Supplementary Information for "How to build Hamiltonians that transport noncommuting charges in quantum thermodynamics"

SUPPLEMENTARY NOTE 1. THE KILLING FORM INDUCES A METRIC ON EVERY SIMPLE LIE ALGEBRA.

Here, we prove a claim made in Sec. II.A of the main text: The Killing form induces a metric on every simple Lie algebra. The proof relies on background material reviewed in Sec. II.B of the main text.

Every inner product defines a metric. Therefore, proving that the Killing form induces an inner product suffices. On a simple Lie algebra, all symmetric bilinear forms equal each other to within a multiplicative constant [1]. The Killing form is one symmetric bilinear form; another is $\mathrm{Tr}(Q_\alpha Q_\beta)$. Hence $(Q_\alpha, Q_\beta) \propto \mathrm{Tr}(Q_\alpha Q_\beta) = \mathrm{Tr}\big(Q_\alpha^\dagger Q_\beta\big)$. The final equality follows from the charges' Hermiticity. The final expression is the Hilbert-Schmidt inner product. Hence the Killing form induces an inner product.

SUPPLEMENTARY NOTE 2. GENERAL HAMILTONIAN THAT TRANSPORTS $\mathfrak{su}(2)$ ELEMENTS LOCALLY WHILE CONSERVING THEM GLOBALLY

Section II.B of the main text illustrated how to construct Hamiltonians that transport $\mathfrak{su}(2)$ elements locally while conserving them globally. The illustration was not maximally general; we restricted a unitary U more than required, for pedagogy. We generalize the construction here. For clarity of presentation, we derive the charges' forms first (Supplementary Note II A) and the ladder operators' forms second (Supplementary Note II B). We then construct the two-body Hamiltonian $H^{(j,j')}$ and a three-body Hamiltonian (Supplementary Note II C).

II A. Preferred basis of charges for $\mathfrak{su}(2)$

The conventional Cartan-Weyl basis contains the Hermitian operator

$$Q_1 = \sigma_z. \tag{26}$$

To identify the next Cartan-Weyl basis, we invoke a general unitary $U \in \mathrm{SU}(2)$. In the Euler parameterization,

$$U = e^{i\sigma_z \phi_1/2} e^{i\sigma_y \phi_2/2} e^{i\sigma_z \phi_3/2}, \tag{27}$$

wherein $\phi_1 \in [0, 2\pi)$, $\phi_2 \in [0, \pi]$, and $\phi_3 \in [0, 2\pi)$. We restrict this general unitary to a U_i that maps Q_2 to a Killing-orthogonal charge $Q_2 = U_\mathrm{i}^\dagger Q_1 U_\mathrm{i}$. For $X, Y \in \mathfrak{su}(D)$, the Killing form evaluates to $(X, Y) = \mathrm{Tr}(XY)$ [1]. Hence the Killing form between the charges is

$$0 = \left(U_\mathrm{i}^\dagger Q_1 U_\mathrm{i},\, Q_1\right) = \mathrm{Tr}\left(U_\mathrm{i}^\dagger Q_1 U_\mathrm{i} Q_1\right) = 2\cos\left(\phi_2^{(\mathrm{i})}\right). \tag{28}$$

The superscript (i), here and below, labels a parameter as belonging to U_i. The equation, with $\phi_2^{(\mathrm{i})} \in [0, \pi]$, implies that $\phi_2^{(\mathrm{i})} = \pi/2$. The unitary and charge assume the forms

$$U_\mathrm{i} = e^{i\sigma_z \phi_1^{(\mathrm{i})}/2} e^{i\sigma_y \pi/4} e^{i\sigma_z \phi_3^{(\mathrm{i})}/2} \quad \text{and} \quad Q_2 = \cos\!\left(\phi_3^{(\mathrm{i})}\right)\sigma_x + \sin\!\left(\phi_3^{(\mathrm{i})}\right)\sigma_y. \tag{29}$$

Having identified the second charge, we identify the final one. We transform Q_1 with a unitary $U_\mathrm{ii} \in \mathrm{SU}(2)$ such that $Q_3 = U_\mathrm{ii}^\dagger Q_1 U_\mathrm{ii}$ is Killing-orthogonal to the first two charges. The first orthogonality constraint has the form of Eq. (28), except that a (ii) replaces the superscript (i). The second orthogonality constraint is

$$0 = \mathrm{Tr}\left(U_\mathrm{ii}^\dagger Q_1 U_\mathrm{ii},\, Q_2\right) = \mathrm{Tr}\left(U_\mathrm{ii}^\dagger Q_1 U_\mathrm{ii} Q_2\right) = 2\cos\left(\phi_3^{(\mathrm{i})} - \phi_3^{(\mathrm{ii})}\right). \tag{30}$$

Hence $\phi_3^{(\mathrm{ii})} = \phi_3^{(\mathrm{i})} + \pi\left(n^{(\mathrm{ii})} - \frac{1}{2}\right)$, wherein $n^{(\mathrm{ii})} \in \mathbb{Z}$. Hence U_ii and Q_3 have the forms

$$U_\mathrm{ii} = e^{i\sigma_z \phi_1^{(\mathrm{ii})}/2} e^{i\sigma_y \pi/4} e^{i\sigma_z [\phi_3^{(\mathrm{i})} + \pi(n - \frac{1}{2})]/2} \quad \text{and} \tag{31}$$

$$Q_3 = (-1)^{n^{(\mathrm{ii})}} \left[\sin\left(\phi_3^{(\mathrm{i})}\right)\sigma_x - \cos\left(\phi_3^{(\mathrm{i})}\right)\sigma_y\right]. \tag{32}$$

Equations (32), (29), and (26) specify the preferred basis of charges for $\mathfrak{su}(2)$.

