Supplementary information. Genuine quantum networks with superposed tasks and addressing

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SUPPLEMENTARY DISCUSSION - IMPOSSIBILITY OF COHERENTLY CONTROLLED MEASUREMENTS

We provide indications that a controlled known measurement performed in a coherent way on some unknown state is not realizable. Given a control and a target qubit, the desired effect of the operation is to obtain a coherent superposition between the target state being measured —in case the control qubit is in $|1\rangle$— and not measured — when control qubit is in $|0\rangle$—. For that purpose, consider two qubit registers. The first one, conceived as control register, is given in the state:

$$|\varphi\rangle_{\mathrm{c}} = \alpha_c |0\rangle + \beta_c |1\rangle \,, \tag{1}$$

where coefficients α_c and β_c define the weights of the desired superposition. The measurement is performed in a controlled way on some target state $|\psi\rangle_t$. The desired effect for an arbitrary projective measurement $\mathrm{M} = \{A_k\}$ is

$$|\varphi\rangle_c|\psi\rangle_t \rightarrow \{p_k, \alpha_0|0\rangle_c|\psi\rangle_t + \alpha_1|1\rangle|\psi_k\rangle\}, \tag{2}$$

with $\sum_k A_k^\dagger A_k = 1$. One sees that different branches arise from the measurement, each one corresponding with each measurement outcome, with probability p_k.

Observation 1. In this first observation, we try to show that the required controlled measurement is not a valid quantum operation, by considering a general initial setting, and the desired final state. We see that, apparently, linearity of the operation is not guaranteed.

Consider again two qubit registers prepared in arbitrary states, $|\varphi\rangle_{\mathrm{c}} = \alpha_c |0\rangle + \beta_c |1\rangle$ and $|\varphi\rangle_{\mathrm{t}} = \alpha_t |0\rangle + \beta_t |1\rangle$ respectively, with $|\alpha_i|^2 + |\beta_i|^2 = 1$. The first qubit is conceived as the control register and the second as the target. One aims to apply a controlled-measurement operation between the two qubits. The desired effect of the controlled measurement is given by Eq. 2. Consider the initial state

$$\rho_0 = |\psi_0\rangle \langle\psi_0| \,, \tag{3}$$

where $|\psi_0\rangle$ is the composite system

$$|\psi_0\rangle = |\varphi\rangle_{\mathrm{c}} |\varphi\rangle_{\mathrm{t}} = \alpha_c\alpha_t |00\rangle + \alpha_c\beta_t |01\rangle + \beta_c\alpha_t |10\rangle + \beta_c\beta_t |11\rangle = a |00\rangle + b |01\rangle + c |10\rangle + d |11\rangle \,, \tag{4}$$

where coefficients have been relabelled for simplicity. We assume a projective measurement performed in the computational basis. Since two different branches arise from the measurement (see Eq. 2), the desired final state has to be described as a classical mixture of two states, each one corresponding to each of the possible outcomes of the projective measurement:

$$\rho_{\mathrm{f}} = p_1 |\psi_1\rangle \langle\psi_1| \otimes |1\rangle_m \langle 1| + p_2 |\psi_2\rangle \langle\psi_2| \otimes |2\rangle_m \langle 2| \,, \tag{5}$$

where register m defines the apparatus state, i.e. the outcome of the measurement. Besides, $p_1 + p_2 = 1$, where

$$|\psi_1\rangle = a |00\rangle + b |01\rangle + \sqrt{cc^* + dd^*} |10\rangle = \alpha_c |0\rangle_c |\varphi\rangle_{\mathrm{t}} + |\beta_c| |1\rangle_c |0\rangle_t \,, \tag{6}$$

$$|\psi_2\rangle = a |00\rangle + b |01\rangle + \sqrt{cc^* + dd^*} |11\rangle = \alpha_c |0\rangle_c |\varphi\rangle_{\mathrm{t}} + |\beta_c| |1\rangle_c |1\rangle_t \,, \tag{7}$$

with $p_1 = \frac{cc^*}{cc^*+dd^*} = |\alpha_t|^2$ and $p_2 = \frac{dd^*}{cc^*+dd^*} = |\beta_t|^2$.

