Supplementary Information: Optimal teleportation via noisy quantum channels without additional qubit resources

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SUPPLEMENTARY NOTE 1: DETAILS OF THE TELEPORTATION PROTOCOL VIA A NOISY QUANTUM CHANNEL

We provide details on the schematic and protocol of the optimal teleportation via a noisy quantum channel. Assume that Alice transmits an unknown qubit $|\psi\rangle_a = \alpha|0\rangle_a + \beta|1\rangle_a$ in the input mode a to Bob via an entangled quantum channel state $|\Psi\rangle_{AB} = \cos(\theta/2)|0\rangle_A|0\rangle_B + \sin(\theta/2)|1\rangle_A|1\rangle_B$ where $0 \leq \theta \leq \pi/2$ (maximally entangled state when $\theta = \pi/2$). Let us consider an example in which decoherence occurs in mode B, see Fig. 1 in the main text. The amplitude damping noise can be described by $\hat{E}^1 = |0\rangle\langle 0| + \sqrt{1-\mathrm{D}}|1\rangle\langle 1|$ and $\hat{E}^2 = \sqrt{\mathrm{D}}|0\rangle\langle 1|$ [1]. If we assume that the joint measurement is performed on the Bell basis, the corresponding effective quantum measurement $\hat{M}^k_{r,a\to B} = {}_{aA}\langle W_r|\hat{E}^k_B|\Psi\rangle_{AB}$ is given as

$$\hat{M}^{k=1}_{r=1,a\to B} = \frac{1}{\sqrt{2}}\cos\frac{\theta}{2}|0\rangle_{B\,a}\langle 0| + \sqrt{\frac{1-D}{2}}\sin\frac{\theta}{2}|1\rangle_{B\,a}\langle 1|, \qquad \hat{M}^{k=2}_{r=1,a\to B} = \sqrt{\frac{D}{2}}\sin\frac{\theta}{2}|0\rangle_{B\,a}\langle 1|, \tag{1a}$$

$$\hat{M}^{k=1}_{r=2,a\to B} = \frac{1}{\sqrt{2}}\cos\frac{\theta}{2}|0\rangle_{B\,a}\langle 0| - \sqrt{\frac{1-D}{2}}\sin\frac{\theta}{2}|1\rangle_{B\,a}\langle 1|, \qquad \hat{M}^{k=2}_{r=2,a\to B} = -\sqrt{\frac{D}{2}}\sin\frac{\theta}{2}|0\rangle_{B\,a}\langle 1|, \tag{1b}$$

$$\hat{M}^{k=1}_{r=3,a\to B} = \frac{1}{\sqrt{2}}\cos\frac{\theta}{2}|0\rangle_{B\,a}\langle 1| + \sqrt{\frac{1-D}{2}}\sin\frac{\theta}{2}|1\rangle_{B\,a}\langle 0|, \qquad \hat{M}^{k=2}_{r=3,a\to B} = \sqrt{\frac{D}{2}}\sin\frac{\theta}{2}|0\rangle_{B\,a}\langle 0|, \tag{1c}$$

$$\hat{M}^{k=1}_{r=4,a\to B} = -\frac{1}{\sqrt{2}}\cos\frac{\theta}{2}|0\rangle_{B\,a}\langle 1| + \sqrt{\frac{1-D}{2}}\sin\frac{\theta}{2}|1\rangle_{B\,a}\langle 0|, \qquad \hat{M}^{k=2}_{r=4,a\to B} = \sqrt{\frac{D}{2}}\sin\frac{\theta}{2}|0\rangle_{B\,a}\langle 0|, \tag{1d}$$

and the optimal reversing operators are defined as

$$\hat{R}_{r=1,B} = \hat{I}\left(\sqrt{1-D}\tan\frac{\theta}{2}|0\rangle\langle 0| + |1\rangle\langle 1|\right), \tag{2a}$$

$$\hat{R}_{r=2,B} = \hat{\sigma}_z\left(\sqrt{1-D}\tan\frac{\theta}{2}|0\rangle\langle 0| + |1\rangle\langle 1|\right), \tag{2b}$$

$$\hat{R}_{r=3,B} = \hat{\sigma}_x\left(\sqrt{1-D}\tan\frac{\theta}{2}|0\rangle\langle 0| + |1\rangle\langle 1|\right), \tag{2c}$$

$$\hat{R}_{r=4,B} = \hat{\sigma}_x\hat{\sigma}_z\left(\sqrt{1-D}\tan\frac{\theta}{2}|0\rangle\langle 0| + |1\rangle\langle 1|\right). \tag{2d}$$