II B. General ladder operators for $\mathfrak{su}(2)$

The conventional Cartan-Weyl basis contains operators that raise and lower σ_z:

$$L_{\pm 1} = \sigma_{\pm z} = \frac{1}{2}(\sigma_x \pm i\sigma_y). \tag{33}$$

Conjugation with U_{i} yields the ladder operators for Q_2, and conjugation with U_{ii} yields the ladder operators for Q_3:

$$L_{\pm 2} = U_{\mathrm{i}}^\dagger L_{\pm 1} U_{\mathrm{i}} = \frac{-e^{\mp i\phi_1^{(\mathrm{i})}}}{2}[\sigma_z \pm i(\sin\{\phi_3^{(\mathrm{i})}\}\sigma_x - \cos\{\phi_3^{(\mathrm{i})}\}\sigma_y)], \quad \text{and} \tag{34}$$

$$L_{\pm 3} = U_{\mathrm{ii}}^\dagger L_{\pm 1} U_{\mathrm{ii}} = \frac{-e^{\mp i\phi_1^{(\mathrm{ii})}}}{2}\left\{\sigma_z \mp i(-1)^{n^{(\mathrm{ii})}}\left[\cos\left(\phi_3^{(\mathrm{i})}\right)\sigma_x + \sin\left(\phi_3^{(\mathrm{i})}\right)\sigma_y\right]\right\}. \tag{35}$$

II C. Two-body and three-body Hamiltonians for $\mathfrak{su}(2)$

To form $H^{(j,j')}$, we substitute for the ladder operators from Eqs. (33) and (34) into Eq. (12). We require that $H^{(j,j')}$ conserve each global charge, imposing Eq. (14). This equation holds, algebra reveals, if and only if the hopping frequencies $J_\alpha^{(j,j')}$ equal each other. The Hamiltonian simplifies to Eq. (15). The final expression does not depend on our choice of $\phi_k^{(\mathrm{i})}$, $\phi_k^{(\mathrm{ii})}$, or $n^{(\mathrm{i})}$.

Let us construct a Hamiltonian $H^{(j,j',j'')}$ that transfers $\mathfrak{su}(2)$ charges between three sites—j, j', and j''—while conserving the charges globally. We multiply three two-body Hamiltonians together cyclically:

$$H^{(j,j',j'')} \propto H^{(j,j')}H^{(j',j'')}H^{(j'',j)} \tag{36}$$

We substitute in from Eq. (13), the $H^{(j,j')}$ expression in which the hopping frequencies have not yet been restricted. The frequencies can assume different values, when $[H^{(j,j',j'')}, Q_\alpha^{\mathrm{tot}}] = 0$, than when $[H^{(j,j')}, Q_\alpha^{\mathrm{tot}}] = 0$. Imposing the first commutator equation yields four sets of solutions for the J_α's, when $J_\alpha \neq 0$ for all α:

1. $J_1 = J_2 = J_3$, $J_4 = J_5 = J_6$, and $J_7 = J_8 = J_9$.
2. $J_1 = J_2 = -J_3$, $J_4 = J_5 = -J_6$, and $J_7 = J_8 = -J_9$.
3. $J_1 = J_2 = \frac{-J_3}{2}$, $J_4 = J_5 = \frac{-J_6}{2}$, and $J_7 = J_8 = \frac{-J_9}{2}$.
4. $\frac{J_2}{J_1} = \frac{J_5}{J_4} = \frac{J_8}{J_7}$, $J_1 + J_2 = -J_3$, $J_4 + J_5 = -J_6$, and $J_7 + J_8 = -J_9$.

We have omitted superscripts for conciseness. The four solutions lead to distinct Hamiltonians.[1]

For concreteness, we detail the first set of solutions, item 1. We collect three of the frequencies to simplify notation: $J^{j,j',j''} = J_1^{(j,j')}J_4^{(j',j'')}J_7^{(j',j'')}$. Substituting the J_α's into the Hamiltonian (36) yields

$$\begin{aligned} H^{(j,j',j'')} \propto J^{(j,j',j'')}\Big\{&3\mathbb{1}\mathbb{1}\mathbb{1} - 2\left(H^{(j,j')} - H^{(j'',j)} + H^{(j',j'')}\right) \\ &+ i[(\sigma_x\sigma_y\sigma_z + \sigma_y\sigma_z\sigma_x + \sigma_z\sigma_x\sigma_y) - (\sigma_z\sigma_y\sigma_x + \sigma_x\sigma_z\sigma_y + \sigma_y\sigma_x\sigma_z)\}. \end{aligned} \tag{37}$$

We have omitted some superscripts to simplify notation. The first term is trivial, terms 2-4 are two-body, and each of terms 1-4 conserves each Q_α^{tot}. Subtracting these terms off yields the solely three-body Hamiltonian, Eq. (18). We have absorbed the i into the coefficient such that $J^{j,j',j''} \in \mathbb{R}$.

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[1] However, each solution contains a little redundancy: Consider picking one of the four solutions, then cycling the indices in $(1,2,3)$ identically to the indices in $(4,5,6)$ and to the indices in $(7,8,9)$. The resulting J_α's specify a Hamiltonian identical to the original.