Observe that the desired superposition is found for each term of the mixture. For each individual term, the measurement is performed if the control qubit is in the state $|1\rangle$ with probability $|\beta_c|^2$, where $\beta_c = \sqrt{|c|^2 + |d|^2}$. The target qubit remains untouched if the control state is $|0\rangle$ (with probability $|\alpha_c|^2$). Therefore, in each branch of the measurement, a coherent superposition of measuring and not measuring the target qubit (with their corresponding probabilities) is found. It is however straightforward to see that the map from Eq. (3) to Eq. (5), i.e. $\xi(\rho_0) = \rho_{\mathrm{f}}$, is not a linear transformation, i.e. $\xi\left(\sum_i p_i \rho_i\right) \neq \sum_i p_i \xi(\rho_i)$. By direct inspection one observes that the off-diagonal terms of Eq. (5) related with the control qubit being in the state $|1\rangle$ are not linear functions of the elements of Eq. (3). This might indicate that the map is not linear and, consequently, it is not a valid quantum operation.

Observation 2. In this second observation, we assume that the controlled measurement is linear and we find inconsistencies that might indicate that the linearity assumption is not correct.

First, consider the desired effect of a controlled measurement on different states. For simplicity, we choose the control state to be in the $|\psi\rangle_{\mathrm{c}} = |+\rangle$ state, i.e. same weights of the desired superposition. Besides, we consider a projective measurement in the computational basis. The initial state reads

$$|\varphi\rangle = |+\rangle_{\mathrm{c}} |\psi\rangle_{\mathrm{t}} = \frac{1}{\sqrt{2}} \left(|0\rangle_{\mathrm{c}} |\psi\rangle_{\mathrm{t}} + |1\rangle_{\mathrm{c}} |\psi\rangle_{\mathrm{t}}\right). \tag{8}$$

The desired effect of the operation for different target states and the different measurement branches is:

- $|\psi\rangle_{\mathrm{t}} = |0\rangle$

$$|\Phi_{00}\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_{\mathrm{c}} |0\rangle_{\mathrm{t}} + |1\rangle_{\mathrm{c}} |0\rangle_{\mathrm{t}}\right) \text{ with } p = 1 \tag{9}$$

$$|\Phi_{01}\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_{\mathrm{c}} |0\rangle_{\mathrm{t}} + |1\rangle_{\mathrm{c}} |1\rangle_{\mathrm{t}}\right) \text{ with } p = 0 \tag{10}$$

- $|\psi\rangle_{\mathrm{t}} = |1\rangle$

$$|\Phi_{10}\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_{\mathrm{c}} |1\rangle_{\mathrm{t}} + |1\rangle_{\mathrm{c}} |0\rangle_{\mathrm{t}}\right) \text{ with } p = 0 \tag{11}$$

$$|\Phi_{11}\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_{\mathrm{c}} |1\rangle_{\mathrm{t}} + |1\rangle_{\mathrm{c}} |1\rangle_{\mathrm{t}}\right) \text{ with } p = 1 \tag{12}$$

- $|\psi\rangle_{\mathrm{t}} = |+\rangle$

$$|\Phi_{+0}\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_{\mathrm{c}} |+\rangle_{\mathrm{t}} + |1\rangle_{\mathrm{c}} |0\rangle_{\mathrm{t}}\right) \text{ with } p = \frac{1}{2} \tag{13}$$

$$|\Phi_{+1}\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_{\mathrm{c}} |+\rangle_{\mathrm{t}} + |1\rangle_{\mathrm{c}} |1\rangle_{\mathrm{t}}\right) \text{ with } p = \frac{1}{2} \tag{14}$$

- $|\psi\rangle_{\mathrm{t}} = |-\rangle$

$$|\Phi_{-0}\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_{\mathrm{c}} |-\rangle_{\mathrm{t}} + |1\rangle_{\mathrm{c}} |0\rangle_{\mathrm{t}}\right) \text{ with } p = \frac{1}{2} \tag{15}$$

$$|\Phi_{-1}\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_{\mathrm{c}} |-\rangle_{\mathrm{t}} - |1\rangle_{\mathrm{c}} |1\rangle_{\mathrm{t}}\right) \text{ with } p = \frac{1}{2} \tag{16}$$

Consider now the state $\rho_t = \frac{1}{2}\mathbb{1}$. The initial state now reads $\rho_0 = |+\rangle_{\mathrm{c}} \langle +| \otimes \frac{1}{2}\mathbb{1}_{\mathrm{t}}$. We analyze two different cases arising from this:

1. Decompose $\rho_t = \frac{1}{2}\mathbb{1} = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$.

From this decomposition and from the initial state ρ_0, assuming that the controlled-measurement we consider is linear, the operation applied to ρ_0 leads to

$$\rho = \frac{1}{2} |\Phi_{00}\rangle\langle\Phi_{00}| \otimes |0\rangle_M \langle 0| + \frac{1}{2} |\Phi_{11}\rangle\langle\Phi_{11}| \otimes |1\rangle_M \langle 1|, \tag{17}$$

where the states $|\Phi_{ij}\rangle$ are the desired actions previously defined, and the register M defines the apparatus state, i.e. the measurement outcome.

1. Decompose $\rho_t = \frac{1}{2}\mathbb{1} = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -|$.

Analogously, for this decomposition and the effect on different target states shown above, the final state reads

$$\rho = \frac{1}{4} |\Phi_{+0}\rangle\langle\Phi_{+0}| \otimes |0\rangle_M \langle 0| + \frac{1}{4} |\Phi_{+1}\rangle\langle\Phi_{+1}| \otimes |1\rangle_M \langle 1| + \frac{1}{4} |\Phi_{-0}\rangle\langle\Phi_{-0}| \otimes |0\rangle_M \langle 0| + \frac{1}{4} |\Phi_{-1}\rangle\langle\Phi_{-1}| \otimes |1\rangle_M \langle 1|. \tag{18}$$

One can easily see that expressions Eq. (17) and Eq. (18) do not match. This contradiction might directly indicate that our assumption is wrong, and the controlled-measurement is indeed not linear.

SUPPLEMENTARY DISCUSSION - CONTROLLED SENDING

In this appendix, we provide the formalization of the controlled-sending introduced in the manuscript by means of quantum teleportation. Crucially, we observe that the coefficients of the superposition do not change throughout the process. The initial state reads

$$|\Psi\rangle_{\rm in} = (\alpha_0 |0\rangle_{\rm c} + \alpha_1 |1\rangle_{\rm c}) |\psi\rangle_{\rm a_1} \left|\Phi^+\right\rangle_{\rm a_2 b} |0\rangle_{\rm ax_1} |+\rangle_{\rm ax_2} , \tag{19}$$

where party A possesses five qubits, the control register c, one qubit in the arbitrary state $|\psi\rangle_{\rm a_1} = \alpha |0\rangle + \beta |1\rangle$ (to be sent), another one a_2 sharing a Bell state with party B, and two auxiliary qubits ($\rm ax_1$ and $\rm ax_2$) initialized in the states $|0\rangle_{\rm ax_1}$ and $|+\rangle_{\rm ax_2}$ respectively. Party B just possesses the qubit b of the Bell state $|\Phi^+\rangle_{\rm a_2 b} = \frac{1}{\sqrt{2}} \left(|00\rangle_{\rm a_2 b} + |11\rangle_{\rm a_2 b}\right)$. First, the controlled-swap operation is applied simultaneously between qubits a_1 and $\rm ax_1$, and qubits a_2 and $\rm ax_2$, such that the global states evolves to

$$\begin{aligned} |\Psi\rangle = {} & \alpha_0 |0\rangle_{\rm c} |0\rangle_{\rm ax_1} |+\rangle_{\rm ax_2} \left|\Phi^+\right\rangle_{\rm a_2 b} |\psi\rangle_{\rm a_1} + \\ & \frac{1}{\sqrt{2}} \alpha_1 |1\rangle_{\rm c} \left(\alpha \left(|000\rangle_{{\rm ax_1} ax_2 b} + |011\rangle_{{\rm ax_1} ax_2 b}\right) + \beta \left(|100\rangle_{{\rm ax_1} ax_2 b} + |111\rangle_{{\rm ax_1} ax_2 b}\right)\right) |0\rangle_{\rm a_1} |+\rangle_{\rm a_2} . \end{aligned} \tag{20}$$