We can obtain the (unnormalized) teleported states after applying the optimal reversing operators for each outcome r as

$$p_B(1,\psi)\rho_{r=1,B} = p_B(2,\psi)\rho_{r=2,B} = \frac{1-D}{2}\sin^2\frac{\theta}{2}|\psi\rangle_{B\,B}\langle\psi| + \frac{D(1-D)|\beta|^2}{2}\sin^2\frac{\theta}{2}\tan^2\frac{\theta}{2}|0\rangle_{B\,B}\langle 0|, \tag{3a}$$

$$p_B(3,\psi)\rho_{r=3,B} = p_B(4,\psi)\rho_{r=4,B} = \frac{1-D}{2}\sin^2\frac{\theta}{2}|\psi\rangle_{B\,B}\langle\psi| + \frac{D(1-D)|\alpha|^2}{2}\sin^2\frac{\theta}{2}\tan^2\frac{\theta}{2}|1\rangle_{B\,B}\langle 1|, \tag{3b}$$

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where $|\psi\rangle_B = \alpha|0\rangle_B + \beta|1\rangle_B$ and $p_B(r,\psi) = {}_a\langle\psi| \sum_k \hat{M}_r^{k\dagger} \hat{R}_r{}^\dagger \hat{R}_r \hat{M}_r^k |\psi\rangle_a$. As a result, the average teleportation fidelity of the teleportation by the optimal reversing operation can be evaluated as

$$F_R = \int d\psi \left[\frac{1 + D|\alpha|^2|\beta|^2 \tan^2\frac{\theta}{2}}{1 + D|\beta|^2 \tan^2\frac{\theta}{2}} \left(|\alpha|^2 \cos^2\frac{\theta}{2} + |\beta|^2 \sin^2\frac{\theta}{2} \right) + \frac{1 + D|\alpha|^2|\beta|^2 \tan^2\frac{\theta}{2}}{1 + D|\alpha|^2 \tan^2\frac{\theta}{2}} \left(|\alpha|^2 \sin^2\frac{\theta}{2} + |\beta|^2 \cos^2\frac{\theta}{2} \right) \right]. \quad (4)$$

We can compare this with the maximum teleportation fidelity of the conventional protocol based on conditional unitary reversal $\hat{U}^r = \{\hat{I}, \hat{\sigma}_z, \hat{\sigma}_x, \hat{\sigma}_x\hat{\sigma}_z\}$ can be calculated as

