Supplementary information for:
Real-time calibration with spectator qubits

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In this document, we present additional details of the calculations used in the article. In Supplementary Section I, we explain in more detail the equations for the case where both data qubit and spectator qubit are subject to linear errors. Supplementary Section II has the derivation of the probability distribution for the estimator of an error parameter undergoing a random walk. Both Supplementary Sections III and IV contain the analytical solutions for the fidelities in the applications for the cases where a spectator qubit is not present, and when it is, respectively. Finally, Supplementary Section V describes the difficulties of applying the Robust Phase Estimation (RPE) for a parameter that is changing in time.

I. DETAILS OF THE LINEAR VS. LINEAR CASE

To derive Eq. (16), we take into account that the gate fidelity $F(\phi) = \cos^2(N\phi)$ for N overrotations of an amount ϕ has the following second and third derivatives:

$$F^{(2)}(\phi) = -2N^2 \cos(2N\phi),$$
$$F^{(3)}(\phi) = 4N^3 \sin(2N\phi).$$

Replacing these expressions on the right-hand side of condition (12), we find:

$$\left|\frac{1}{F^{(2)}(0)} R_2(\phi_s)\right| = \left|2N \int_0^{\phi_s} \mathrm{d}x\ (x-\phi_s)^2 \sin(2Nx)\right|,$$

where $R_2(\phi)$ is two times the Lagrange remainder:

$$R_2(\phi) = \int_0^{\phi} \mathrm{d}x\ (x-\phi)^2 F^{(3)}(x). \tag{1}$$

After integration, this becomes:

$$\left|\frac{1}{F^{(2)}(0)} R_2(\phi_s)\right| = \left|\phi_s^2 + \frac{\cos(2N\phi_s)-1}{2N^2}\right|.$$

For large N, the term of the order of ϕ_s^2 predominates. If $\phi_s^2 = (4N)^{-1}$, this expression becomes the same as Eq. (16). How these terms become similar as N grows can be seen in Fig. 1.

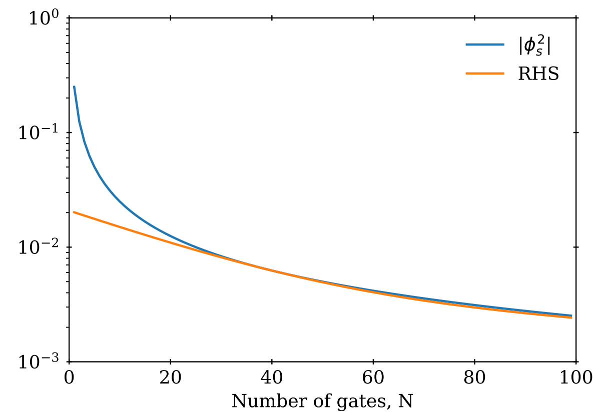

FIG. 1. $|\phi_s^2|$ and right-hand side (RHS) of condition (12) for $\phi_s = (4N)^{-1}$, as in Eq. (16). After ~ 10 gates, the two sides are already of the same order of magnitude, and condition (12) is no longer satisfied.

II. PROBABILITY DISTRIBUTION OF THE ESTIMATOR

In Eq. (23), we defined a random variable $\Theta_{kM}^{(\mathrm{est})}$ that was the average of the Gaussian random variables Θ_{kM}, Θ_{kM-1}, ..., $\Theta_{kM-(M-1)}$. The probability distribution of this random variable will therefore also be a Gaussian, and can be fully characterized by its first two cumulants, μ_k and σ_k^2. To find these two, we just need the means, variances and covariances of the individual Gaussian random variables Θ_k, as can be seen from the following:

$$\mu_k = \frac{1}{M}\sum_{n=0}^{M-1} \langle \Theta_{kM-n}\rangle,$$
$$\sigma_k^2 = \frac{1}{M^2}\sum_{n,q=0}^{M-1} \left[\langle \Theta_{kM-n}\Theta_{kM-q}\rangle - \langle \Theta_{kM-n}\rangle\langle \Theta_{kM-q}\rangle\right].$$

Therefore, we just need the probability distribution for the individual positions in the random walk. As long as space is isotropic, the conditional probability distribution for Θ_{kM-n} will be proportional to the product of two random walks starting at $\Theta_0 = \theta_0$ and Θ the extremities and ending at θ_{kM-n}:

$$p(\Theta_{kM-n} = \theta_{kM-n}|\Theta_0 = \theta_0, \Theta_{kM} = \theta_{kM})$$
$$\propto p(\Theta_{kM-n} = \theta_{kM-n}|\Theta_0 = \theta_0) p(\Theta_n = \theta_n|\Theta_0 = \theta_{kM}).$$