SUPPLEMENTARY NOTE 3. SIMPLE FORM TO WHICH A TWO-BODY HAMILTONIAN MAY COLLAPSE

In the $\mathfrak{su}(2)$ example, $H^{(j,j')}$ collapsed to the simple form (16). The $\mathfrak{su}(3)$ $H^{(j,j')}$ collapses to an analogous form, we shown in Sec. II.D. This form generalizes to

$$\sum_{\alpha=1}^{c} Q_\alpha^{(j)} Q_\alpha^{(j')}. \tag{38}$$

This expression generally conserves noncommuting charges globally, and transport the charges locally, as proved below. However, the expression's equality with a two-body Hamiltonian that clearly, overtly transports local charges from site to site is proved only in the $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ examples.

Proposition 1. *Consider any Lie algebra whose structure constants have the antisymmetry property*

$$f_{\alpha\beta}^{\gamma} = -f_{\gamma\beta}^{\alpha}. \tag{39}$$

A two-body Hamiltonian of the form (38) *conserves the algebra's elements globally.*

Every compact semisimple Lie algebra has such structure constants [2].

Proof. First, we substitute from Eq. (38) into the conservation law. Then, we invoke the commutator's linearity and the arguments' tensor-product forms:

$$0 = \left[H^{(j,j')},\, Q_\alpha^{\text{tot}}\right] = \left[\sum_{\beta=1}^{c} Q_\beta^{(j)} Q_\beta^{(j')},\, Q_\alpha^{(j)} \otimes \mathbb{1}^{(j')} + \mathbb{1}^{(j)} \otimes Q_\alpha^{(j')}\right] \tag{40}$$

$$= \sum_{\beta=1}^{c} \left(\left[Q_\beta^{(j)} Q_\beta^{(j')},\, Q_\alpha^{(j)} \otimes \mathbb{1}^{(j')}\right] + \left[Q_\beta^{(j)} Q_\beta^{(j')},\, \mathbb{1}^{(j)} \otimes Q_\alpha^{(j')}\right] \right) \tag{41}$$

$$= \sum_{\beta=1}^{c} \left(\left[Q_\beta^{(j)},\, Q_\alpha^{(j)}\right] Q_\beta^{(j')} + Q_\beta^{(j)} \left[Q_\beta^{(j')},\, Q_\alpha^{(j')}\right] \right). \tag{42}$$

Let $f_{\alpha\beta}^{\gamma}$ denote the Lie algebra's structure constants. The f's dictate how a Lie bracket decomposes as a linear combination of the algebra's elements:

$$[Q_\alpha,\, Q_\beta] = \sum_{\gamma=1}^{c} f_{\alpha\beta}^{\gamma} Q_\gamma. \tag{43}$$

We substitute into Eq. (42), then pull the sums and constants out front:

$$0 = \sum_{\beta=1}^{c} \left[\left(\sum_{\gamma=1}^{c} f_{\beta\alpha}^{\gamma} Q_\gamma^{(j)} \right) Q_\beta^{(j')} + Q_\beta^{(j)} \left(\sum_{\gamma=1}^{c} f_{\beta\alpha}^{\gamma} Q_\gamma^{(j')} \right) \right] = \sum_{\beta,\gamma=1}^{c} f_{\beta\alpha}^{\gamma} \left(Q_\gamma^{(j)} Q_\beta^{(j')} + Q_\beta^{(j)} Q_\gamma^{(j')} \right). \tag{44}$$

The final equation holds if $f_{\beta\alpha}^{\gamma} = -f_{\gamma\alpha}^{\beta}$. Consider relabeling the index α as β and vice versa. Equation (39) results. □

Having proved that the simple operator (38) conserves noncommuting charges globally, we prove that it transports charges locally.

Proposition 2. *The simple two-body Hamiltonian* (38) *transports the charges Q_α locally.*

Proof. Charge Q_α is transported locally if it satisfies Eq. (3), having a nonzero commutator

$$\left[H^{(j,j')}, Q_\alpha^{(j)}\right] = \left[\sum_{\beta=1}^{c} Q_\beta^{(j)} Q_\beta^{(j')}, Q_\alpha^{(j)}\right] = \sum_{\beta=1}^{c} \left[Q_\beta^{(j)}, Q_\alpha^{(j)}\right] Q_\beta^{(j')} = \sum_{\beta,\gamma=1}^{c} f_{\beta\alpha}^{\gamma}\, Q_\gamma^{(j)} Q_\beta^{(j')}. \tag{45}$$

The final expression vanishes if Q_α commutes with all the other charges Q_γ in the preferred basis. If a Lie algebra has a basis of which one element commutes with the others, the algebra is Abelian, by definition [1]. We assume that the algebra $\mathcal{A}$ is non-Abelian (Sec. II.A of the main text). Therefore, the right-hand side of (45) is nonzero, and the Hamiltonian transports the charges locally. □

SUPPLEMENTARY NOTE 4. PROOF OF PROPOSITION 1

Proposition 1 states that the algebra $\mathcal{A}$ has an integer ratio c/r, wherein c denotes the algebra's dimension and r denotes the rank.

Proof. For every finite-dimensional complex Lie algebra, there exists a corresponding connected Lie group that is unique to within finite coverings. The Lie algebra has the same dimension and rank as each of the corresponding Lie groups. Thus, if Proposition 1 holds for all semisimple Lie groups, it holds for all semisimple Lie algebras. We prove the group claim.

