

Supplementary Information to "Experimental study of quantum uncertainty from lack of information"

Yuan-Yuan Zhao,^{1,2,3,*} Filip Rozpędek,^{4,5,6,*} Zhibo Hou,^{1,3} Kang-Da Wu,^{1,3} Guo-Yong Xiang,^{1,3,†} Chuan-Feng Li,^{1,3} and Guang-Can Guo^{1,3}

¹Key Laboratory of Quantum Information, University of Science and Technology of China, CAS, Hefei, 230026, China

²Research Center of Circuits and Systems, Peng Cheng Laboratory, Shenzhen 518055, Guangdong, China

³CAS Center for Excellence in Quantum Information and Quantum Physics,
University of Science and Technology of China, Hefei, 230026, China

⁴QuTech, Delft University of Technology, Lorentzweg 1, 2628 CJ Delft, The Netherlands

⁵Kavli Institute of Nanoscience, Delft University of Technology, Lorentzweg 1, 2628 CJ Delft, The Netherlands

⁶Pritzker School of Molecular Engineering, University of Chicago, Chicago, IL 60637, USA

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Supplementary Note 1. The guessing game

1. Quantifying lack of information

In the general guessing game considered, the state of the register R is given by

$$\rho_R = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1| + \gamma(|0\rangle\langle 1| + |1\rangle\langle 0|)), \quad (1)$$

where $\gamma \in [0, 1]$, and states $|0\rangle$ and $|1\rangle$ of R are associated with the measurement of S and T respectively. We note that we assume that any possible complex phase in ρ_R is also known to Bob and therefore γ can be restricted to a real-valued parameter as shown in [1].

To better understand the meaning of γ , which determines the coherence of ρ_R , and its relation to Bob's lack of information about the system P which purifies R , let us recall how we defined those systems. Here we will follow the definitions and interpretations presented in [1]. Specifically, even though Bob is given access to R , we emphasize that he does not have access to P in our guessing game. Hence, we can think of P as representing Bob's lack of information.

For example, for the classical game in which Bob sees the choice of the measurement basis as a random coin flip, $\rho_R = \mathbb{I}/2$. Then the purification of R is a maximally entangled state such as

$$|\xi_{RP}\rangle = \frac{1}{\sqrt{2}}(|0\rangle_R |0\rangle_P + |1\rangle_R |1\rangle_P). \quad (2)$$

If ρ_R is pure, then P is in a tensor product with R i.e.,

$$|\xi_{RP}\rangle = |\xi_R\rangle \otimes |\xi_P\rangle. \quad (3)$$

Since in the classical game both S and T are measured with equal probability, a natural extension when the purification of the coin is included in R is to set $|\xi_R\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Clearly the case when the initial state is maximally entangled, corresponds to $\rho_R = \mathbb{I}/2$ and so in this case P , to which Bob does not have access, holds the maximal amount of information useful to Bob. Of course if R is already pure then P does not contain any additional information that Bob could use.

Here we recall how [1] interpolates between these two extremes. Let C denote a classical coin. Then clearly C must be part of R . However, additionally R and P consist of many environmental subsystems $E_1, \dots, E_n$, each of which holds a small amount of information that will be useful to Bob. Then Bob's lack of information can be quantified by the number of the environment systems that are part of P instead of part of R .

That is $R = CE_1 \dots E_j$ and $P = E_{j+1} \dots E_n$. In [1] it is then shown that the continuous parameter $\gamma \in [0, 1]$ can be used to quantify the number of environmental subsystems included in R in the limit $n \rightarrow \infty$.

*These authors contributed equally.