After normalization, we find a Gaussian distribution of the form given by Eq. (21):

$$p(\Theta_{kM-n} = \theta_{kM-n}|\Theta_0 = \theta_0, \Theta_{kM} = \theta_{kM})$$
$$= \mathcal{N}\left(\theta_{kM-n}; \theta_{kM} + n\frac{\theta_0-\theta_{kM}}{kM}, \frac{n(kM-n)}{kM}(\Delta\theta)^2\right).$$

The mean of the estimator $\Theta_{kM}^{(\mathrm{est})}$ can then be found from the average of the random variables Θ_{kM-n}:

$$\mu_k = \frac{1}{M}\sum_{n=0}^{M-1}\left(\theta_{kM} + n\frac{\theta_0-\theta_{kM}}{kM}\right),$$

which trivially yields Eq. (25).

To find σ_k^2, we will also need the covariances between any two random variables Θ_{kM-n} and Θ_{kM-q}, which can be found from the correlation function ρ:

$$\begin{aligned}\langle\Theta_{kM-n}\Theta_{kM-q}\rangle &- \langle\Theta_{kM-n}\rangle\langle\Theta_{kM-q}\rangle \\ &= \sqrt{\frac{n(kM-n)}{kM}}\sqrt{\frac{q(kM-q)}{kM}}(\Delta\theta)^2\rho.\end{aligned}$$

The correlation function can be found from the coefficients that multiply the quadratic terms in the exponents of the joint Gaussian distribution:

$$\begin{aligned}p(\Theta_{kM-n} = \theta_{kM-n}|\Theta_0 = \theta_0)&p(\Theta_q = \theta_{kM}|\Theta_0 = \theta_{kM-q}) \\ \times\, p(\Theta_{n-q} &= \theta_{kM-q}|\Theta_0 = \theta_{kM-n}) \\ \propto \exp&\left\{-\frac{kM}{(1-\rho^2)n(kM-n)}\frac{\theta_{kM-n}^2}{2(\Delta\theta)^2}\right\} \\ &\times\exp\left\{-\frac{kM}{(1-\rho^2)q(kM-q)}\frac{\theta_{kM-q}^2}{2(\Delta\theta)^2}\right\},\end{aligned}$$

where we assumed $n \geq q$, without loss of generality.

Comparing with the known probability distributions for the random walk, we find:

$$\rho^2 = 1 - \frac{kM(n-q)}{n(kM-q)} = \frac{q(kM-n)}{n(kM-q)}.$$

Then, we find a simple expression for the covariance of the individual random variables, which can be translated into the variance of the estimator:

$$\sigma_k^2 = \frac{1}{M^2}\sum_{n=0}^{M-1}\left[\frac{n(kM-n)}{kM} + 2\sum_{q=0}^{n-1}\frac{q(kM-n)}{kM}\right](\Delta\theta)^2.$$

Using Faulhaber's formulas to calculate $\sum n^2$ and $\sum n^3$, it is straightforward to recover the result from Eq. (26).

III. ANALYTICAL EXPRESSIONS FOR THE FIDELITY WITHOUT SPECTATOR QUBITS

A. Pairs of perpendicular pulses

A sequence of four pulses in the direction $\hat{\mathbf{e}}_{\mathbf{x}}$ spaced by time intervals τ, while a system undergoes a rotation by a classical magnetic field $\mathbf{B}$, will result in the following process fidelity:

$$F = \frac{1}{4}\left|\mathrm{Tr}\left\{\left[e^{-i\mathbf{B}\cdot\boldsymbol{\sigma}\tau}(\hat{\mathbf{e}}_{\mathbf{x}}\cdot\boldsymbol{\sigma})\right]^4\right\}\right|^2.$$

The average of this fidelity can then be written in terms of components that are parallel ($B_{\parallel}$), and perpendicular ($B_{\perp}$) to $\hat{\mathbf{e}}_{\mathbf{x}}$:

$$\begin{aligned}\langle F_{\mathrm{nospec}}\rangle = 1 - 16\langle B_{\parallel}^2\rangle\tau^2 + \frac{16}{3}\langle B_{\parallel}^2 B^2\rangle\tau^4 + 16\langle B_{\parallel}^4\rangle\tau^4 \\ + O(B^6\tau^6).\end{aligned}$$