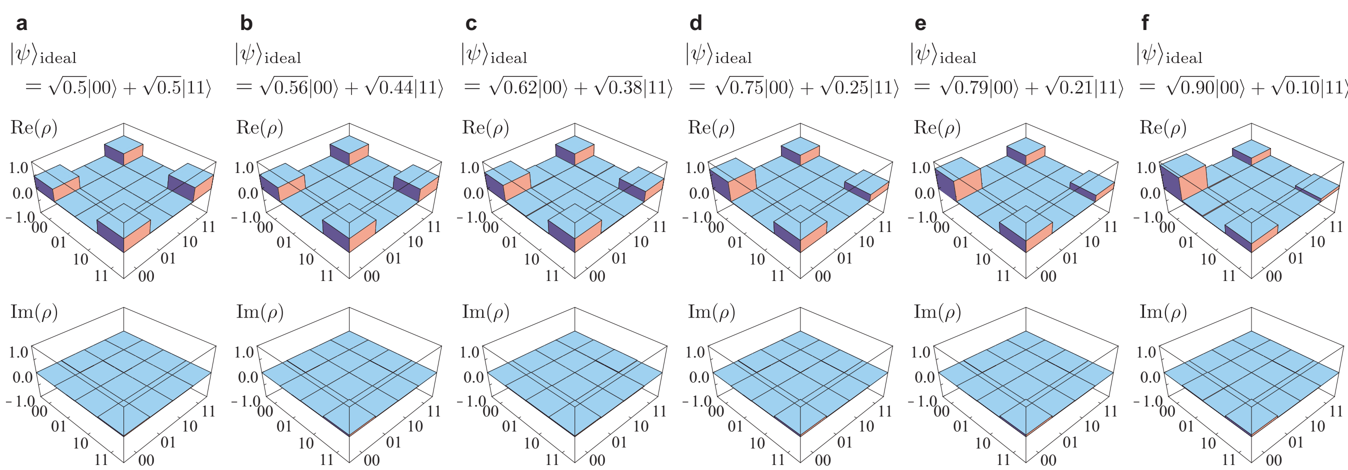
$$\begin{aligned} F_U &= \max \int d\psi \sum_r \sum_k |{}_a\langle\psi|\hat{U}^r \hat{M}^k_{r,a\to B}|\psi\rangle_a|^2 \\ &= \frac{4 - D + \cos\theta - (1-D)\cos\theta + 2\sqrt{1-D}\sin\theta}{6}. \end{aligned} \quad (5)$$

Similarly, we can also consider other noise models such as the decoherence in mode A and also for the dephasing and depolorizing noises (see Ref. [2]).

