Supplementary information for

Distillation of Gaussian Einstein-Podolsky-Rosen steering with noiseless linear amplification

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Supplementary Note 1: Details of the theoretical calculation of measurement-based NLA

The ideal NLA operation with gain g can be described as $g^{\hat{n}}$, where $\hat{n} = \hat{a}^{\dagger}\hat{a}$ is the photon number operator. If Bob's state is ρ_B, after the NLA, the probability of the heterodyne measurement outcome α is

$$\begin{aligned} P_g(\alpha) &= \frac{1}{\pi}\langle\alpha| g^{\hat{n}}\rho_B g^{\hat{n}}|\alpha\rangle \\ &= \frac{1}{\pi}e^{\left(g^2-1\right)|\alpha|^2}\langle g\alpha|\rho_B|g\alpha\rangle, \end{aligned} \tag{1}$$

where we used $g^{\hat{n}}|\alpha\rangle = e^{\frac{\left(g^2-1\right)|\alpha|^2}{2}}|g\alpha\rangle$. When ρ_B is measured with a heterodyne detection system directly, Bob will get the measurement outcome β with probability $P(\beta) = \frac{1}{\pi}\langle\beta|\rho_B|\beta\rangle$. If β is rescaled as $\alpha = \beta/g$ and assigned to a probabilistic filter function $P_\beta = e^{\left(1-g^{-2}\right)|\beta|^2}$, Bob's measurement outcome follows the same distribution given by Eq. (1) and thus emulates the operation $g^{\hat{n}}$. Notably, the probabilistic filter function is always greater than 1 for $g > 1$. So it is not a legitimate weighting probability which means that it is impossible to implement a ideal NLA. To implement a good approximation to the ideal NLA $g^{\hat{n}}$, we can introduce a finite cutoff $|\beta_c|$ and renormalize the probabilistic filter function P_β with respect to P_{β_c}, which gives the acceptance probability $P_{acc}(\beta)$ in Eq. (1) in the main text.

For a two-mode state, its density operator can be expressed as the following Weyl representation, i.e.,

$$\hat{\rho}_{in} = \int \frac{d^2\alpha d^2\beta}{\pi^2}\chi_{in}(\alpha,\beta) D(-\alpha) D(-\beta), \tag{2}$$

where χ_{in} and $D(\alpha) = \exp\left\{\alpha\hat{a}^{\dagger} - \alpha^*\hat{a}\right\}$ are the characteristic function corresponding to ρ_{in} and the displacement operator. When each mode of ρ_{in} goes through a noiseless amplifier, denoted as $g_1^{\hat{a}^{\dagger}\hat{a}}$ and $g_2^{\hat{b}^{\dagger}\hat{b}}$, the output state can be shown as

$$\begin{aligned} \hat{\rho}_{out} &= N g_2^{\hat{b}^{\dagger}\hat{b}} g_1^{\hat{a}^{\dagger}\hat{a}} \rho_{in} g_1^{\hat{a}^{\dagger}\hat{a}} g_2^{\hat{b}^{\dagger}\hat{b}} \\ &= N\int \frac{d^2\alpha d^2\beta}{\pi^2}\chi_{in}(\alpha,\beta) g_2^{\hat{b}^{\dagger}\hat{b}} g_1^{\hat{a}^{\dagger}\hat{a}} D_a(-\alpha) D_b(-\beta) g_1^{\hat{a}^{\dagger}\hat{a}} g_2^{\hat{b}^{\dagger}\hat{b}}, \end{aligned} \tag{3}$$

where N is the normalization factor, which is determined by $\mathrm{tr}\hat{\rho}_{out} = 1$. The characteristic function of the amplified state is given by