The extant sum of averages of components of $\mathbf{B}$ are replaced by the appropriate moments of the random walks of B_x, B_y, and B_y, which correspond to Gaussians centered in the initial value of $\mathbf{B}$ and with standard deviations that grow with $\sqrt{N}$. $\hat{\mathbf{e}}_{\mathbf{x}}$ is chosen to be proportional to the cross product of the current estimate of the magnetic field, $\tilde{\mathbf{B}}$ and some random vector, being therefore:

$$\begin{aligned}\hat{\mathbf{e}}_{\mathbf{x}} = \frac{3}{2\tilde{B}}&\left(\sqrt{1-u^2}\sin\phi\tilde{B}_z - u\tilde{B}_y\right)\hat{\mathbf{x}} \\ &+ \frac{3}{2\tilde{B}}\left(u\tilde{B}_x - \sqrt{1-u^2}\cos\phi\tilde{B}_z\right)\hat{\mathbf{y}} \\ +\, \frac{3}{2\tilde{B}}&\left(\sqrt{1-u^2}\cos\phi\tilde{B}_y - \sqrt{1-u^2}\sin\phi\tilde{B}_x\right)\hat{\mathbf{z}},\end{aligned}$$

where u and ϕ are sampled uniformly from intervals $[-1, 1]$ and $[0, 2\pi]$, respectively.

B. Tailored XY-4 dynamical decoupling

Given alternated pulses in the $\hat{\mathbf{e}}_{\mathbf{x}}$ and $\hat{\mathbf{e}}_{\mathbf{y}}$ directions, the average process fidelity of a qubit under a classical field $\hat{\mathbf{B}}$ will be, after a sequence of four pulses spaced by a time τ;

$$F = \left|1 - 8(\hat{\mathbf{e}}_{\mathbf{x}}\cdot\mathbf{B})^2(\hat{\mathbf{e}}_{\mathbf{y}}\cdot\mathbf{B})^2\frac{\sin^4(B\tau)}{B^4}\right|^2.$$

The unit vectors $\hat{\mathbf{e}}_{\mathbf{x}}$ and $\hat{\mathbf{e}}_{\mathbf{y}}$ are chosen to be perpendicular to each other and the current estimate of the magnetic field, $\tilde{\mathbf{B}}$. The vector $\hat{\mathbf{e}}_{\mathbf{x}}$ is proportional to the cross product between $\tilde{\mathbf{B}}$ and some random unit vector $\hat{\mathbf{n}}$, while $\hat{\mathbf{y}}$ is proportional to the cross product between $\tilde{\mathbf{B}}$ and $\hat{\mathbf{e}}_{\mathbf{x}}$, so that average the process fidelity will be:

$$\langle F_{\mathrm{nospec}}\rangle = 1 - 2\left\langle\left\|\mathbf{B} - \frac{1}{\tilde{B}^2}\tilde{\mathbf{B}}(\mathbf{B}\cdot\tilde{\mathbf{B}})\right\|^4\right\rangle\tau^4 + O(B^6\tau^6).$$

The extant moments of $\mathbf{B}$ are then replaced as in the case of perpendicular pulses.

C. Pointing instability

The process fidelity of an X gate affected by a pointing instability δ, when we estimate this parameter to be $\bar{\delta}$, is given by the exact expression:

$$F = \left| \mathrm{Tr}\left\{ \exp\left\{ -i\frac{\pi}{2}\left(\frac{1-\delta^2}{1-\bar{\delta}^2}-1\right)\sigma_x \right\}\right\}\right|^2$$
$$= \frac{1}{2}+\frac{1}{2}\cos\left(\pi\frac{\bar{\delta}^2-\delta^2}{1-\bar{\delta}^2}\right).$$

Multiplying by the probability distribution and integrating, we find the following expression for the average process fidelity:

$$\langle F_{\mathrm{nospec}}\rangle = \frac{1}{2}+\frac{1}{2}\sqrt{\frac{1-\bar{\delta}^2}{1-\bar{\delta}^2+2iN(\Delta\delta)^2\pi}}$$
$$\times \mathrm{Re}\left\{ \exp\left\{ \frac{-i\delta_0^2\pi}{1-\bar{\delta}^2+2iN(\Delta\delta)^2\pi}+\frac{i\bar{\delta}^2\pi}{1-\bar{\delta}^2}\right\}\right\}.$$

and where δ_0 is the initial value, and the $\Delta\delta$ is the standard deviation of the Gaussian steps of the random walk.