SUPPLEMENTARY NOTE 2: EXPERIMENTAL OF OPTIMAL TELEPORTATION VIA A NOISELESS QUANTUM CHANNEL

Quantum state tomography (QST) of pure entangled states

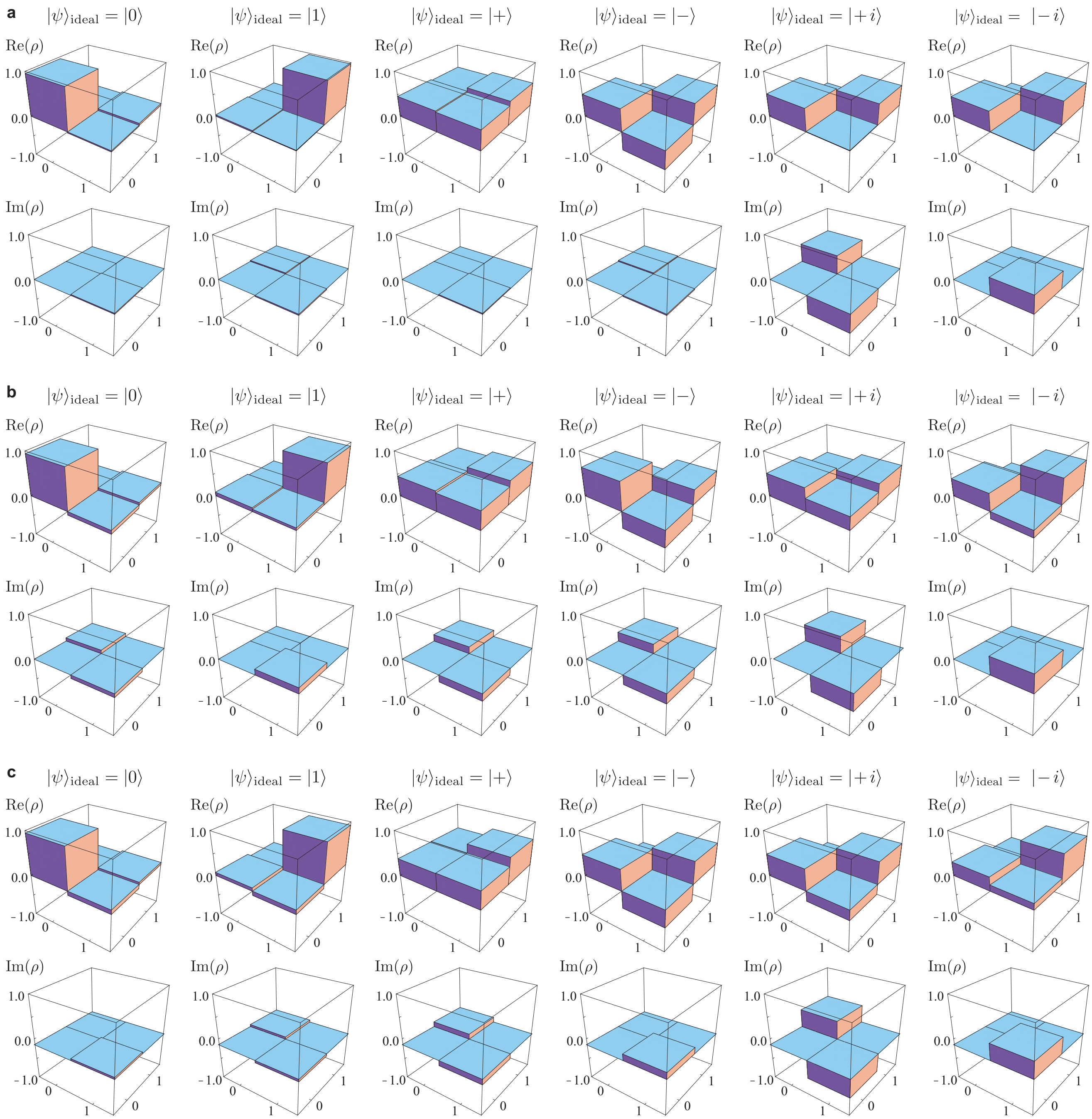
A pure non-maximally entangled state is prepared by inserting a partial polarizing beam splitter (PPBS1), which partially reflects the vertical polarization with transmittance T_{V1} and transmits the horizontal polarization with transmittance 1, in the path of photon B as shown in Fig. 2 of the main text. After passing through the PPBS1, a maximally entangled state becomes a pure non-maximally entangled state, $\sqrt{1/(1+T_{V1})}|00\rangle_{AB} + \sqrt{T_{V1}/(1+T_{V1})}|11\rangle_{AB}$. We generate five different pure non-maximally entangled states with PPBS1 of $T_{V1} = 0.78$, 0.608, 0.333, 0.26, and 0.111. Supplementary Figure 1 shows the results of quantum state tomography (QST) of the prepared entangled states.



Supplementary Figure 1: QST of a maximally entangled state and pure non-maximally entangled states. The bar graphs show the reconstructed density matrices ρ for a maximally entangled state and five different pure non-maximally entangled states which are shared between Alice and Bob. The fidelities between the reconstructed two-qubit states and the ideal states are (a) 0.984, (b) 0.986, (c) 0.983, (d) 0.983, (e) 0.978, and (f) 0.983.

Optimal teleportation and QST of teleported states

The teleported states in Eq. (3a) and (3b) via a noiseless quantum channel become $\rho_{r,B} = \frac{1}{2}\sin^2\frac{\theta}{2}|\psi\rangle_{BB}\langle\psi|$. To demonstrate the teleportation fidelity of unity, QST of teleported state is performed for a maximally entangled state and two pure non-maximally entangled states with PPBS1 of $T_{V1} = 0.78$, and 0.333. Here, the joint measurement basis is chosen to be $|W_1\rangle_{aA} = (|00\rangle + |11\rangle)/\sqrt{2}$. To realize the corresponding reversing operation in Eq. (2a), the angle φ_2 of HWP2 in Fig. 2 in the main text is set at $\frac{1}{2}\cos^{-1}(\frac{1}{T_{V1}})$ for the polarization dependent loss and two additional HWPs at 0° are used for $\hat{I}$. The results are shown in Supplementary Figure 2.