†Electronic address: gyxiang@ustc.edu.cn

2. Optimal guessing probability

The full evolution of the quantum states on registers B and R is provided in [1]. Here we provide the key information that allows us to pose the optimization problem for finding the optimal guessing probability. After Alice's measurement, the quantum-classical state between the register R and the outcome X is expressed as

$$\rho_{RX}(\gamma, d, \rho_B) = \sum_x \tilde{\rho}_R^x(\gamma, d, \rho_B) \otimes |x\rangle\langle x|_x, \quad (4)$$

where

$$\tilde{\rho}_R^x(\gamma, d, \rho_B) = \frac{1}{2} \begin{bmatrix} \langle x | \rho_B | x \rangle & \gamma \langle x | \rho_B F^\dagger | x \rangle \\ \gamma \langle x | F \rho_B | x \rangle & \langle x | F \rho_B F^\dagger | x \rangle \end{bmatrix} \quad (5)$$

is the sub-normalized post-measurement state of the register R . Let us denote the corresponding normalized state as $\rho_R^x(\gamma, d, \rho_B) = \tilde{\rho}_R^x(\gamma, d, \rho_B)/p_x(d, \rho_B)$, where $p_x(d, \rho_B) = \text{Tr}(\tilde{\rho}_R^x(\gamma, d, \rho_B))$. Bob then tries to guess the outcome $X = x$ after determining which state $\rho_R^x(\gamma, d, \rho_B)$ he has received. Now the guessing problem becomes a state discrimination problem. Finally, the maximal guessing probability is achieved by optimizing the input state ρ_B and the corresponding measurement on R :

$$P_{\text{guess}}^{\max}(\gamma, d) = \max_{\rho_B} \max_{\{M_x\}} \sum_{x=0}^{d-1} p_x(d, \rho_B) \text{Tr}[M_x \rho_R^x(\gamma, d, \rho_B)]. \quad (6)$$

For $d = 2$, Helstrom has found the optimal measurements and the corresponding maximum probability of correctly distinguishing between two quantum states analytically [2]. His result makes it possible to easily find the optimal input state of Bob and hence to analytically calculate $P_{\text{guess}}^{\max}$ as shown in [1]. For $d > 2$ no analytical solution to the optimisation problem (6) is known due to its non-convex nature. Therefore for higher-dimensional games that involve distinguishing more than two states we use numerical techniques described in [1] that unfortunately cannot guarantee the global optimality of the found solution. Nevertheless, an analytical argument described in [1] shows that $P_{\text{guess}}^{\max}(\gamma, d > 2) < 1$.

Supplementary Note 2. Device settings for the implementation of the guessing game

In this section we provide the numerical values of the settings of the optical components in our experimental setup. All the components are referred to according to their labels in FIG. 3 of the main text.

1. Settings for the $d = 2$ game

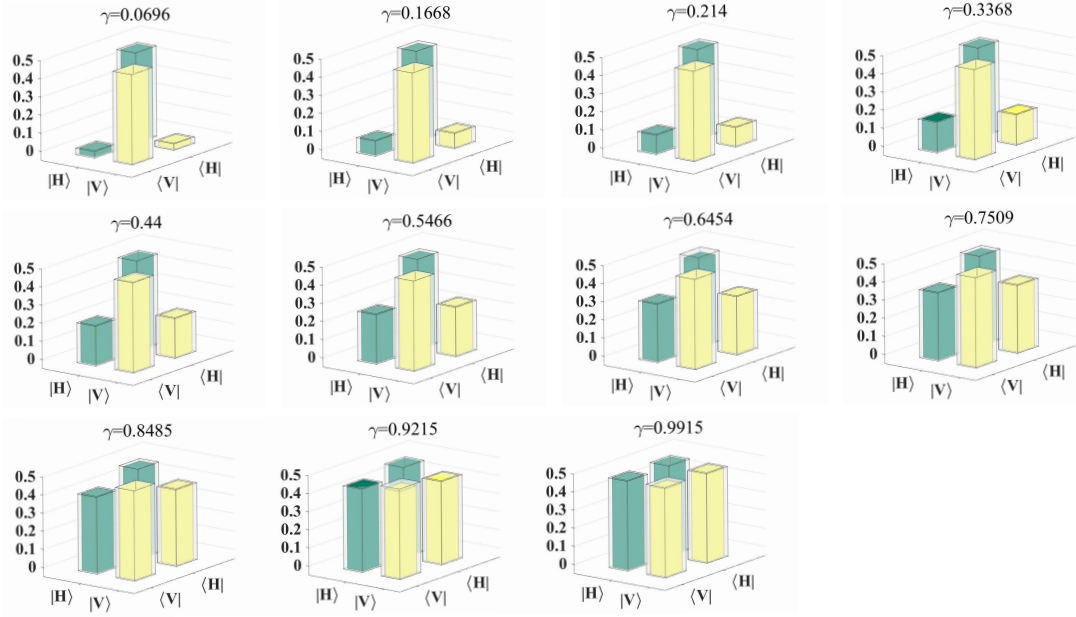
In the $d = 2$ guessing game, the wave plate H1 is rotated by θ_1 to prepare the optimal input state $|\psi\rangle_B$ in basis $|H\rangle$ and $|V\rangle$. The HBD is used to encode the polarization state into the path DoF by displacing the V component into path-0, and the H component into path-1 with a 4-mm lateral displacement. Then H2 unifies the polarization of the photon in different paths, and the system state becomes