Every Lie group has a maximal torus $\mathbb{T}^r$, which is the group generated by a Cartan subalgebra of the Lie algebra. The torus' dimensionality equals the group's rank, r. A torus is an r-fold Cartesian product of $\mathbb{S}^1$ manifolds [equivalently, of the group U(1)]. Quotienting out the torus' action from the Lie group yields a finite-dimensional coset space. Every finite-dimensional coset space's dimensionality is a positive integer $n \in \mathbb{Z}_{>0}$. Thus, the semisimple Lie group's dimension is $c = rn$. □

SUPPLEMENTARY NOTE 5. MATHEMATICAL DETAILS: CONSTRUCTION OF A TWO-BODY HAMILTONIAN THAT TRANSPORTS $\mathfrak{su}(3)$ ELEMENTS LOCALLY WHILE CONSERVING THEM GLOBALLY

Section II.D illustrated the Hamiltonian-construction prescription with $\mathfrak{su}(3)$. We flesh out the explanation here. Appendix V A reviews the conventional Cartan-Weyl basis for $\mathfrak{su}(3)$. Appendix V B identifies the preferred basis of charges for $\mathfrak{su}(3)$. Appendix V C presents the ladder operators from which we construct a Hamiltonian.

V A. Conventional Cartan-Weyl basis for $\mathfrak{su}(3)$

$\mathfrak{su}(3)$ has dimension $c = 8$ and rank $r = 2$. The conventional Cartan-subalgebra generators are denoted by $t_z = \lambda_3/2$ and $y = \lambda_8/\sqrt{3}$, wherein λ_3 and λ_8 denote Gell-mann matrices [3]. These generators, in the three-dimensional representation of $\mathfrak{su}(3)$, manifest as

$$T_z = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Y = \frac{1}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}. \tag{46}$$

t_z and y are orthogonal relative to the Killing form. They (more precisely, rescaled versions of them) belong in our preferred basis of charges: $Q_1 \propto t_z$, and $Q_2 \propto y$.

These charges are raised and lowered by $c - r = 8 - 2 = 6$ ladder operators, $t_\pm = (\lambda_1 \pm i\lambda_2)/2$, $v_\pm = (\lambda_4 \pm i\lambda_5)/2$, and $u_\pm = (\lambda_6 \pm i\lambda_7)/2$. In the three-dimensional representation of $\mathfrak{su}(3)$, the ladder operators manifest as

$$T_+ = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad T_- = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V_+ = \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V_- = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \tag{47}$$

$$U_+ = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad U_- = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{48}$$

The ladder operators participate in the following commutation relations with the charges:

$$[t_z, t_\pm] = \pm t_\pm, \quad [y, t_\pm] = 0, \tag{49}$$

$$[t_z, v_\pm] = \pm\frac{1}{2}v_\pm, \quad [y, v_\pm] = \pm v_\pm \tag{50}$$

$$[t_z, u_\pm] = \mp\frac{1}{2}u_\pm, \quad \text{and} \quad [y, u_\pm] = \pm u_\pm. \tag{51}$$

These relations imply that (i) $t_\pm$ raises and lowers t_z, whereas (ii) $v_\pm$ raises or lowers both t_z and y, as does $u_\pm$. We can prove this physical significance easily: Let $L_\pm$ denote a ladder operator (a $t_\pm$, a $v_\pm$, or a $u_\pm$) that raises/lowers a charge Q. Let $|\psi\rangle$ denote a Q eigenstate associated with the eigenvalue q: $Q\,|\psi\rangle = q\,|\psi\rangle$. Consider operating on the state with the ladder operator: $L_\pm |\psi\rangle$. Suppose, for notational convenience, that, (i) if L_+ operates, q is not the greatest Q eigenvalue and (ii) if L_- operates, q is not the least Q eigenvalue. The resulting state is a Q eigenstate associated with the eigenvalue $q \pm a$, wherein $a = 1$ or $1/2$. To prove this claim, we operate on the new state with the charge: $Q(L_\pm |\psi\rangle)$. Invoking the appropriate commutation relation [Eqs. (49)-(51)] yields

$$QL_\pm |\psi\rangle = (L_\pm Q \pm L_\pm) |\psi\rangle = L_\pm (Q \pm a\mathbb{1}) |\psi\rangle = L_\pm (q \pm a) |\psi\rangle = (q \pm a) L_\pm |\psi\rangle \,. \tag{52}$$

By Eqs. (49)-(52), $t_\pm$ raises/lowers the t_z charge by one quantum and preserves y. $u_\pm$ lowers/raises t_z by half a quantum and raises/lowers y by one quantum. $v_\pm$ raises/lowers each of t_z and y by one quantum.

Having reviewed the conventional Cartan-Weyl basis for $\mathfrak{su}(3)$, we dispense with the conventional notation (t_z, $t_\pm$, etc.). We revert to the notation introduced in the main text (Q_α and $L_{\pm\alpha}$).