Supplementary Figure 2: QST of teleported states for optimal teleportation via a noiseless quantum channel. The bar graphs show the reconstructed density matrices ρ of teleported states for six input quantum states $|0\rangle$, $|1\rangle$, $|+\rangle$, $|-\rangle$, $|+i\rangle$, and $|-i\rangle$ when the polarization-entangled qubits are prepared in (a) the pure maximally entangled state, (b) the pure non-maximally entangled state $\sqrt{0.56}|00\rangle + \sqrt{0.44}|00\rangle$, and (c) $\sqrt{0.75}|00\rangle + \sqrt{0.25}|00\rangle$. The fidelities between the reconstructed teleported state ρ and the ideal input states are (a) 0.956, 0.970, 0.938, 0.930, 0.934, and 0.901, (b) 0.946, 0.913, 0.912, 0.897, 0.906, and 0.917, and (c) 0.949, 0.926, 0.901, 0.904, 0.924, and 0.883.

SUPPLEMENTARY NOTE 3: EXPERIMENTAL OPTIMAL TELEPORTATION VIA A NOISY QUANTUM CHANNEL

Optimal teleportation with a maximally entangled state under amplitude damping decoherence

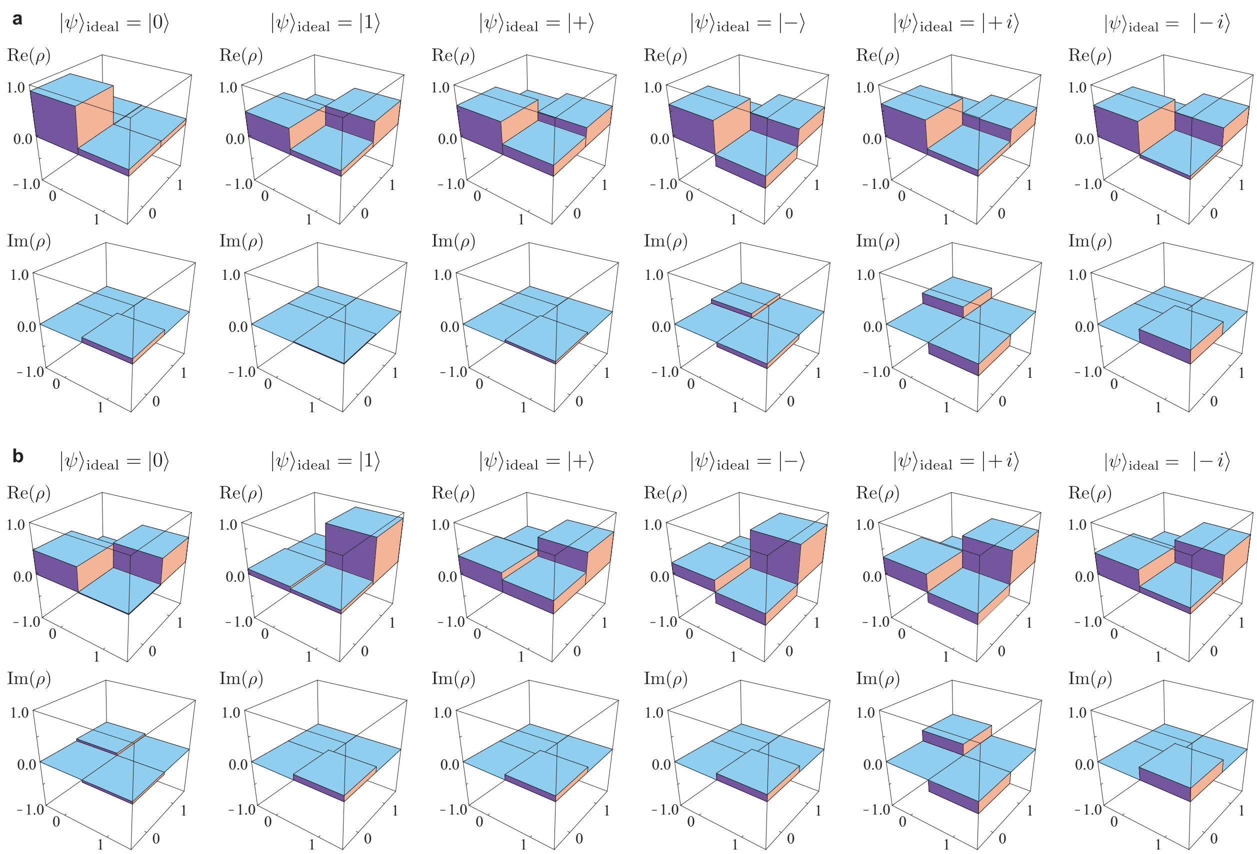
The teleported states in Eq. (3a) and (3b) with a maximally entangled state under amplitude damping decoherence become