Second, a Bell measurement is always applied by party A between systems $\rm ax_1$ and $\rm ax_2$. Assuming the outcome of this measurement is Φ_i, and taking into account the decomposition

$$|00\rangle = \frac{1}{\sqrt{2}} \left(\left|\Phi^+\right\rangle + \left|\Phi^-\right\rangle\right) , \; |11\rangle = \frac{1}{\sqrt{2}} \left(\left|\Phi^+\right\rangle - \left|\Phi^-\right\rangle\right) , \tag{21}$$

the state evolves to

$$|\Psi\rangle = \alpha_0 |0\rangle_{\rm c} \frac{1}{2} |\Phi_i\rangle_{{\rm ax_1} ax_2} \left|\Phi^+\right\rangle_{\rm a_2 b} |\psi\rangle_{\rm a_1} + \alpha_1 |1\rangle_{\rm c} \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} \alpha |\Phi_i\rangle_{{\rm ax_1} ax_2} |0\rangle_{\rm b} + \frac{1}{\sqrt{2}} \beta |\Phi_i\rangle_{{\rm ax_1} ax_2} |1\rangle_{\rm b}\right) |0\rangle_{\rm a_1} |+\rangle_{\rm a_2} . \tag{22}$$

In case another measurement outcome is found, a controlled unitary correction operation on qubit b of the form $|0\rangle_c \langle 0| \otimes \mathbb{1} + |1\rangle_c \langle 1| \otimes \sigma_i^{\rm b}$ is necessary. Note that, since the control register belongs to party A, this correction can always be introduced by party B applying locally the unitary σ_i on qubit b, followed by party A applying the controlled unitary $|0\rangle_c \langle 0| \otimes \sigma_i^{\intercal} + |1\rangle_c \langle 1| \otimes \mathbb{1}$ on qubit a_2. Finally, after re-normalization the desired state is found:

$$|\Psi\rangle_{\rm f} = \Big(\alpha_0 |0\rangle_{\rm c} |\psi\rangle_{\rm a_1} \left|\Phi^+\right\rangle_{\rm a_2 b} + \alpha_1 |1\rangle_{\rm c} |0\rangle_{\rm a_1} |+\rangle_{\rm a_2} \sigma_i |\psi\rangle_{\rm b}\Big) |\Phi_i\rangle_{\rm ax_1 ax_2} . \tag{23}$$

Therefore, the weights of the superposition remain unchanged and the desired target state is generated, where the state $|\psi\rangle$ is teleported in case of the control qubit is in $|1\rangle$, and the state $|\psi\rangle$ and the shared Bell pair are kept untouched otherwise.

Additionally, if one is not interested on keeping the initial Bell state between parties A and B, qubit a_2 can be measured in the computational basis. Assuming the outcome $|0\rangle$ is obtained, the resulting state is

$$|\Psi\rangle_{\rm f} = \left(\alpha_0 |0\rangle_{\rm c} |0\rangle_{\rm b} |\psi\rangle_{\rm a_1} + \alpha_1 |1\rangle_{\rm c} |\psi\rangle_{\rm b} |0\rangle_{\rm a_1}\right) \left|\Phi^+\right\rangle_{\rm ax_1 ax_2} |0\rangle_{\rm a_2} , \tag{24}$$

where the state has been re-normalized.

SUPPLEMENTARY DISCUSSION - CONTROLLED MERGING

In this appendix, we provide a detailed mathematical description of the controlled-merging of graph states. The initial state consists of two graph states, an auxiliary qubit, and a control register (see Fig. ??)

$$|\Psi\rangle_{\rm in} = \alpha_0 |0\rangle_{\rm c} |{\rm G}_1\rangle |{\rm G}_2\rangle |+\rangle_{\rm aux} + \alpha_1 |1\rangle_{\rm c} |{\rm G}_1\rangle |{\rm G}_2\rangle |+\rangle_{\rm aux} , \tag{25}$$

where $|{\rm G}_i\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_{a_i} |{\rm G}_i/a_i\rangle + |1\rangle_{a_i} \prod \sigma_z^{{\rm N}_{a_i}} |{\rm G}_i/a_i\rangle\right)$. Qubits a_1, a_2 and aux belong to the same device A, which also possesses the control register. We analyze both branches of Eq. (25) independently. First, consider the case where the control register is in $|0\rangle_{\rm c}$ the state. In this case, a swap operation is applied between qubits a_2 and aux. Note that, in the rest of the paper, we consider that the swap is performed if the control register is in $|1\rangle_{\rm c}$. We change here this criteria for convenience. After the swap operation is applied, the state of the $|0\rangle_{\rm c}$ branch reads:

$$\frac{1}{\sqrt{2}} \alpha_0 |0\rangle_{\rm c} |{\rm G}_1\rangle \left(|+\rangle_{\rm a_2} |{\rm G}_2/{\rm aux}\rangle |0\rangle_{\rm aux} + |+\rangle_{\rm a_2} \prod \sigma_z^{{\rm N}_{a_{\rm aux}}} |{\rm G}_2/{\rm aux}\rangle |1\rangle_{\rm aux}\right) .$$