V B. Preferred basis of charges for $\mathfrak{su}(3)$

The first two charges appear in Eqs. (20). We construct two new charges from Q_1, Q_2, and a unitary $U \in \mathrm{SU}(2)$. The general form of such a U, appears, in the Euler parameterization, in Eq. (21). We constrain U with the Killing-orthogonality conditions (9), obtaining a unitary U_{i}. The transformed charges have the forms $Q_3 = U_{\mathrm{i}}^\dagger Q_1 U_{\mathrm{i}}$ and $Q_4 = U_{\mathrm{i}}^\dagger Q_2 U_{\mathrm{i}}$. The new charges are Killing-orthogonal to each other by unitarity: $0 = \mathrm{Tr}\left(\left[U_{\mathrm{i}}^\dagger Q_1 U_{\mathrm{i}}\right]\left[U_{\mathrm{i}}^\dagger Q_2 U_{\mathrm{i}}\right]\right) = \mathrm{Tr}\,(Q_1 Q_2) = 0$. Killing-orthogonality to the old charges, Eq. (20), with the form of the $\mathfrak{su}(D)$ Killing form [1], implies

$$0 = \mathrm{Tr}\left(\left[U_{\mathrm{i}}^\dagger Q_1 U_{\mathrm{i}}\right] Q_2\right) = -\cos(\phi_2)/3, \qquad 0 = \mathrm{Tr}\left(\left[U_{\mathrm{i}}^\dagger Q_1 U_{\mathrm{i}}\right] Q_1\right) = -\frac{1}{2\sqrt{3}}\cos(\phi_3 + \phi_5), \tag{53}$$

$$0 = \mathrm{Tr}\left(\left[U_{\mathrm{i}}^\dagger Q_2 U_{\mathrm{i}}\right] Q_2\right) = \frac{1}{2}\left[\cos(\phi_4) + \frac{1}{3}\right], \qquad \text{and} \qquad 0 = \mathrm{Tr}\left(\left[U_{\mathrm{i}}^\dagger Q_2 U_{\mathrm{i}}\right] Q_1\right) = -\cos(\phi_6)/3. \tag{54}$$

Since $\phi_2, \phi_4, \phi_6 \in [0, \pi]$ and $\phi_3, \phi_5 \in [0, 2\pi)$, $\phi_2 = \frac{\pi}{2}$, $\phi_4 = \mathrm{acos}(-1/3)$, $\phi_6 = \frac{\pi}{2}$ and $\phi_5 = \pi(n - 1/2) - \phi_3$, for $n \in \{1, 2, 3, 4\}$.

Transforming Q_1 and Q_2 with a $U_{\mathrm{ii}} \in \mathrm{SU}(3)$ yields the charges Q_5 and Q_6, and transforming Q_1 and Q_2 with a $U_{\mathrm{iii}} \in \mathrm{SU}(3)$ yields Q_7 and Q_8. These last four charges are Killing-orthogonal to Q_1 and Q_2, like Q_3 and Q_4. So U_{ii} and U_{iii} share the form of U_{i}. However, parameters $a^{(\mathrm{ii})}$ and $b^{(\mathrm{ii})}$, or $a^{(\mathrm{iii})}$ and $b^{(\mathrm{iii})}$, replace the $a^{(\mathrm{i})}$ and $b^{(\mathrm{i})}$. The later unitaries' parameters are more constrained than the U_{i} parameters. Similarly, Q_5 through Q_8 share the forms of Q_3 and Q_1, apart from their more-constrained parameters.

Evaluating the restrictions on all the charges simultaneously will prove useful. First, the conditions for Q_5 to be orthogonal to Q_3 and Q_4 are

$$0 = \mathrm{Tr}(Q_5 Q_3) \propto (-1)^{n^{(\mathrm{i})}+n^{(\mathrm{ii})}} \cos\left(a^{(\mathrm{i})} - a^{(\mathrm{ii})} - b^{(\mathrm{i})} + b^{(\mathrm{ii})}\right) + \cos\left(a^{(\mathrm{i})} - a^{(\mathrm{ii})}\right) + \cos\left(b^{(\mathrm{i})} - b^{(\mathrm{ii})}\right) \quad \text{and} \tag{55}$$

$$0 = \mathrm{Tr}(Q_5 Q_4) \propto (-1)^{n^{(\mathrm{i})}+n^{(\mathrm{ii})}} \sin\left(a^{(\mathrm{i})} - a^{(\mathrm{ii})} - b^{(\mathrm{i})} + b^{(\mathrm{ii})}\right) - \sin\left(a^{(\mathrm{i})} - a^{(\mathrm{ii})}\right) + \sin\left(b^{(\mathrm{i})} - b^{(\mathrm{ii})}\right). \tag{56}$$

The orthogonality conditions for Q_6 impose the same constraints, since $\mathrm{Tr}(Q_6 Q_3) \propto \mathrm{Tr}(Q_5 Q_4)$ and $\mathrm{Tr}(Q_6 Q_4) \propto \mathrm{Tr}(Q_5 Q_3)$ (as can be checked explicitly). Similarly, the orthogonality conditions on Q_7 evaluate to

$$0 = \mathrm{Tr}(Q_7 Q_3) \propto (-1)^{n^{(\mathrm{i})}+n^{(\mathrm{iii})}} \cos\left(a^{(\mathrm{i})} - a^{(\mathrm{iii})} - b^{(\mathrm{i})} + b^{(\mathrm{iii})}\right) + \cos\left(a^{(\mathrm{i})} - a^{(\mathrm{iii})}\right) + \cos\left(b^{(\mathrm{i})} - b^{(\mathrm{iii})}\right), \tag{57}$$