$$p_B(1,\psi)\rho_{r=1,B} = p_B(2,\psi)\rho_{r=2,B} = \frac{1-D}{4}|\psi\rangle_{BB}\langle\psi| + \frac{D(1-D)|\beta|^2}{4}|0\rangle_{BB}\langle 0|, \tag{6a}$$

$$p_B(3,\psi)\rho_{r=3,B} = p_B(4,\psi)\rho_{r=4,B} = \frac{1-D}{4}|\psi\rangle_{BB}\langle\psi| + \frac{D(1-D)|\alpha|^2}{4}|1\rangle_{BB}\langle 1|, \tag{6b}$$

and the teleportation fidelity in Eq. (4) is calculated as $F_R = \frac{1}{2}\int d\psi\left[\frac{1+D|\alpha|^2|\beta|^2}{1+D|\beta|^2} + \frac{1+D|\alpha|^2|\beta|^2}{1+D|\alpha|^2}\right]$. To demonstrate our protocol, teleportation is performed for strong decoherence values of $D = 0.74$ with $T_{V2} = 0.26$ and $D = 0.92$ with $T_{V2} = 0.087$, where T_{V2} is the transmittance of vertical polarization of PPBS2, shown in Fig. 2 of the main text. The average fidelities for six input states and four joint measurement bases are presented in Fig. 4 of the main text.

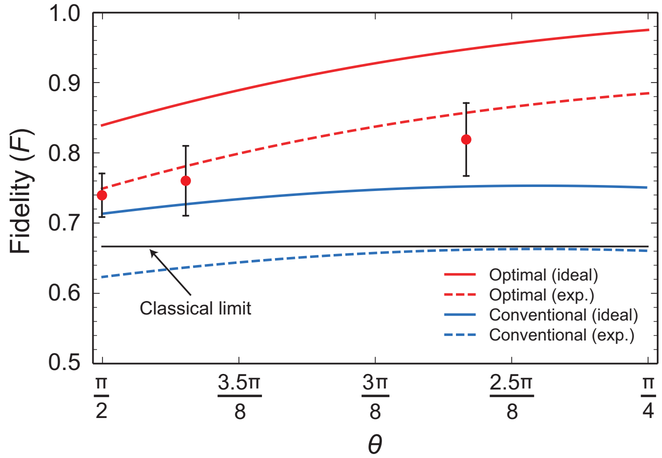
Here, we present the QST results of the teleported states for the joint measurement bases $|W_1\rangle_{aA} = (|00\rangle+|11\rangle)/\sqrt{2}$ and $|W_3\rangle_{aA} = (|01\rangle + |10\rangle)/\sqrt{2}$ for the decoherence value of $D = 0.92$, see Supplementary Figure 3. For $|W_1\rangle_{aA}$, the corresponding reversing operation in Eq. (2a) is realized with $\varphi_2 = \frac{1}{2}\cos^{-1}(\frac{1}{T_{V2}})$ and two additional HWPs at $0°$ for $\hat{I}$. For $|W_3\rangle_{aA}$, the corresponding reversing operation in Eq. (2c) is realized with $\varphi_2 = \frac{1}{2}\cos^{-1}(\frac{1}{T_{V2}})$ and one additional HWP at $45°$ for $\hat{\sigma}_x$.