$$\begin{aligned}
&\chi_{out}\left(\bar{\alpha},\bar{\beta}\right)\\
&= N\int\frac{d^2\alpha d^2\beta}{\pi^2}\chi_{in}\left(\alpha,\beta\right)\\
&\mathtt{tr}\left[g_2^{\hat{b}^\dagger\hat{b}}g_1^{\hat{a}^\dagger\hat{a}}D_a\left(-\alpha\right)D_b\left(-\beta\right)g_1^{\hat{a}^\dagger\hat{a}}g_2^{\hat{b}^\dagger\hat{b}}D_a\left(\bar{\alpha}\right)D_b\left(\bar{\beta}\right)\right]\\
&= \frac{N}{\left(1-g_1^2\right)\left(1-g_2^2\right)}\int\frac{d^2\alpha d^2\beta}{\pi^2}\chi_{in}\left(\alpha,\beta\right)\\
&\times\exp\left\{A\left(|\alpha|^2+|\bar{\alpha}|^2\right)+C\left(\bar{\alpha}\alpha^*+\bar{\alpha}^*\alpha\right)\right\}\\
&\times\exp\left\{B\left(|\beta|^2+\left|\bar{\beta}\right|^2\right)+D\left(\bar{\beta}\beta^*+\bar{\beta}^*\beta\right)\right\},
\end{aligned}\tag{4}$$

where $A=\frac{g_1^2+1}{2\left(g_1^2-1\right)},\ C=\frac{g_1}{1-g_1^2},\ B=\frac{g_2^2+1}{2\left(g_2^2-1\right)},\ D=\frac{g_2}{1-g_2^2}$. For simplification, taking $\alpha=x_1+ip_1,\ \beta=x_2+ip_2,\ \bar{\alpha}=\bar{x}_1+i\bar{p}_1,\ \bar{\beta}=\bar{x}_2+i\bar{p}_2$, and $X=(x_1,p_1,x_2,p_2)^T$, $\bar{X}=(\bar{x}_1,\bar{p}_1,\bar{x}_2,\bar{p}_2)^T$, then Eq. (4) becomes

$$\begin{aligned}
\chi_{out}\left(\bar{X}\right)&=\frac{N}{\left(1-g_1^2\right)\left(1-g_2^2\right)}\exp\left\{\bar{X}^TG_1\bar{X}\right\}\\
&\times\int\frac{d^4X}{\pi^2}\chi_{in}\left(X\right)\exp\left\{X^TG_1X+\bar{X}^TG_2X\right\},
\end{aligned}\tag{5}$$

where

$$G_1=\begin{pmatrix}A&0&0&0\\0&A&0&0\\0&0&B&0\\0&0&0&B\end{pmatrix},G_2=\begin{pmatrix}2C&0&0&0\\0&2C&0&0\\0&0&2D&0\\0&0&0&2D\end{pmatrix}.\tag{6}$$

For any Gaussian state, with mean d and covariance matrix σ_{AB}, their characteristic function can be expressed as [1]

$$\chi_{in}\left(X\right)=\exp\left\{-\frac{1}{2}X^T\left(\Omega\sigma_{AB}\Omega^T\right)X-i\left(\Omega d\right)^TX\right\},\tag{7}$$

where $\Omega=\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}$. Then substituting Eq. (7) into Eq. (5) and using the integration formula

$$\int d^n X \exp\left\{-\frac{1}{2}X^T M X + X^T v\right\} = \frac{\pi^{n/2}}{\sqrt{|M|}} \exp\left\{\frac{1}{2}v^T M^{-1} v\right\}, \tag{8}$$

we have

$$\begin{aligned}
&\chi_{out}\left(\bar{X}\right) \\
&\to \exp\left\{-\frac{1}{2}\bar{X}^T\left[G_2\left(2G_1 - \Omega\sigma_{AB}\Omega^T\right)^{-1}G_2 - 2G_1\right]\bar{X}\right\} \\
&\exp\left\{\frac{1}{2}\left(G_2\bar{X}\right)^T\left(\Omega\sigma_{AB}\Omega^T - 2G_1\right)^{-1}\left(-i\left(\Omega d\right)\right)\right\} \\
&\exp\left\{\frac{1}{2}\left(-i\left(\Omega d\right)\right)^T\left(\Omega\sigma_{AB}\Omega^T - 2G_1\right)^{-1}\left(G_2\bar{X} - i\left(\Omega d\right)\right)\right\},
\end{aligned} \tag{9}$$

which indicates that the covariance matrix of the amplified state is

$$\begin{aligned}
\sigma_{nla} &= \Omega^{-1}\left[G_2\left(2G_1 - \Omega\sigma_{AB}\Omega^T\right)^{-1}G_2 - 2G_1\right]\left(\Omega^T\right)^{-1} \\
&= G_2\left(2G_1 - \sigma_{AB}\right)^{-1}G_2 - 2G_1
\end{aligned} \tag{10}$$

