

$$\mathrm{Tr}[\rho^B\sigma_z] = \langle\psi_1|\sigma_z|\psi_1\rangle = \langle\psi_2|\sigma_z|\psi_2\rangle, \quad (69)$$

Such a measurement always exists, see the discussion above Eqs. (64).

Depending on the outcome i, Bob then applies a stochastic incoherent operation to convert the state $|\psi_i\rangle^B$ into the desired state σ^B. This conversion protocol gives a lower bound on our figure of merit:

$$\begin{aligned} P_{\mathrm{a}}\left(|\psi\rangle^{AB} \to \sigma^B\right) \geq & qP\left(|\psi_1\rangle^B \to \sigma^B\right) \\ & + [1-q]P\left(|\psi_2\rangle^B \to \sigma^B\right) \\ = & \min\left\{1, (1-|r_z|)\,\frac{1+\sqrt{1-s^2}}{s^2}\right\}, \end{aligned} \quad (70)$$

where in the last equality we used Eqs. (66) and (69). Noting that the lower bound (70) coincides with the upper bound (68), we conclude that the presented protocol achieves the claimed conversion probability (63) and that this probability is optimal. □

Next we prove our results concerning two-qubit Werner states.

Theorem 7. *The optimal probability $P_{\mathrm{a}}\left(\rho_{\mathrm{w}}^{AB} \to \sigma^B\right)$ for converting ρ_{w}^{AB} into the qubit state σ^B via one-way LQICC is given by,*

$$P_{\mathrm{a}}\left(\rho_{\mathrm{w}}^{AB} \to \sigma^B\right) = \begin{cases} 1 & \text{if } q_{\mathrm{w}} \geq \frac{s^2}{\sqrt{1-s_z^2}}, \\ 0 & \text{otherwise.} \end{cases} \quad (71)$$

Proof. Suppose that Alice performs a general local measurement with POVM elements $\{M_i^A\}$. Conditioned on the measurement outcome i, Bob finds his system in the state $\rho_i^B = \mathrm{Tr}_A[M_i^A\rho_{\mathrm{w}}^{AB}]/p_i$, where $p_i = \mathrm{Tr}[M_i^A\rho_{\mathrm{w}}^{AB}]$ is the corresponding probability. To determine the set of states that Bob can achieve in this setting with nonzero probability, recall from the discussion below Thm. 5 in the main text that Bob can transform a qubit state ρ into another qubit state $\tilde{\rho}$ via stochastic incoherent operations if and only if

$$C_{\Delta,R}(\rho) \geq C_{\Delta,R}(\tilde{\rho}), \quad (72)$$

where $C_{\Delta,R}$ is the Δ-robustness of coherence [4, 5, 8], which, for a single-qubit state ρ, can be expressed as

$$C_{\Delta,R}(\rho) = \frac{r}{\sqrt{1-r_z^2}}. \quad (73)$$

Thus, for characterizing the set of states achievable with nonzero probability, we need to evaluate the maximal Δ-robustness $C_{\Delta,R}$ for any possible post-measurement state of Bob. Noting that $C_{\Delta,R}$ is convex, which follows directly from Eq. (59), we can restrict ourselves to rank-one POVMs on Alice's side.

In the next step, note that for any rank one POVM element M^A, the corresponding post-measurement state of Bob has the form

$$\rho^B = q_{\mathrm{w}}|\eta\rangle\langle\eta|^B + (1-q_{\mathrm{w}})\frac{\mathbb{1}^B}{2}. \quad (74)$$

While the probability q_{w} here is fixed via the initial Werner state

$$\rho_{\mathrm{w}}^{AB} = q_{\mathrm{w}}|\phi^+\rangle\langle\phi^+| + (1-q_{\mathrm{w}})\frac{\mathbb{1}}{4}, \quad (75)$$

we can arbitrarily vary the state $|\eta\rangle$ by suitable adjusting Alice's POVM elements. Among all such states, the maximal Δ-robustness $C_{\Delta,R}$ is attained for

$$\mu^B = q_{\mathrm{w}}|+\rangle\langle+|^B + (1-q_{\mathrm{w}})\frac{\mathbb{1}^B}{2}, \quad (76)$$

i.e., it holds $C_{\Delta,R}(\rho^B) \leq C_{\Delta,R}(\mu^B)$. To see this, note that for single-qubit states the Δ-robustness does not increase under incoherent operations [4, 5, 8], and moreover for any $|\eta\rangle^B$ there exists an incoherent operation converting μ^B into ρ^B [9]. Combining these arguments, we see that any post-measurement state of Bob has not more Δ-robustness of coherence than the state μ^B. Thus, we obtain the following condition for assisted state conversion of the Werner state (75) into a state σ^B on Bob's side:

$$P_{\mathrm{a}}\left(\rho_{\mathrm{w}}^{AB} \to \sigma^B\right) > 0 \;\Rightarrow\; C_{\Delta,R}(\sigma^B) \leq C_{\Delta,R}(\mu^B). \quad (77)$$