Supplementary Figure 3: QST of teleported states for optimal teleportation with a maximally entangled state under amplitude damping decoherence. The bar graphs show the reconstructed density matrices ρ of teleported states for six input quantum states $|0\rangle$, $|1\rangle$, $|+\rangle$, $|-\rangle$, $|+i\rangle$, and $|-i\rangle$ for the joint measurement bases (a) $|W_1\rangle_{aA} = (|00\rangle + |11\rangle)/\sqrt{2}$ and (b) $|W_3\rangle_{aA} = (|01\rangle + |10\rangle)/\sqrt{2}$. Decoherence in the channel is set to be very high at $D = 0.92$. The fidelities between the reconstructed states ρ and ideal states are (a) 0.890, 0.497, 0.704, 0.734, 0.728, and 0.758, and (b) 0.487, 0.892, 0.744, 0.721, 0.734, and 0.750.

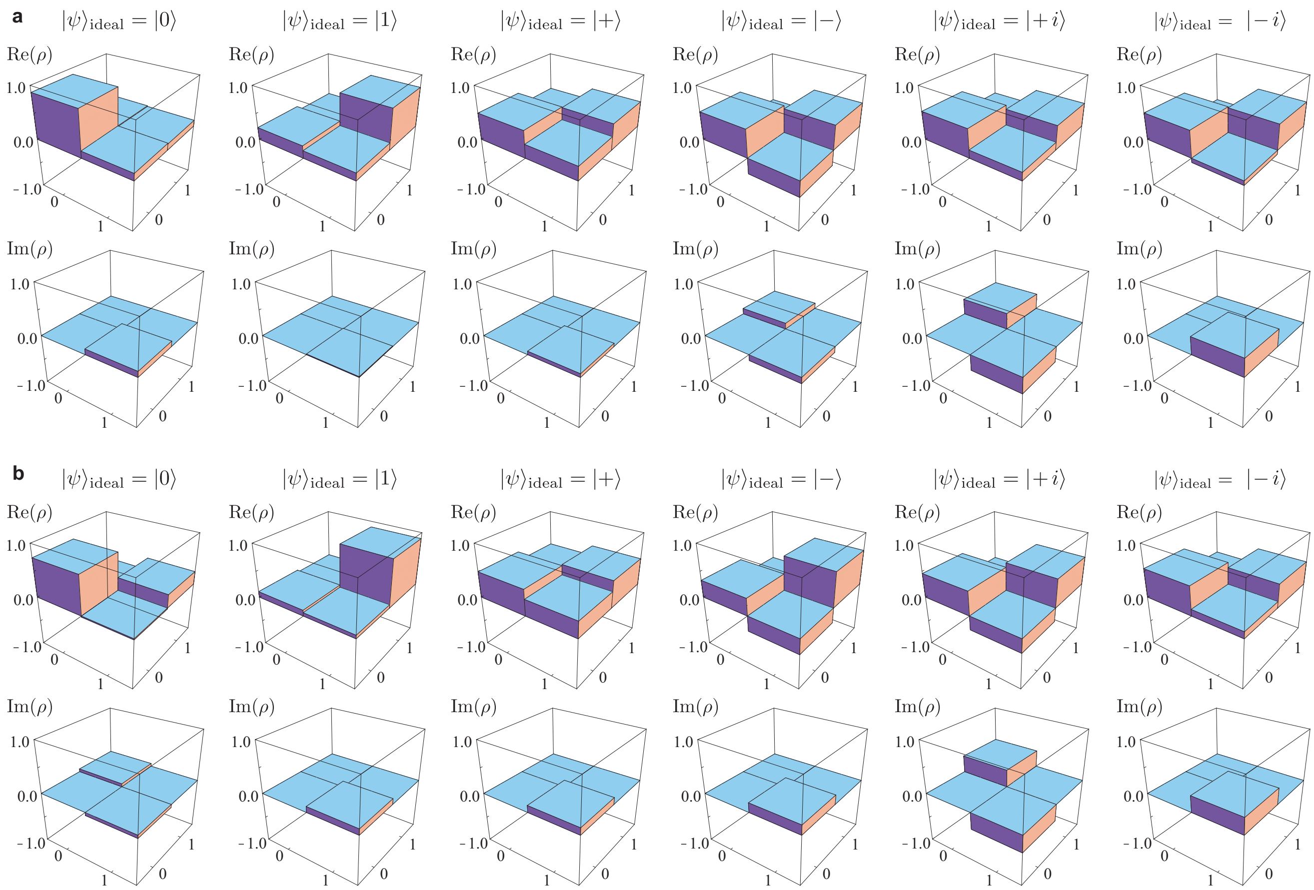
Optimal teleportation with a non-maximally entangled state under amplitude damping decoherence