$$0 = \mathrm{Tr}(Q_7 Q_4) \propto (-1)^{n^{(\mathrm{i})}+n^{(\mathrm{iii})}} \sin\left(a^{(\mathrm{i})} - a^{(\mathrm{iii})} - b^{(\mathrm{i})} + b^{(\mathrm{iii})}\right) - \sin\left(a^{(\mathrm{i})} - a^{(\mathrm{iii})}\right) + \sin\left(b^{(\mathrm{i})} - b^{(\mathrm{iii})}\right), \tag{58}$$

$$0 = \mathrm{Tr}(Q_7 Q_5) \propto (-1)^{n^{(\mathrm{ii})}+n^{(\mathrm{iii})}} \cos\left(a^{(\mathrm{ii})} - a^{(\mathrm{iii})} - b^{(\mathrm{ii})} + b^{(\mathrm{iii})}\right) + \cos\left(a^{(\mathrm{ii})} - a^{(\mathrm{iii})}\right) + \cos\left(b^{(\mathrm{ii})} - b^{(\mathrm{iii})}\right), \text{ and} \tag{59}$$

$$0 = \mathrm{Tr}(Q_7 Q_6) \propto (-1)^{n^{(\mathrm{ii})}+n^{(\mathrm{iii})}} \sin\left(a^{(\mathrm{ii})} - a^{(\mathrm{iii})} - b^{(\mathrm{ii})} + b^{(\mathrm{iii})}\right) - \sin\left(a^{(\mathrm{ii})} - a^{(\mathrm{iii})}\right) + \sin\left(b^{(\mathrm{ii})} - b^{(\mathrm{iii})}\right). \tag{60}$$

The orthogonality conditions for Q_8 impose the same constraints [Eqs. (57)-(60)].

We now identify sets of $a^{(\ell)}, b^{(\ell)}$, and $n^{(\ell)}$ that are solutions for all six constraints, Eqs. (55)-(60). First, we define $x_{\ell m} := a^{(\ell)} - a^{(m)}$ and $y_{\ell m} := b^{(\ell)} - b^{(m)}$, for $(\ell, m) = (2,3), (2,4), (3,4)$. By these definitions, $x_{24} = x_{23} + x_{34}$, and $y_{24} = y_{23} + y_{34}$. Second, the values of the $n^{(\ell)}$ themselves are irrelevant. Only whether $n^{(\ell)} + n^{(m)}$ is even or odd matters. Only four unique possibilities for the $n^{(\ell)}$ exist: All the $n^{(\ell)} + n^{(m)}$ are even; or one $n^{(\ell)} + n^{(m)}$ is even, while the other two sums are odd. A solution can therefore be expressed in terms of just four quantities: x_{23}, x_{34}, y_{23}, and y_{34}. Each solution is periodic:

$$(x_{23}, x_{34}, y_{23}, y_{34}) \equiv (x_{23}, x_{34}, y_{23}, y_{34}) + (2\pi n, 2\pi n, 2\pi n, 2\pi n), \tag{61}$$

wherein $n \in \mathbb{Z}$. Therefore, we omit the $2\pi n$ when listing the solutions below.

First, suppose that all the $n^{(\ell)} + n^{(m)}$ are even. The constraints (55)-(60) admit of 18 solutions. The first ten are

$$\begin{aligned}(x_{23}, x_{34}, y_{23}, y_{34}) = &\left(0, \pm\frac{2\pi}{3}, \mp\frac{2\pi}{3}, \pm\frac{2\pi}{3}\right), \left(0, 0, \pm\frac{2\pi}{3}, \pm\frac{2\pi}{3}\right), \left(0, \pm\frac{2\pi}{3}, \pm\frac{2\pi}{3}, 0\right), \left(\pm\frac{2\pi}{3}, 0, \pm\frac{2\pi}{3}, \mp\frac{2\pi}{3}\right),\\ &\left(\pm\frac{2\pi}{3}, \pm\frac{2\pi}{3}, \pm\frac{2\pi}{3}, \pm\frac{2\pi}{3}\right).\end{aligned} \tag{62}$$

The next eight solutions are identical to the first eight, except that each $x_{\ell m}$ is swapped with the corresponding $y_{\ell m}$.

Second, $n^{(\mathrm{i})} + n^{(\mathrm{iii})}$ can be even while $n^{(\mathrm{i})} + n^{(\mathrm{ii})}$ and $n^{(\mathrm{ii})} + n^{(\mathrm{iii})}$ are odd. The constraints (55)-(60) admit of another 18 solutions. The first ten are

$$\begin{aligned}(x_{23}, x_{34}, y_{23}, y_{34}) = &\left(\pi, \pm\frac{\pi}{3}, \mp\frac{\pi}{3}, \pm\frac{\pi}{3}\right), \left(\pi, \pi, \pm\frac{\pi}{3}, \pm\frac{\pi}{3}\right), \left(\pi, \pm\frac{\pi}{3}, \pm\frac{\pi}{3}, \pi\right), \left(\pm\frac{\pi}{3}, \pi, \pm\frac{\pi}{3}, \mp\frac{\pi}{3}\right),\\ &\left(\pm\frac{\pi}{3}, \pm\frac{\pi}{3}, \pm\frac{\pi}{3}, \pm\frac{\pi}{3}\right).\end{aligned} \tag{63}$$

The next eight solutions are identical to the first eight, except that each $x_{\ell m}$ is swapped with the corresponding $y_{\ell m}$.