D. Amplitude instability

If we apply an X gate affected by the ε parameter via an SK1 pulse sequence, the process fidelity will be:

$$F = \left[\frac{7}{8}+\frac{1}{8}\cos^2\left(\pi\frac{\bar{\varepsilon}-\varepsilon}{1-\bar{\varepsilon}}\right)\right]^2\cos^2\left(\frac{\pi}{2}\frac{\bar{\varepsilon}-\varepsilon}{1-\bar{\varepsilon}}\right)$$
$$+\frac{1}{4}\sin\left(2\pi\frac{\bar{\varepsilon}-\varepsilon}{1-\bar{\varepsilon}}\right)\sin\left(\pi\frac{\bar{\varepsilon}-\varepsilon}{1-\bar{\varepsilon}}\right)\left[\frac{7}{8}+\frac{1}{8}\cos^2\left(\pi\frac{\bar{\varepsilon}-\varepsilon}{1-\bar{\varepsilon}}\right)\right]$$
$$+\frac{1}{16}\sin^2\left(2\pi\frac{\bar{\varepsilon}-\varepsilon}{1-\bar{\varepsilon}}\right)\sin^2\left(\frac{\pi}{2}\frac{\bar{\varepsilon}-\varepsilon}{1-\bar{\varepsilon}}\right),$$

where $\bar{\varepsilon}$ is the current estimate of the error parameter. Multiplying by the probability distribution, writing all the trigonometric functions in terms of complex exponentials, and integrating, we find the following final expression:

$$\langle F_{\mathrm{nospec}}\rangle = \frac{9}{2^{11}}C_5-\frac{15}{2^{10}}C_4-\frac{155}{2^{11}}C_3+\frac{15}{2^8}C_2+\frac{585}{2^{10}}C_1+\frac{467}{2^{10}},$$

where the C_q are defined as the real part of a product of exponentials:

$$C_q = \mathrm{Re}\left\{ e^{-q^2N\pi^2(\Delta\varepsilon)^2/(2(1-\bar{\varepsilon})^2)}e^{-iq\pi(\bar{\varepsilon}-\varepsilon_0)/(1-\bar{\varepsilon})}\right\}.$$

As usual, ε_0 is the initial value, and the $\Delta\varepsilon$ is the standard deviation of the Gaussian steps of the random walk.

IV. ANALYTICAL EXPRESSIONS FOR THE FIDELITY WITH SPECTATOR QUBITS

A. Pointing instability

We find an approximate analytical expression for the fidelity after k spectator cycles by integrating the formula for the fidelity given in Supplementary Section III.C according to Eq. (26). To calculate the integrals, we need the Fisher information, which we find from Eq. (3) with $\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}/2$ replaced with the terms that multiply δ in Eq. (19):

$$\frac{\Omega t}{c}\pm 2\sqrt{\ln c} = \pm\frac{4\pi\sqrt{\ln c}}{c},$$

where we choose $\Omega t = 4(\pi/2)$, which is the ideal duration of four X gates in sequence. We then find the following value for the Fisher information after M measurements of two qubits:

$$f_\delta = 2M\ln c\left(\frac{8\pi}{c}\frac{1}{1-\bar{\delta}^2_{(k-1)M}}\right)^2 \approx 2M\ln c\left(\frac{8\pi}{c}\right)^2,$$

where we are neglecting the effect that the quadratic correction of our previous estimate $\bar{\delta}_{(k-1)M}$ will have in the Fisher information. Calling:

$$J = \frac{1}{2kM(\Delta\delta)^2},$$
$$K = \frac{M-1}{2kM},$$
$$W = 1+2iN\sigma^2\pi,$$

we find the following approximate analytical expression for small δ:

$$\langle F_{\mathrm{spec}}\rangle = \frac{1}{2}+E\sqrt{J}e^{\alpha}e^{-\delta_0^2J}\sqrt{(J-\gamma)^{-1}}$$
$$\times\exp\left\{\frac{(J\delta_0+\beta/2)^2}{J-\gamma}\right\}\left\{A+B\left[\frac{J\delta_0+\beta/2}{J-\gamma}\right]\right.$$
$$\left.+C\left[\frac{1}{2(J-\gamma)}+\left(\frac{J\delta_0+\beta/2}{J-\gamma}\right)^2\right]\right\}+O(\delta_{kM}^3),$$