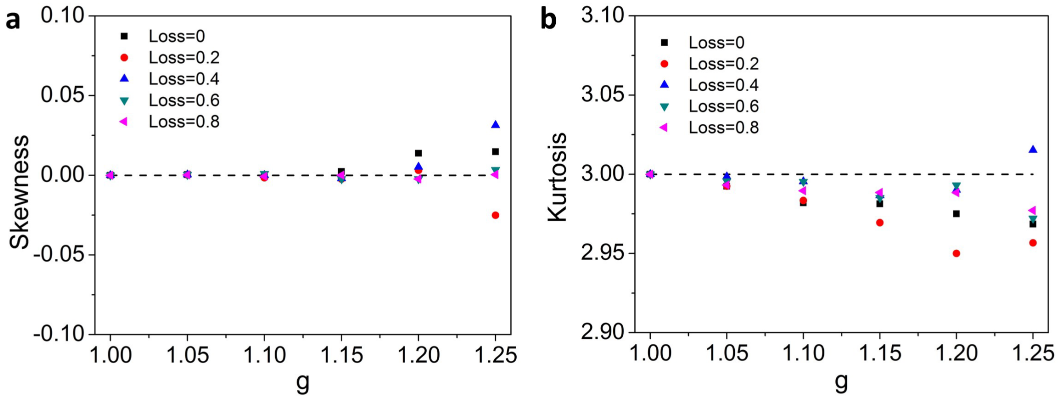
where $\Omega^T G_1 \Omega = G_1,\ \Omega^T G_2 \Omega = G_2,\ \Omega^{-1} = \Omega^T$. When only mode $\hat{B}$ of the two-mode state go through noiseless amplifier, we can take limit $g_1 \to 1$.

Supplementary Table 1 The optimal cutoff $|\beta_c|$ at different loss and g

	g=1.05	g=1.10	g=1.15	g=1.20	g=1.25
Loss=0.0	4.25	4.75	5.25	5.50	6.00
Loss=0.2	4.00	4.25	4.50	4.75	5.25
Loss=0.4	3.75	4.00	4.25	4.50	5.00
Loss=0.6	3.50	3.75	4.00	4.25	4.50
Loss=0.8	3.00	3.25	3.50	3.75	4.00

Supplementary Note 2: The selection of cutoff value

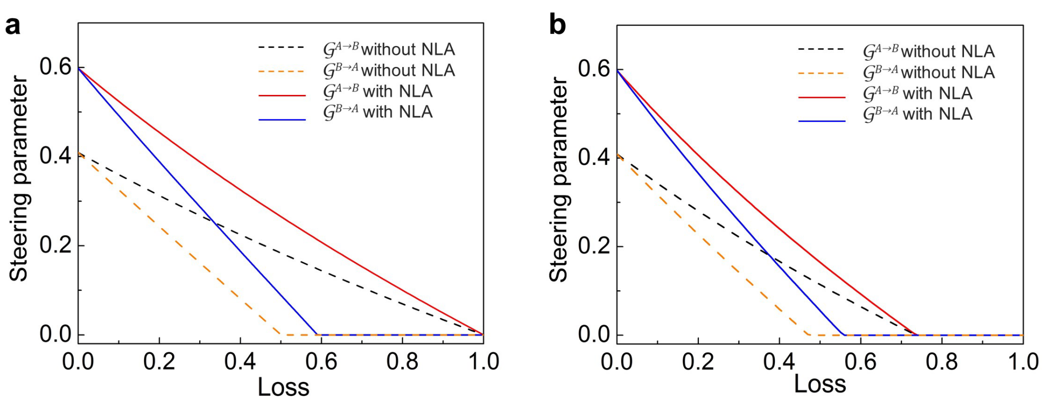
The optimal cutoff $|\beta_c|$ is selected to ensure high acceptance rate and high fidelity of the emulated NLA at the same time. However, we find that the post-selection rates is inversely proportional to cutoff $|\beta_c|$, and the fidelity of truncated filter with the ideal NLA is proportional to $|\beta_c|$. Here, we select the optimal $|\beta_c|$ according to the numerical simulation, in which the initial state is the same with that in the experiment. We choose the smallest $|\beta_c|$ to ensure the high fidelity of measurement-based NLA and ideal implementation $g^{\hat{n}}$ with two conditions: 1. the accepted data