In the case of amplitude damping decoherence, it is possible to further improve the teleportation fidelity by starting with a pure non-maximally entangled state, as we have shown in section . In principle, the teleportation fidelity could be arbitrarily close to unity with the success probability approaching to zero. Supplementary Figure 4 represents both the theoretically and experimentally obtained average teleportation fidelities by varying θ for $D = 0.74$. Here, we use PPBS1 with $T_{V1} = 0.78$ ($\theta = 0.46\pi$) and 0.333 ($\theta = 0.33\pi$) to prepare the pure non-maximally entangled state, and use PPBS2 with $T_{V2} = 0.26$ for $D = 0.74$.



Supplementary Figure 4: Average teleportation fidelities by changing θ for $D = 0.74$ with a pure non-maximally entangled state under amplitude damping decoherence. The solid lines represent the theoretical upper bounds of the average fidelities for our protocol (red) and the conventional teleportation protocol (blue) based on the unitary reversing operation, respectively, while the dashed lines are drawn including experimental imperfections. The red solid circles are the experimental data based on our protocol, with the error bars representing one standard deviation. The results clearly demonstrate that the teleportation fidelity could be arbitrarily close to unity by using a pure non-maximally entangled state for a quantum channel under amplitude damping decoherence.

QST results of the teleported states are also presented for the joint measurement bases $|W_1\rangle_{aA} = (|00\rangle + |11\rangle)/\sqrt{2}$ and $|W_3\rangle_{aA} = (|01\rangle + |10\rangle)/\sqrt{2}$ for $\theta = 0.33\pi$ with $D = 0.92$ in Supplementary Figure 5. For $|W_1\rangle_{aA}$, we set the angle of HWP2 $\varphi_2 = \frac{1}{2}\cos^{-1}(\frac{1}{T_{V1}T_{V2}})$ and use two additional HWPs at 0° for $\hat{I}$ to realize the corresponding reversing operation in Eq. (2a). For $|W_3\rangle_{aA}$, we set the angle of HWP2 $\varphi_2 = \frac{1}{2}\cos^{-1}(\frac{1}{T_{V2}T_{V2}})$ and use one additional HWP at 45° for $\hat{\sigma}_x$ to realize the corresponding reversing operation in Eq. (2c).



Supplementary Figure 5: QST of teleported states for optimal teleportation with a non-maximally entangled state under amplitude damping decoherence. The bar graphs show the reconstructed density matrices ρ of teleported states for six input quantum states $|0\rangle$, $|1\rangle$, $|+\rangle$, $|-\rangle$, $|+i\rangle$, and $|-i\rangle$ for the joint measurement bases (a) $|W_1\rangle_{aA} = (|00\rangle + |11\rangle)/\sqrt{2}$ and (b) $|W_3\rangle_{aA} = (|01\rangle + |10\rangle)/\sqrt{2}$ for $\theta = 0.33\pi$ with $D = 0.92$. The fidelities between the reconstructed states ρ and the ideal states are (a) 0.878, 0.769, 0.759, 0.807, 0.808, and 0.837, and (b) 0.745, 0.891, 0.811, 0.797, 0.892, and 0.813.

SUPPLEMENTARY REFERENCES

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[2] Lee, S.-W., Im, D.-G., Kim, Y.-H., Nha, H. & Kim, M. S. Quantum teleportation is a reversal of quantum measurement. Preprint at arXiv:2104.12178 (2021).