We will now show that the state μ^B is in fact achievable with unit probability: $P_{\mathrm{a}}\left(\rho_{\mathrm{w}}^{AB} \to \mu^B\right) = 1$. For this, Alice first performs a local measurement on the Werner state in the $\{|+\rangle, |-\rangle\}$ basis. Conditioned on the measurement outcome Bob finds his system either in the desired state μ^B, or in the state $\sigma_z\mu^B\sigma_z$. In the latter case, Bob performs an incoherent σ_z rotation, thus obtaining μ^B with unit probability. Note that via local incoherent operations Bob can transform the state μ^B into another state σ^B with unit probability if and only if $C_{\Delta,R}(\sigma^B) \leq C_{\Delta,R}(\mu^B)$. This follows from the discussion of the geometrical interpretation of Thm. 3 in the main text. All states reachable from μ^B with non-zero probability are inside the ellipse through μ^B, which is equivalent to $C_{\Delta,R}(\sigma^B) \leq C_{\Delta,R}(\mu^B)$. These states are reachable with certainty, since $\mathrm{Tr}(\mu^B\sigma_z) = 0$. Combining these arguments, we obtain the following condition:

$$C_{\Delta,R}(\sigma^B) \leq C_{\Delta,R}(\mu^B) \;\Rightarrow\; P_{\mathrm{a}}\left(\rho_{\mathrm{w}}^{AB} \to \sigma^B\right) = 1. \quad (78)$$

Both conditions (77) and (78) imply the following:

$$C_{\Delta,R}(\sigma^B) \leq C_{\Delta,R}(\mu^B) \quad \Leftrightarrow \quad P_{\mathrm{a}}\left(\rho_{\mathrm{w}}^{AB} \rightarrow \sigma^B\right) = 1, \tag{79a}$$

$$C_{\Delta,R}(\sigma^B) > C_{\Delta,R}(\mu^B) \quad \Leftrightarrow \quad P_{\mathrm{a}}\left(\rho_{\mathrm{w}}^{AB} \rightarrow \sigma^B\right) = 0. \tag{79b}$$

The proof of the Theorem is complete by noting that $C_{\Delta,R}(\mu^B) = q_{\mathrm{w}}$. □

We will continue with the proof of our claim from the main text that correlations enhance conversion probabilities, as long as the global state is not quantum-incoherent.

Theorem 8. *If Bob's system is a qubit, then for any state ρ^{AB} which is correlated and not quantum-incoherent the set of accessible states for Bob via stochastic one-way LQICC is strictly larger, when compared to $\rho^A \otimes \rho^B$.*

Proof. As is shown in Lem. 2, for any correlated state $\rho^{AB} \neq \rho^A \otimes \rho^B$ there exists a two-element POVM $\{M_1^A, M_2^A\}$ on Alice's side such that $\rho_1^B \neq \rho_2^B$, where ρ_i^B is the state of Bob conditioned on the measurement outcome of Alice:

$$\rho_i^B = \frac{\mathrm{Tr}_A\left[M_i^A \rho^{AB}\right]}{p_i}, \tag{80}$$

and $p_i = \mathrm{Tr}\left[M_i^A \rho^{AB}\right]$ is the corresponding probability for obtaining the outcome i. Noting that

$$\rho^B = p_1 \rho_1^B + p_2 \rho_2^B, \tag{81}$$

we conclude that – whenever the state ρ^{AB} is not quantum-incoherent – either ρ_1^B or ρ_2^B must be outside of the reachable ellipsoid of Bob's reduced state ρ^B. This completes the proof of the Theorem. □

Next we continue with our statements about asymptotic conversion rates.