Supplementary Figure 1 The skewness and kurtosis. a The skewness of the post-selected ensembles as a function of the g for different losses. **b** The kurtosis of the post-selected ensembles as a function of the g for different losses.

satisfy Gaussian distribution (skewness approaches to 0 and kurtosis approaches to 3); 2. the EPR steering calculated from the accepted data is close to that of the state after the ideal implementation $g^{\hat{n}}$. The optimal cutoff $|\beta_c|$ for different losses and g are given in Supplementary Table 1. With the increase of losses, the optimal cutoff value becomes smaller.

Supplementary Note 3: Distillation of EPR steering for a pure two-mode squeezed state



Supplementary Figure 2 The distillation of Gaussian EPR steering for a pure TMSS. The dependence of Gaussian EPR steering on loss for a pure two-mode squeezed state in a lossy channel **a** and in a noisy channel **b** with and with out the NLA implemented based on Bob's measurement results. The black and yellow dashed lines are the steerability of $\mathcal{G}^{A\to B}$ and $\mathcal{G}^{B\to A}$ without NLA, respectively. The red and blue solid lines are the steerability of $\mathcal{G}^{A\to B}$ and $\mathcal{G}^{B\to A}$ with NLA, respectively.

The purity of a two-mode squeezed state can be described by [3]

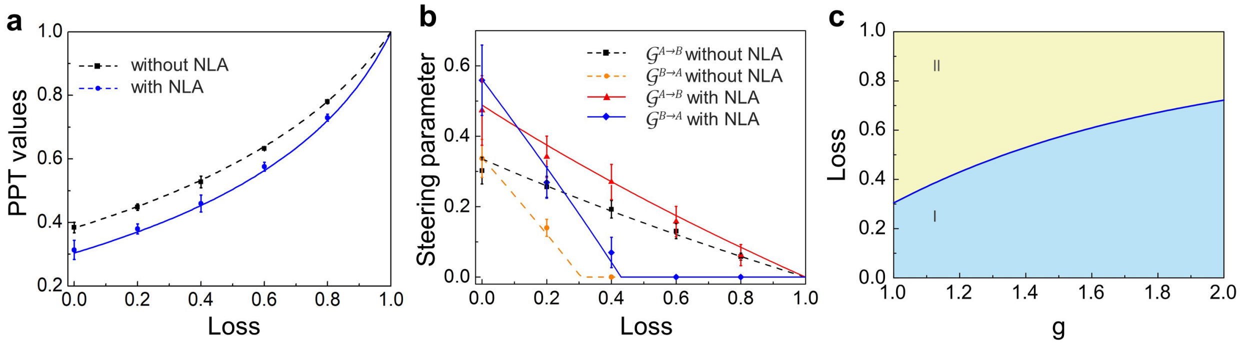
$$\mathcal{P} = \frac{1}{\sqrt{\det(\sigma_{AB})}}, \tag{11}$$