Third, $n^{(\mathrm{i})} + n^{(\mathrm{ii})}$ can be even while $n^{(\mathrm{i})} + n^{(\mathrm{iii})}$ and $n^{(\mathrm{ii})} + n^{(\mathrm{iii})}$ are odd. The constraints (55)-(60) admit of another 18 solutions. The first ten are

$$\begin{aligned}(x_{23}, x_{34}, y_{23}, y_{34}) = &\left(0, \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}, \pm\frac{\pi}{3}\right), \left(0, \pi, \pm\frac{2\pi}{3}, \mp\frac{\pi}{3}\right), \left(0, \mp\frac{\pi}{3}, \pm\frac{2\pi}{3}, \pi\right), \left(\pm\frac{2\pi}{3}, \pi, \pm\frac{2\pi}{3}, \pm\frac{\pi}{3}\right),\\ &\left(\pm\frac{2\pi}{3}, \mp\frac{\pi}{3}, \pm\frac{2\pi}{3}, \mp\frac{\pi}{3}\right).\end{aligned} \tag{64}$$

The next eight solutions are identical to the first eight, except that each $x_{\ell m}$ is swapped with the corresponding $y_{\ell m}$.

Fourth, suppose that $n^{(\mathrm{ii})} + n^{(\mathrm{iii})}$ is even while $n^{(\mathrm{i})} + n^{(\mathrm{ii})}$ and $n^{(\mathrm{i})} + n^{(\mathrm{iii})}$ are odd. The constraints (55)-(60) admit of another 18 solutions. The first ten are

$$\begin{aligned}(x_{23}, x_{34}, y_{23}, y_{34}) = &\left(\pi, \pm\frac{2\pi}{3}, \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}\right), \left(\pi, 0, \pm\frac{\pi}{3}, \mp\frac{2\pi}{3}\right), \left(\pi, \mp\frac{2\pi}{3}, \pm\frac{\pi}{3}, 0\right), \left(\pm\frac{\pi}{3}, 0, \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}\right),\\ &\left(\pm\frac{\pi}{3}, \mp\frac{2\pi}{3}, \pm\frac{\pi}{3}, \mp\frac{2\pi}{3}\right).\end{aligned} \tag{65}$$

The next eight solutions are identical to the first eight, except that each $x_{\ell m}$ is swapped with the corresponding $y_{\ell m}$.

One can check explicitly that the tuple $(x_{23} + y_{23},\, x_{34} + y_{34})$ has three possible values: $(x_{23} + y_{23},\, x_{34} + y_{34}) = (\pm 2\pi/3,\, \pm 2\pi/3), (\pm 4\pi/3,\, \pm 4\pi/3), (\pm 2\pi/3,\, \mp 4\pi/3)$. Three sets of solutions follow. For example, the first set of solutions is $(x_{23} + y_{23},\, x_{34} + y_{34}) = (\pm 2\pi/3,\, \pm 2\pi/3)$. Hence

$$a^{(\mathrm{i})} - a^{(\mathrm{ii})} + b^{(\mathrm{i})} - b^{(\mathrm{ii})} = \pm\frac{2\pi}{3}, \qquad a^{(\mathrm{ii})} - a^{(\mathrm{iii})} + b^{(\mathrm{ii})} - b^{(\mathrm{iii})} = \pm\frac{2\pi}{3}, \tag{66}$$

$$a^{(\ell)} - a^{(m)} \in \left\{0, \frac{\pm\pi}{3}, \frac{\pm 2\pi}{3}, \pi\right\}, \quad \text{and} \quad b^{(\ell)} - b^{(m)} \in \left\{0, \frac{\pm\pi}{3}, \frac{\pm 2\pi}{3}, \pi\right\}, \tag{67}$$

for $(\ell, m) = (2, 3)$ and $(3, 4)$. All the solutions lead to the same Hamiltonian, Eq. (25).

V C. Ladder operators for $\mathfrak{su}(3)$

The conventional Cartan-Weyl basis contains six ladder operators [Eqs. (24)]. We transform $L_{\pm 1,2,3}$ with the unitaries U_{i}, U_{ii}, and U_{iii} of Sec. V B, to construct the rest of the ladder operators: $L_{\pm 4} = U_{\mathrm{i}}^\dagger L_{\pm 1} U_{\mathrm{i}}$, $L_{\pm 5} = U_{\mathrm{i}}^\dagger L_{\pm 2} U_{\mathrm{i}}$, and $L_{\pm 6} = U_{\mathrm{i}}^\dagger L_{\pm 3} U_{\mathrm{i}}$. Substituting in for $L_{\pm 1,2,3}$ from Eq. (24) yields

$$\begin{aligned}L_{\pm 4} = \frac{i e^{\mp i \phi_1^{(\mathrm{i})}}}{6} \Big\{ &2i \cos\left(a^{(\mathrm{i})} - b^{(\mathrm{i})}\right) \lambda_1 - 2i \sin\left(a^{(\mathrm{i})} - b^{(\mathrm{i})}\right) \lambda_2 \mp \left[\sqrt{3} \mp i(-1)^{n^{(\mathrm{i})}}\right] \left[\cos\left(a^{(\mathrm{i})}\right) \lambda_4 - \sin\left(a^{(\mathrm{i})}\right) \lambda_5\right]\\ &\pm \left[\sqrt{3} \pm i(-1)^{n^{(\mathrm{i})}}\right] \left[\cos\left(b^{(\mathrm{i})}\right) \lambda_6 - \sin\left(b^{(\mathrm{i})}\right) \lambda_7\right] \mp \sqrt{3}(-1)^{n^{(\mathrm{i})}} \lambda_3 - \sqrt{3} i \lambda_8 \Big\},\end{aligned} \tag{68}$$