$$|\psi\rangle_B = \cos 2\theta_1 |0\rangle - \sin 2\theta_1 |1\rangle. \quad (7)$$

Here θ_1 is set to 11.3° to prepare the optimal input state.

Then, 22.5° oriented half-wave plate H3 prepares the state of the register R into the state $1/\sqrt{2}(|H\rangle + |V\rangle)$. After that the polarization of the photon is coupled by QP to its frequency distribution realizing a dephasing noise to vary γ in $\rho_R(\gamma)$. Subsequently, the first VBD in FIG. 3(b) directs the H photon to the upper layer and V photon to the lower layer to prepare the control state in the basis $|u\rangle$ and $|l\rangle$. To analyze the specific form of the experimentally generated state ρ_R^{exp} , the optical axis of the H1 is horizontally placed to make all the photons pass through path "0". Then a standard quantum state tomography process is performed with a QWP and an HWP inserted before VBD, which behaves as a PBS now, to implement the three Pauli measurements. After the VBD, the photons are reflected out of the setup by a temporarily placed mirror and detected by the single-photon detectors. The detailed form of the tomographic state ρ_R^{exp} can be found in Supplementary Figure. 1.

For each experimentally generated state ρ_R^{exp} , we calculate its fidelity with the state $\rho_R(\gamma)$ given in Eq. (1) for every $\gamma \in [0, 1]$ (with the step length 10^{-4}). Here the fidelity between quantum states ρ and σ is given by $F = \text{Tr}(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}})$.



Supplementary Figure 1. Quantum state tomography for the experimentally generated states ρ_R^{exp} . The solid bars show the real parts of the tomographic states $\rho_R^{\text{exp}}(\gamma)$, whereas the transparent bars correspond to the theoretical values.

We choose the γ of the state $\rho_R(\gamma)$ which gives the highest fidelity and assign this value of γ to the experimental state ρ_R^{exp} . In our experiment, the values of γ and the corresponding thicknesses of the QP are given in Supplementary Table 1. For every state ρ_R^{exp} , the corresponding fidelity is higher than 0.9995.

After the preparation of the states in registers B and R a 22.5° rotated half-wave plate H6 and two HBDs before and after it are used to implement the Hadamard operation, just as the HBD-HWP-HBD structure shown in Supplementary Figure 2. Other wave plates H4, H5, and H7 are 45° rotated altering the polarization of the photon to make the corresponding beams combine coherently in the right places.

At last, the second VBD converts the path DoF of the two layers to the polarization DoF and a QWP (Q1), a HWP (H8) and a PBS are used to perform a measurement on the register R that aims to distinguish the quantum states ρ_R^x in order for Bob to guess Alice's measurement outcome X . This is a projective measurement with POVM elements $\{|M_0\rangle\langle M_0|, |M_1\rangle\langle M_1|\}$ such that

$$\begin{aligned} |M_0\rangle &= (\sin \alpha \cos \beta + i \cos \alpha \sin \beta) |0\rangle + (\cos \alpha \cos \beta - i \sin \alpha \sin \beta) |0\rangle, \\ |M_1\rangle &= (i \cos \alpha \cos \beta - \sin \alpha \sin \beta) |0\rangle - (\cos \alpha \sin \beta + i \sin \alpha \cos \beta) |1\rangle, \end{aligned} \quad (8)$$