where in all integrals we discarded terms of higher order than the third power of the variable being integrated. The parameters A, B, C, E, α, β, and γ are:

$$A = 1-i\frac{N(\Delta\delta)^2\pi}{W}\left\{\frac{1}{2z_1}+\frac{1}{8(\Delta\delta)^4}\frac{1}{z_1^2}\left[w_1+\frac{w_1^2}{2\sigma_k^4}J^2\delta_0^2\right]\right\},$$
$$B = -i\frac{N(\Delta\delta)^2\pi}{W}\left[\frac{f_\delta^2}{8}\frac{1}{z_1^2}\frac{w_1^2}{\sigma_k^4}(1-K)K\delta_0\right],$$
$$C = \frac{1}{2}\left[\frac{w_2}{w_1}-\frac{z_2}{z_1}\right]\left\{1-i\frac{N(\Delta\delta)^2\pi}{W}\left\{\frac{1}{2z_1}\right.\right.$$
$$\left.\left.+\frac{f_\delta^2}{8}\frac{1}{z_1^2}\left[w_1+\frac{w_1^2}{2\sigma_k^4}K^2\delta_0^2\right]\right\}\right\}$$
$$-i\frac{N(\Delta\delta)^2\pi}{W}\left\{-\frac{z_2}{2z_1^2}+\frac{f_\delta^2}{8}\frac{1}{z_1^2}\left[\frac{w_1^2}{2\sigma_k^4}(1-K)^2\right.\right.$$
$$\left.\left.-2w_1\frac{z_2}{z_1}\left(1+\frac{w_1K^2\delta_0^2}{2\sigma_k^4}\right)+w_2\left(1+\frac{w_1}{\sigma_k^4}K^2\delta_0^2\right)\right]\right\},$$

$$\begin{aligned}
E =& \frac{f_\delta^2}{4\bar{\sigma}}\sqrt{\frac{w_1}{Wz_1}},\\
\alpha =& \left(\frac{w_1}{4\sigma_k^4} - \frac{1}{2\sigma_k^2}\right) K^2\delta_0^2,\\
\beta =& 2\left(\frac{w_1}{4\sigma_k^4} - \frac{1}{2\sigma_k^2}\right)(1-K)\,K\delta_0,\\
\gamma =& -\frac{i\pi}{Z} + \left(\frac{w_1}{4\sigma_k^4} - \frac{1}{2\sigma_k^2}\right)(1-K)^2 + \frac{w_2}{4\sigma_k^2}K^2\delta_0^2,
\end{aligned}$$

where σ_k^2 is the variance of the estimator given in Eq. (26), and we defined the complex numbers w_1, w_2, z_1, and z_2 as:

$$\begin{aligned}
w_1 =& \frac{4\sigma_k^2 z_1}{2z_1 + 2z_1\sigma_k^2 f_\delta^2 - f_\delta^4},\\
w_2 =& \frac{4\sigma_k^4 f_\delta^4 z_2}{(2z_1 + 2z_1\sigma_k^2 f_\delta^2 - \sigma_k^2 f_\delta^4)^2},
\end{aligned}$$

and:

$$z = z_1 + z_2\delta_{kM}^2 = \left(\frac{f_\delta^2}{2} - i\pi\right) + \left(\frac{i\pi}{W^2}\right)\delta_{kM}^2.$$

B. Amplitude instability

For the case of ε errors, we follow the same procedure, so that the Fisher information for two qubits after M measurements is obtained using Eqs. (3) and (19), and assuming a single X gate ($\Omega t = \pi/2$):

$$f_\varepsilon = 2M\left(\frac{\pi}{c}\frac{1}{1-\bar{\varepsilon}_{(k-1)M}}\right)^2 \approx 2M\left(\frac{\pi}{c}\frac{1}{1-\bar{\varepsilon}_{kM}}\right)^2.$$

Here, $\bar{\varepsilon}_{(k-1)M}$ is the estimate of the error parameter in the previous cycle. For purposes of calculating the Fisher information, we assume that the estimate is not very different from the value in the current cycle, $\bar{\varepsilon}_{kM}$.