Theorem 9. *Assume qubit states ρ and σ obey*

$$s_z^2 \leq r_z^2 \text{ and } s = r. \tag{82}$$

Then we have $R(\rho \rightarrow \sigma) = 1$.

Proof. In the first step of the proof note that $P(\rho \rightarrow \sigma) = 1$ for any two states ρ and σ fulfilling Eqs. (82), which follows directly from Eqs. (3a) and (3b) in [3]. This proves that $R(\rho \rightarrow \sigma) \geq 1$ in this case.

In the next step we will show that states fulfilling Eqs. (82) have equal coherence cost:

$$C_{\mathrm{c}}(\rho) = C_{\mathrm{c}}(\sigma). \tag{83}$$

Since $C_{\mathrm{c}}(\rho)/C_{\mathrm{c}}(\sigma)$ is an upper bound on the conversion rate, this will then complete the proof of the Theorem. For proving Eq. (83), note that $r^2 = r_x^2 + r_y^2 = 4|\rho_{01}|^2$. Thus, Eqs. (82) directly imply the equality $|\rho_{01}|^2 = |\sigma_{01}|^2$. Now note that for any single-qubit state ρ the coherence cost is a simple function of $|\rho_{01}|^2$, see Eq. (19) in the main text. This completes the proof of Eq. (83) and also the proof of the Theorem. □

As we show now, the above Theorem cannot be formulated as an if and only if statement. From Eq. (18) in the main text follows that the coherence cost of a pure state ψ is equal to $S(\Delta[\psi])$ and therefore equal to its distillable coherence (compare Eq. (17) in the main text). Now there exist pure states ψ and mixed states ρ such that

$$0 < C_{\mathrm{d}}(\psi) = C_{\mathrm{d}}(\rho). \tag{84}$$

Using Eq. (16) from the main text, we can conclude

$$1 = \frac{C_{\mathrm{d}}(\rho)}{C_{\mathrm{d}}(\psi)} = \frac{C_{\mathrm{d}}(\rho)}{C_{\mathrm{c}}(\psi)} \leq R(\rho \rightarrow \psi) \leq \frac{C_{\mathrm{d}}(\rho)}{C_{\mathrm{d}}(\psi)} = 1, \tag{85}$$

which proves $R(\rho \rightarrow \psi) = 1$. However, this case is not covered by Thm. 9.

We conclude this section with the proof of Prop. 10.

Proposition 10. *Among all single-qubit states, the family of states*

$$\sigma = q|+\rangle\langle+| + (1-q)|-\rangle\langle-| \quad (0 \leq q \leq 1) \tag{86}$$

has the minimal distillable coherence C_{d} for a fixed coherence cost C_{c} and vice versa maximal C_{c} for fixed C_{d}.

Proof. In the first step, we recall that for any single-qubit state ρ the coherence cost depends only on the absolute value of the offdiagonal element $|\rho_{01}| = |\langle 0|\rho|1\rangle|$, see Eq. (19) in the main text. In particular, C_{c} is a strictly monotonically increasing function of $|\rho_{01}|$. Moreover, recall that $|\rho_{01}|$ is directly related to the Euclidian distance of the state to the incoherent axis in the Bloch space: $r_x^2 + r_y^2 = 4|\rho_{01}|^2$ [10]. This means that all states with a fixed coherence cost have the same distance to the incoherent axis in the Bloch space.

In the next step, we note that for any single-qubit state ρ with Bloch vector $\boldsymbol{r} = (r_x, r_y, r_z)^T$ we can introduce the state $\tilde{\rho}$ having the Bloch coordinates

$$\tilde{r}_x = \sqrt{r_x^2 + r_y^2}, \quad \tilde{r}_y = 0, \quad \tilde{r}_z = r_z. \tag{87}$$

The state $\tilde{\rho}$ can be obtained from ρ via an incoherent unitary, and thus both states have the same coherence cost and distillable coherence. In the next step, we introduce the state τ as follows:

$$\tau = \frac{1}{2}\tilde{\rho} + \frac{1}{2}\sigma_x \tilde{\rho} \sigma_x. \tag{88}$$

Note that τ has the same distance to the incoherent axis – and thus the same coherence cost – as ρ and $\tilde{\rho}$, i.e.,

$$C_{\mathrm{c}}(\tau) = C_{\mathrm{c}}(\tilde{\rho}) = C_{\mathrm{c}}(\rho). \tag{89}$$

Moreover, it is straightforward to see that τ lies on the maximally coherent plane, i.e., the plane spanned by Bloch vectors corresponding to maximally coherent states. By construction, the Bloch vector of τ also lies in the x-z plane, which implies that τ has the desired form (86).