$$
\begin{aligned}
L_{\pm 5} = \frac{ie^{\mp\frac{i}{2}(\phi_3^{(\mathrm{i})}+\phi_1^{(\mathrm{i})})}}{6}\Big(& i\left[\cos\Big(a^{(\mathrm{i})}-b^{(\mathrm{i})}\Big)-\sqrt{3}(-1)^{n^{(\mathrm{i})}}\sin\Big(a^{(\mathrm{i})}-b^{(\mathrm{i})}\Big)\right]\lambda_1 \\
&- i\left[\sin\Big(a^{(\mathrm{i})}-b^{(\mathrm{i})}\Big)+\sqrt{3}(-1)^{n^{(\mathrm{i})}}\cos\Big(a^{(\mathrm{i})}-b^{(\mathrm{i})}\Big)\right]\lambda_2 \pm \frac{1}{2}\left\{(-1)^{n^{(\mathrm{i})}}\left[3\sin\Big(a^{(\mathrm{i})}\Big)\pm i\cos\Big(a^{(\mathrm{i})}\Big)\right]+\sqrt{3}e^{\pm ia^{(\mathrm{i})}}\right\}\lambda_4 \\
&\pm \frac{1}{2}\left\{(-1)^{n^{(\mathrm{i})}}\left[3\cos\Big(a^{(\mathrm{i})}\Big)\mp i\sin\Big(a^{(\mathrm{i})}\Big)\right]\pm i\sqrt{3}e^{\pm ia^{(\mathrm{i})}}\right\}\lambda_5 \\
&\pm \frac{1}{2}\left\{(-1)^{n^{(\mathrm{i})}}\left[3\sin\Big(b^{(\mathrm{i})}\Big)\pm i\cos\Big(b^{(\mathrm{i})}\Big)\right]-\sqrt{3}e^{\pm ib^{(\mathrm{i})}}\right\}\lambda_6 \\
&\pm \frac{1}{2}\left\{(-1)^{n^{(\mathrm{i})}}\left[3\cos\Big(b^{(\mathrm{i})}\Big)\mp i\sin\Big(b^{(\mathrm{i})}\Big)\right]\mp i\sqrt{3}e^{\pm ib^{(\mathrm{i})}}\right\}\lambda_7 \mp \sqrt{3}(-1)^{n^{(\mathrm{i})}}\lambda_3+\sqrt{3}i\lambda_8\Big), \quad \text{and}
\end{aligned} \tag{69}
$$

$$
\begin{aligned}
L_{\pm 6} = \frac{ie^{\mp\frac{i}{2}\left(\phi_3^{(\mathrm{i})}-\phi_1^{(\mathrm{i})}\right)}}{6}\Big(& - i\left[\cos\Big(a^{(\mathrm{i})}-b^{(\mathrm{i})}\Big)+\sqrt{3}(-1)^{n^{(\mathrm{i})}}\sin\Big(a^{(\mathrm{i})}-b^{(\mathrm{i})}\Big)\right]\lambda_1 \\
&+ i\left[\sin\Big(a^{(\mathrm{i})}-b^{(\mathrm{i})}\Big)-\sqrt{3}(-1)^{n^{(\mathrm{i})}}\cos\Big(a^{(\mathrm{i})}-b^{(\mathrm{i})}\Big)\right]\lambda_2 \mp \frac{1}{2}\left\{(-1)^{n^{(\mathrm{i})}}\left[3\sin\Big(a^{(\mathrm{i})}\Big)\pm i\cos\Big(a^{(\mathrm{i})}\Big)\right]-\sqrt{3}e^{ia^{(\mathrm{i})}}\right\}\lambda_4 \\
&\mp \frac{1}{2}\left\{(-1)^{n^{(\mathrm{i})}}\left[3\cos\Big(a^{(\mathrm{i})}\Big)\mp i\sin\Big(a^{(\mathrm{i})}\Big)\right]\mp i\sqrt{3}e^{ia^{(\mathrm{i})}}\right\}\lambda_5 \mp \frac{1}{2}\left\{(-1)^{n^{(\mathrm{i})}}\left[3\sin\Big(b^{(\mathrm{i})}\Big)\pm i\cos\Big(b^{(\mathrm{i})}\Big)\right]+\sqrt{3}e^{ib^{(\mathrm{i})}}\right\}\lambda_6 \\
&\mp \frac{1}{2}\left\{(-1)^{n^{(\mathrm{i})}}\left[3\cos\Big(b^{(\mathrm{i})}\Big)\mp i\sin\Big(b^{(\mathrm{i})}\Big)\right]\pm i\sqrt{3}e^{ib^{(\mathrm{i})}}\right\}\lambda_7 \mp \sqrt{3}(-1)^{n^{(\mathrm{i})}}\lambda_3-\sqrt{3}i\lambda_8\Big).
\end{aligned} \tag{70}
$$

$L_{\pm 7}$, $L_{\pm 8}$, and $L_{\pm 9}$ have the same forms. However, (ii)'s replace the superscripts (i)'s. $L_{\pm 10}$, $L_{\pm 11}$, and $L_{\pm 12}$ likewise have the same form, except that (iii)'s replace the (i)'s.

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