To integrate the expression derived in Supplementary Section III.C according to Eq. (26), we can notice that all we need are the integrals of the C_q. Defining C_q' as the complex number of which C_q is the real part ($C_q = \mathrm{Re}\{C_q'\}$), we find a solution of the form:

$$\begin{aligned}
\langle C_q'\rangle_{k>0} =& \frac{e^{F'}e^{E'^2/(4D')}}{\sqrt{2kMD'(\Delta\varepsilon)^2}}\frac{1}{\sqrt{\alpha'}}\\
&\times\left[1 - \frac{1}{2}\frac{\beta'}{\alpha'}\frac{E'}{2D'} + \frac{3}{8}\frac{\beta^2}{\alpha^2}\left(\frac{1}{2D} + \frac{E'^2}{4D'^2}\right)\right] + O(\varepsilon_{kM}^3),
\end{aligned}$$

where, once again, we discard terms in the integrand of higher order than the third power of the variable that we are integrating. The parameters α', β', D', E', and F' are defined differently from the previous appendix:

$$\begin{aligned}
\alpha' =& 1 + \left[f_\varepsilon^{-2} - \tilde{\sigma}^2\right]\left[3n(q\pi\sigma)^2 + 2iq\pi\right],\\
\beta' =& -2iq\pi\left[f_\varepsilon^{-2} - \tilde{\sigma}^2\right],\\
D' =& \frac{1}{2kM\Delta\varepsilon^2} - \sum_{i=1}^{5}\frac{1}{d_i}\left[c_i + a_i\left(\frac{e_i^2}{d_i^2} - \frac{f_i}{d_i}\right) - \frac{b_i e_i}{d_i}\right],\\
E' =& \frac{\varepsilon_0}{kM\Delta\varepsilon^2} + \sum_{i=0}^{5}\frac{1}{d_i}\left[b_i - \frac{a_i e_i}{d_i}\right],\\
F' =& -\frac{\varepsilon_0^2}{2kM\Delta\varepsilon^2} + \sum_{i=0}^{5}\frac{a_i}{d_i},
\end{aligned}$$

and the a_i, b_i, c_i, d_i, e_i, and f_i, are:

$$\begin{aligned}
a_1 =& -\frac{N}{2}(q\pi\Delta\varepsilon)^2,\\
b_1 =& iq\pi,\\
d_1 =& 1,\\
c_1 =& 0\\
e_1 =& 0\\
f_1 =& 0,
\end{aligned}$$

and

$$\begin{aligned}
a_2 =& \frac{1}{2}(f_\varepsilon^{-1})^2\left[N(q\pi\Delta\varepsilon)^2 + iq\pi\right]^2,\\
b_2 =& (f_\varepsilon^{-1})^2\left[N(q\pi\Delta\varepsilon)^2 + iq\pi\right](-iq\pi),\\
c_2 =& \frac{(-iq\pi)^2}{2}(f_\varepsilon^{-1})^2,\\
d_2 =& 1 + (f_\varepsilon^{-1})^2\left[3N(q\pi\Delta\varepsilon)^2 + 2iq\pi\right],\\
e_2 =& -2iq\pi(f_\varepsilon^{-1})^2,\\
f_2 =& 0,
\end{aligned}$$

and

$$\begin{aligned}
a_3 =& \frac{1}{2}(K\varepsilon_0)^2\left[3N(q\pi\Delta\varepsilon)^2 + 2iq\pi\right],\\
b_3 =& (K\varepsilon_0)(1-K)\left[3N(q\pi\Delta\varepsilon)^2 + 2iq\pi\right]\\
&+ \frac{1}{2}(K\varepsilon_0)^2(-2iq\pi),\\
c_3 =& \frac{1}{2}(1-K)^2\left[3N(q\pi\Delta\varepsilon)^2 + 2iq\pi\right]\\
&+ (K\varepsilon_0)(1-K)(-2iq\pi),\\
d_3 =& 1 + \left[(f_\varepsilon^{-1})^2 - \sigma_k^2\right]\left[3N(q\pi\Delta\varepsilon)^2 + 2iq\pi\right],\\
e_3 =& -2iq\pi\left[(f_\varepsilon^{-1})^2 - \sigma_k^2\right],\\
f_3 =& 0,
\end{aligned}$$