In the final step, recall that the distillable coherence is convex, and thus

$$C_d(\tau) \leq \frac{1}{2}C_d(\tilde{\rho}) + \frac{1}{2}C_d(\sigma_x\tilde{\rho}\sigma_x) = C_d(\tilde{\rho}) = C_d(\rho), \quad (90)$$

where we used the facts that the Pauli matrix σ_x is an incoherent unitary, and thus preserves C_d, and that ρ and $\tilde{\rho}$ have the same distillable coherence. This completes the proof. □

III. EXPERIMENTAL ASPECTS

In this section, we describe the experimental details concerning the implementation of non-unitary incoherent operations as illustrated in Fig. 1(d) in the main text. The combination of $BD_{1,2,3}$ and $H_{4,5,6,7,8,9}$ allows us to implement strictly incoherent operations given by Kraus operators of the form

$$K_1 = \begin{pmatrix} \cos\theta_0 & 0 \\ 0 & \cos\theta_1 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & \sin\theta_1 \\ \sin\theta_0 & 0 \end{pmatrix}. \quad (91)$$

For experimentally realizing these operators, the angles of $H_{6,8,9}$ are set to 45° for applying a bit flip σ_x on the polarization, the angles of $H_{4,5}$ are set to $\frac{\theta_0}{2}$ and $\frac{\theta_1+90°}{2}$, and H_7 is used for phase compensation, respectively.

Without loss of generality, we suppose initially we have a qubit state

$$\rho_0 = \frac{1}{2}\left(\mathbb{1} + r_x\sigma_x + r_z\sigma_z\right) \quad (92)$$

in the basis $\{|H\rangle, |V\rangle\}$. Considering the path degree of freedom, which is a two dimensional system e_0, e_1, the overall state can be written as

$$\rho_0 \otimes |e_0\rangle\langle e_0|, \quad (93)$$

where we assume the initial state is in path e_0. Then BD_1 displaces the horizontally polarized component of a photon from the vertical component to a distance of about 6 mm. Accordingly, the quantum state is entangled by a controlled-NOT gate (the polarization encoded qubit is the contolling qubit) acting on the whole state, resulting in

$$\rho_1 = \begin{pmatrix} \frac{1+r_z}{2} & 0 & 0 & \frac{r_x}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{r_x}{2} & 0 & 0 & \frac{1-r_z}{2} \end{pmatrix}. \quad (94)$$

Then, using $H_{4,5}$, we implement a controlled-rotation operation on the whole state,

$$\begin{aligned} |H\rangle\langle H| \otimes |e_0\rangle\langle e_0| &\xrightarrow{H_4} |\theta_0^+\rangle\langle\theta_0^+| \otimes |e_0\rangle\langle e_0|, \\ |V\rangle\langle V| \otimes |e_0\rangle\langle e_0| &\xrightarrow{H_4} |\theta_0^-\rangle\langle\theta_0^-| \otimes |e_0\rangle\langle e_0|, \\ |H\rangle\langle H| \otimes |e_1\rangle\langle e_1| &\xrightarrow{H_5} |\theta_1^+\rangle\langle\theta_1^+| \otimes |e_1\rangle\langle e_1|, \\ |V\rangle\langle V| \otimes |e_1\rangle\langle e_1| &\xrightarrow{H_5} |\theta_1^-\rangle\langle\theta_1^-| \otimes |e_1\rangle\langle e_1|, \end{aligned} \quad (95)$$

where we have

$$\begin{aligned} |\theta_0^+\rangle &= \cos\theta_0|H\rangle + \sin\theta_0|V\rangle, \\ |\theta_0^-\rangle &= \sin\theta_0|H\rangle - \cos\theta_0|V\rangle, \\ |\theta_1^+\rangle &= \cos\theta_1|H\rangle + \sin\theta_1|V\rangle, \\ |\theta_1^-\rangle &= \sin\theta_1|H\rangle - \cos\theta_1|V\rangle. \end{aligned} \quad (96)$$