Finally, the merging measurement $\{P_0 = |0\rangle\langle 00| + |1\rangle\langle 11|, P_1 = |0\rangle\langle 01| + |1\rangle\langle 10|\}$ is performed between qubits a_1 and a_2. By expanding $|G_1\rangle$, one finds, for both measurement outcomes:

$$\frac{1}{2\sqrt{2}}\alpha_0 |0\rangle_c \left(|0\rangle_{\tilde{a_1}} |G_1/\tilde{a_1}\rangle + |1\rangle_{\tilde{a_1}} \prod \sigma_z^{N_{\tilde{a_1}}} |G_1/\tilde{a_1}\rangle\right) \left(|0\rangle_{aux} |G_2/aux\rangle + |1\rangle_{aux} \prod \sigma_z^{N_{a_{aux}}} |G_2/aux\rangle\right), \tag{26}$$

where $\tilde{a_1}$ is the resulting qubit of merging a_1 and a_2.

On the other hand, for the branch of Eq. (25) corresponding to $|1\rangle_c$, the swap operation is not applied.Therefore, the next step is the merging operation. The state before the measurement is

$$\frac{1}{2}\alpha_1 |1\rangle_c \left(|0\rangle_{a_1} |G_1/a_1\rangle + |1\rangle_{a_1} \prod \sigma_z^{N_{a_1}} |G_1/a_1\rangle\right) \left(|0\rangle_{a_2} |G_2/a_2\rangle + |1\rangle_{a_2} \prod \sigma_z^{N_{a_2}} |G_2/a_2\rangle\right) |+\rangle_{aux}.$$

After merging qubits a_1 and a_2 into $\tilde{a_1}$, the resulting state for the outcome corresponding to P_0 is

$$\frac{1}{2}\alpha_1 |1\rangle_c \left(|0\rangle_{\tilde{a_1}} |G_1 \cup G_2/\tilde{a_1}\rangle + |1\rangle_{\tilde{a_1}} \prod \sigma_z^{N_{\tilde{a_1}}} |G_1 \cup G_2/\tilde{a_1}\rangle\right) |+\rangle_{aux}, \tag{27}$$

where $N_{\tilde{a_1}} = N_{a_1} \cup N_{a_2}$. In case the measurement outcome corresponds to P_1, the state is

$$\frac{1}{2}\alpha_1 |1\rangle_c \left(\prod \sigma_z^{N_{a_2}} |0\rangle_{\tilde{a_1}} |G_1 \cup G_2/\tilde{a_1}\rangle + |1\rangle_{\tilde{a_1}} \prod \sigma_z^{N_{a_1}} |G_1 \cup G_2/\tilde{a_1}\rangle\right) |+\rangle_{aux}.$$

Therefore, in case one obtains the merging measurement outcome corresponding to P_1, a controlled correction operation of the form $|0\rangle_c\langle 0| \otimes \mathbb{1} + |1\rangle_c\langle 1| \otimes \prod \sigma_z^{N_{a_2}}$ is required to recover the state of Eq. (27). Note that the controlled correction is intended to have no influence on the other branch of the superposition, such that the probabilities of pretending to measure each branch individually are such that the weights in the superposition do not change. This is possible due to the orthogonality of the constituents of the superposition.

Taking into account Eq. (26) and Eq. (27), and by simply relabelling qubits aux $\rightarrow a_2$ and $\tilde{a_1} \rightarrow a_1$, the final global state after re-normalization reads

$$|\Psi\rangle_f = \alpha_0 |0\rangle_c |G_1\rangle |G_2\rangle + \alpha_1 |1\rangle_c |G_1 \cup G_2\rangle |+\rangle_{a_2}, \tag{28}$$

where a superposition between two graph states unaltered and two graph states merged is found. Observe that, crucially, superposition weights remain unchanged through the process. Note also that qubit a_2 of the second branch is now some auxiliary qubit that has been relabelled for convenience.