and

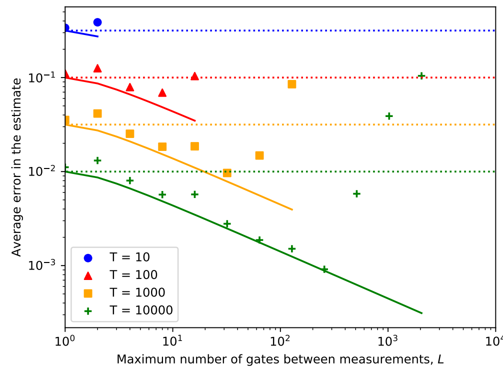

FIG. 2. **Analysis of RPE under time constraints.** If we measure the spectator qubits after each gate during the whole period, our average estimate is lower bounded by the shot-noise limit of $1/\sqrt{T}$, represented by the dashed lines. Using RPE, the Cramér–Rao bound (solid curves) is saturated as the size of L increases, up to the point when the number of individual experiments becomes too small and the performance becomes as bad or worse than the shot noise limit. Numerical simulations for RPE (scattered points) were calculated from an average over 10000 runs.

$$
\begin{aligned}
a_4 =& -\frac{1}{2}(K\varepsilon_0)\left[N(q\pi\Delta\varepsilon)^2 + iq\pi\right],\\
b_4 =& -(1-K)\left[N(q\pi\Delta\varepsilon)^2 + iq\pi\right] - (K\varepsilon_0)(-iq\pi),\\
c_4 =& -(1-K)(-iq\pi),\\
d_4 =& 1 + \left[(f_\varepsilon^{-1})^2 - \sigma_k^2\right]\left[3N(q\pi\Delta\varepsilon)^2 + 2iq\pi\right],\\
e_4 =& -2iq\pi\left[(f_\varepsilon^{-1})^2 - \sigma_k^2\right],\\
f_4 =& 0,
\end{aligned}
$$

and

$$
\begin{aligned}
a_5 =& \frac{\sigma_k^2}{2}\left[N(q\pi\Delta\varepsilon)^2 + iq\pi\right]^2,\\
b_5 =& \sigma_k^2\left[N(q\pi\Delta\varepsilon)^2 + iq\pi\right](-iq\pi),\\
c_5 =& \frac{\sigma_k^2}{2}(-iq\pi)^2,\\
d_5 =& \left[1 + \left[(f_\varepsilon^{-1})^2 - \sigma_k^2\right]\left[3N(q\pi\Delta\varepsilon)^2 + 2iq\pi\right]\right]\\
&\times\left[1 + \left[(f_\varepsilon^{-1})^2\right]\left[3N(q\pi\Delta\varepsilon)^2 + 2iq\pi\right]\right],\\
e_5 =& \left[1 + \left[(f_\varepsilon^{-1})^2 - \sigma_k^2\right]\left[3N(q\pi\Delta\varepsilon)^2\right.\right.\\
&\left.\left.+2iq\pi\right]\right](f_\varepsilon^{-1})^2(-2iq\pi)\\
&+\left[1 + (f_\varepsilon^{-1})^2\left[3N(q\pi\Delta\varepsilon)^2\right.\right.\\
&\left.\left.+2iq\pi\right]\right]\left[(f_\varepsilon^{-1})^2 - \sigma_k^2\right](-2iq\pi),\\
f_5 =& (f_\varepsilon^{-1})^2\left[(f_\varepsilon^{-1})^2 - \sigma_k^2\right](-2iq\pi)^2.
\end{aligned}
$$

In these expressions, the parameter K is defined as in Supplementary Section IV.A, and σ_k^2 is given in Eq. (26).

V. PERFORMANCE OF RPE UNDER TIME CONSTRAINTS

Here we present results for the average imprecision in the estimate of an over-rotation via RPE during a limited time slot in which at most T gates can be applied – after this period the angle is assumed to change and a new estimate will be required. The angle of the overrotation is chosen from a normal distribution centered at zero and with $\pi/32$ standard deviation, and happens during a $\pi/2$ X rotation.

In the situation where the time is limited because the overrotation parameters themselves are changing, we see in Fig. 2 that increasing the size of the concatenations in RPE improves the precision in the limit of the Cramér–Rao bound up until the point where the number of measurements becomes too small. After that, the scheme breaks down, and the precision of the estimate becomes as bad or worse than the approach without RPE. Although Fig. 2 shows that RPE has potential to improve the precision of estimates for overrotations that change in time, it also shows that its benefits are not as clear as the case where the values are static.