where σ_{AB} is the covariance matrix of the two-mode squeezed state. For a two-mode squeezed state with −4.2 dB squeezing and 4.2 dB antisqueezing, we have purity $\mathcal{P}_p = 1$. The theoretical results for the distillation of EPR steering with a pure two-mode squeezed state are shown in Supplementary Figure 2. After measurement-based NLA based on Bob's measurement results with gain $g = 1.2$ is implemented, the steerability for both directions are enhanced and the tolerance of $\mathcal{G}^{B\to A}$ on loss is extended in both lossy and noisy channels. Please note that the steerability from Bob to Alice $\mathcal{G}^{B\to A}$ is equal to $\mathcal{G}^{A\to B}$ when there is no loss and $\mathcal{G}^{B\to A}$ never surpasses $\mathcal{G}^{A\to B}$ for a pure two-mode squeezed state after NLA.

In our experiment, a two-mode squeezed state with −4.2 dB squeezing and 7.3 dB antisqueezing is prepared with purity $\mathcal{P} = 0.55$, which is not pure. The impurity of two-mode squeezed state in the experiment is caused by losses in the experimental system. The resson for steerabilities $\mathcal{G}^{B\to A}$ and $\mathcal{G}^{A\to B}$ are different after the NLA in Figures 2b, 3c and 3d in the main text is that the state becomes asymmetric i.e., the submatrices of Alice's and Bob's states are different after the NLA for the impure TMSS. While this is not happened after the NLA for a pure TMSS since the state remains symmetric, as shown in Supplementary Figure 2.

Supplementary Note 4: The distillation in a lossy channel based on Alice's measurement results

Here we show the theoretical results when the NLA based on Alice's measurement results is implemented in a lossy channel. As shown in Supplementary Figure 3a, the entanglement between Alice and Bob is enhanced after performing the NLA based on Alice's measurement results. For the distillation of EPR steering, both the steerablities of $\mathcal{G}^{A\to B}$ and $\mathcal{G}^{B\to A}$ are increased by the distillation, but the steerable regions are not changed which is different from the results with the NLA based Bob's measurement results [Figure 2b and 2c in the main text]. This is because the NLA based on Alice's measurement results can extent the steerable region of $\mathcal{G}^{A\to B}$ (as shown in the results of the NLA based on Alice's measurement results in a noisy channel [Figure 3d in the main text]), while the steerability of $\mathcal{G}^{A\to B}$ is robust against loss in a lossy channel.



Supplementary Figure 3 The distillation results in a lossy channel based on Alice's measurement results. The PPT values **a** and the Gaussian EPR steering **b** with and without the NLA based on Alice's measurement results, respectively. **c** The EPR steerable regions parameterized by loss and gain with the NLA based on Alice's measurement results in a lossy channel. The blue and yellow regions are the two-way steerable region (I) and one-way steerable region (II), respectively.

Supplementary Note 5: The theoretical calculation of secret key rate for 1sDI QKD

For a pure two-mode squeezed state, the variance of the amplitude and phase quadratures can be expressed by $V_X = V_P = (e^{2r} + e^{-2r})/2 = V$. When Alice performs homodyne detection and Bob performs heterodyne detection, there will be a 50% loss introduced by a 50 : 50 beam splitter in the heterodyne detection. So we have $V_{X_A} = V_{P_A} = V, V_{X_B} = V_{P_B} = VT + 1 - T$, $X_A X_B = -P_A P_B = \sqrt{T}(e^{-2r} - e^{2r})/2$, where $T = 0.5$.

In the case of reverse reconciliation, the secret key rate for this 1sDI QKD protocol is bounded by Eq. (4) in the main text, i.e., $K^{\blacktriangleleft} \geq \log_2 \frac{2}{e\sqrt{V_{P_B|P_A} V_{X_B|X_A}}}$, in which $V_{X_B|X_A} = V_{X_B} - \frac{\langle X_A X_B \rangle^2}{V_{X_A}} = V_{P_B|P_A}$. And $K^{\blacktriangleleft}$ is positive only for $V_{X_B|X_A} V_{P_B|P_A} \leqslant (\frac{2}{e})^2 \approx 0.55$, which implies a squeezing parameter of $r \geqslant 0.692$ (about -6 dB squeezing).

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