Then the overall state after this transformation is given by

$$\rho_2 = \begin{pmatrix} \frac{1+r_z}{2}\cos^2\theta_0 & \frac{r_x}{2}\cos\theta_0\sin\theta_1 & \frac{1+r_z}{4}\sin 2\theta_0 & -\frac{r_x}{2}\cos\theta_0\cos\theta_1 \\ \frac{r_x}{2}\cos\theta_0\sin\theta_1 & \frac{1-r_z}{2}\sin^2\theta_1 & \frac{r_x}{2}\sin\theta_0\sin\theta_1 & -\frac{1-r_z}{4}\sin 2\theta_1 \\ \frac{1+r_z}{4}\sin 2\theta_0 & \frac{r_x}{2}\sin\theta_0\sin\theta_1 & \frac{1+r_z}{2}\sin^2\theta_0 & -\frac{r_x}{2}\sin\theta_0\cos\theta_1 \\ -\frac{r_x}{2}\cos\theta_0\cos\theta_1 & -\frac{1-r_z}{4}\sin 2\theta_1 & -\frac{r_x}{2}\sin\theta_0\cos\theta_1 & \frac{1-r_z}{2}\cos^2\theta_1 \end{pmatrix} \quad (97)$$

in the basis $\{|H\rangle \otimes |e_0\rangle, |H\rangle \otimes |e_1\rangle, |V\rangle \otimes |e_0\rangle, |V\rangle \otimes |e_1\rangle\}$.

BD_2 has length equal to $\frac{2}{3}$ the length of $BD_{1,3}$, which displace the horizontal component 4 mm away from the vertical component, resulting in the following transformations by extending the system to a higher-dimensional space,

$$\begin{aligned} |H\rangle \otimes |e_0\rangle &\xrightarrow{BD_2} |H\rangle \otimes |d_0\rangle, \\ |V\rangle \otimes |e_0\rangle &\xrightarrow{BD_2} |V\rangle \otimes |d_1\rangle, \\ |H\rangle \otimes |e_1\rangle &\xrightarrow{BD_2} |H\rangle \otimes |d_2\rangle, \\ |V\rangle \otimes |e_1\rangle &\xrightarrow{BD_2} |V\rangle \otimes |d_3\rangle. \end{aligned} \quad (98)$$

Then the quantum state after BD_2 will be

$$\rho_3 = \begin{pmatrix} \frac{1+r_z}{2}\cos^2\theta_0 & 0 & \frac{r_x}{2}\cos\theta_0\sin\theta_1 & 0 & 0 & \frac{1+r_z}{4}\sin 2\theta_0 & 0 & -\frac{r_x}{2}\cos\theta_0\cos\theta_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{r_x}{2}\cos\theta_0\sin\theta_1 & 0 & \frac{1-r_z}{2}\sin^2\theta_1 & 0 & 0 & \frac{r_x}{2}\sin\theta_0\sin\theta_1 & 0 & -\frac{1-r_z}{4}\sin 2\theta_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1+r_z}{4}\sin 2\theta_0 & 0 & \frac{r_x}{2}\sin\theta_0\sin\theta_1 & 0 & 0 & \frac{1+r_z}{2}\sin^2\theta_0 & 0 & -\frac{r_x}{2}\sin\theta_0\cos\theta_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{r_x}{2}\cos\theta_0\cos\theta_1 & 0 & -\frac{1-r_z}{4}\sin 2\theta_1 & 0 & 0 & -\frac{r_x}{2}\sin\theta_0\cos\theta_1 & 0 & \frac{1-r_z}{2}\cos^2\theta_1 \end{pmatrix} \tag{99}$$

in the basis $\{|H\rangle\otimes|d_0\rangle, |H\rangle\otimes|d_1\rangle, |H\rangle\otimes|d_2\rangle, |H\rangle\otimes|d_3\rangle, |V\rangle\otimes|d_0\rangle, |V\rangle\otimes|d_1\rangle, |V\rangle\otimes|d_2\rangle, |V\rangle\otimes|d_3\rangle\}$. As $H_{6,8}$ are set to 45°, performing a σ_x operation on the polarization state in path $d_{0,3}$, and H_7 is set to 0° for applying a σ_z operation on the polarization state in path $d_{1,2}$, the state afterwards is given by

$$\rho_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1-r_z}{2}\sin^2\theta_1 & -\frac{1-r_z}{4}\sin 2\theta_1 & \frac{r_x}{2}\cos\theta_0\sin\theta_1 & -\frac{r_x}{2}\sin\theta_0\sin\theta_1 & 0 & 0 \\ 0 & 0 & -\frac{1-r_z}{4}\sin 2\theta_1 & \frac{1-r_z}{2}\cos^2\theta_1 & -\frac{r_x}{2}\cos\theta_0\cos\theta_1 & -\frac{r_x}{2}\sin\theta_0\cos\theta_1 & 0 & 0 \\ 0 & 0 & \frac{r_x}{2}\cos\theta_0\sin\theta_1 & -\frac{r_x}{2}\cos\theta_0\cos\theta_1 & \frac{1+r_z}{2}\cos^2\theta_0 & \frac{1+r_z}{4}\sin 2\theta_0 & 0 & 0 \\ 0 & 0 & -\frac{r_x}{2}\sin\theta_0\sin\theta_1 & -\frac{r_x}{2}\sin\theta_0\cos\theta_1 & \frac{1+r_z}{4}\sin 2\theta_0 & \frac{1+r_z}{2}\sin^2\theta_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{100}$$

The action of BD_3 will coherently combine the paths $d_{0,2}$, $d_{1,3}$, followed by H_9 with rotation angle 45° which flips the polarization state in path e_0. The final quantum state after that two Hs will be

$$\rho_5 = \begin{pmatrix} \frac{1+r_z}{2}\cos^2\theta_0 & \frac{r_x}{2}\cos\theta_0\cos\theta_1 & \frac{r_x}{2}\cos\theta_0\sin\theta_1 & \frac{1+r_z}{4}\sin 2\theta_0 \\ \frac{r_x}{2}\cos\theta_0\cos\theta_1 & \frac{1-r_z}{2}\cos^2\theta_1 & -\frac{1-r_z}{4}\sin 2\theta_1 & -\frac{r_x}{2}\sin\theta_0\cos\theta_1 \\ \frac{r_x}{2}\cos\theta_0\sin\theta_1 & -\frac{1-r_z}{4}\sin 2\theta_1 & \frac{1-r_z}{2}\sin^2\theta_1 & \frac{r_x}{2}\sin\theta_0\sin\theta_1 \\ \frac{1+r_z}{4}\sin 2\theta_0 & -\frac{r_x}{2}\sin\theta_0\cos\theta_1 & \frac{r_x}{2}\sin\theta_0\sin\theta_1 & \frac{1+r_z}{2}\sin^2\theta_0 \end{pmatrix}. \tag{101}$$

Note that we can tilt BD_3 for phase compensation, removing the negative signs in the interference terms between the horizontal and vertical component in each arm. When tracing over the path degree, we obtain

$$\rho_f = \begin{pmatrix} \frac{1+r_z}{2}\cos^2\theta_0 + \frac{1-r_z}{2}\sin^2\theta_1 & \frac{r_x}{2}\cos\theta_0\cos\theta_1 + \frac{r_x}{2}\sin\theta_0\sin\theta_1 \\ \frac{r_x}{2}\cos\theta_0\cos\theta_1 + \frac{r_x}{2}\sin\theta_0\sin\theta_1 & \frac{1+r_z}{2}\sin^2\theta_0 + \frac{1-r_z}{2}\cos^2\theta_1 \end{pmatrix}, \tag{102}$$

which is exactly the state after implementing K_1 and K_2. Combination of two BDs (BD_1 and BD_3) results in a natural robust interference, and the phase between d_0, d_2 and d_1, d_3 can be removed either by adjusting the position of $BD_{1,3}$ or tilting the Hs in each arm.

Note that in our experiments, if we obtain a final state with Bloch coordinates $\tilde{r}_x > 0$ and $\tilde{r}_z > 0$, then we can also obtain the state $(-\tilde{r}_x, 0, \tilde{r}_z)$, $(\tilde{r}_x, 0, -\tilde{r}_z)$, and $(-\tilde{r}_x, 0, -\tilde{r}_z)$ via a σ_z operation, a σ_x operation and the combination of a σ_z operation and a σ_x operation. In Fig. 1(d), H_{10} is set to 0° for implementing a σ_z operation, and H_{11} is set to 45° for implementing a σ_x operation. Namely, we only need to focus on the states with Bloch coordinates $\tilde{r}_x > 0$ and $\tilde{r}_z > 0$, and complete the whole boundary via simple incoherent operations.

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[10] Compare also the proof of Thm. 9.