where $\alpha = \theta_q$ and $\beta = \theta_q - 2\theta_h$, and θ_q and θ_h are the rotated angles for QWP (Q1) and HWP (H8), respectively. We note that the measurement on the system B takes place simultaneously with the measurement on R and corresponds to the measurement of the path degree of freedom of the photon as shown in FIG. 3 of the main text. The settings of Q1 and H8 together with the corresponding guessing probabilities are also shown in Supplementary Table 1.

2. Settings for the $d = 3$ guessing game

For the $d = 3$ guessing game, the input state ρ_B is prepared by rotating the wave plates H1 and H2, and the phases between different path modes are generated by slightly tuning the first two HBDs. For our chosen relative phases the input state can be written as:

$$|\psi\rangle_B = e^{1.41i} \cos 2\theta_1 \cos 2\theta_2 |0\rangle - e^{1.66i} \cos 2\theta_1 \sin 2\theta_2 |1\rangle + \sin 2\theta_1 |2\rangle, \quad (9)$$

where θ_1 and θ_2 are the rotated angles for wave plates H1 and H2.

In the best known strategy, θ_1 and θ_2 are set to be 26.6° and 5.9° , respectively. Moreover, we also test other input states around the optimal one, and the detailed settings of θ_1 and θ_2 are shown in FIG. 5 of the main text and Supplementary Table 2. In the $d = 3$ guessing game, the corresponding optimal measurements used to distinguish

γ	0.9918	0.9221	0.8493	0.7509	0.6458	0.5466	0.4396	0.3369	0.2138	0.1662	0.0686
n	0	58 λ	97 λ	125 λ	154 λ	183 λ	209 λ	237 λ	266 λ	290 λ	305 λ
Q1($^\circ$)	-22.4	-21.3	-20.2	-18.5	-16.4	-14.3	-11.9	-9.3	-6.0	-4.7	-2.0
H8($^\circ$)	33.8	34.3	34.9	35.8	36.8	37.8	39.1	40.3	42.0	42.6	44.0
$P_{\text{guess}}^{\text{max}}$	0.9980	0.9809	0.9639	0.9421	0.9209	0.9029	0.8862	0.8731	0.8615	0.8584	0.8544
$P_{\text{guess}}^{\text{exp}}$	0.9953	0.9776	0.9550	0.9301	0.9079	0.8891	0.8844	0.8702	0.8618	0.8610	0.8531
	± 0.0003	± 0.0007	± 0.001	± 0.0012	± 0.0015	± 0.0016	± 0.0016	± 0.0017	± 0.0016	± 0.0017	± 0.0017

Supplementary Table 1. Settings and results for $d = 2$ guessing game. The parameter n gives the corresponding thickness of the quartz plate used to generate state $\rho_R(\gamma)$, with $\lambda = 808$ nm. The settings of the wave plates Q1 and H8, which are used to implement Bob's measurement (see FIG. 3(a) of the main text), are given in the middle lines. The last two lines $P_{\text{guess}}^{\text{max}}$ and $P_{\text{guess}}^{\text{exp}}$ give the optimal guessing probability predicted theoretically and the values obtained experimentally.

state	1	2	3	4	5	6	7	8
H1($^\circ$)	22.6	24.6	26.6	28.6	30.6	26.6	26.6	26.6
H2($^\circ$)	5.9	5.9	5.9	5.9	5.9	1.9	9.9	17.9
Q1($^\circ$)	55.0	50.4	45.0	-50.6	-56.0	-47.0	46.2	46.1
H12($^\circ$)	12.0	9.0	6.0	36.3	33.4	38.0	6.5	6.5
P_{guess}	0.9669	0.9731	0.9753	0.9731	0.9664	0.9701	0.9702	0.9326
$P_{\text{guess}}^{\text{exp}}$	0.9521	0.9466	0.9611	0.9628	0.9455	0.9419	0.9282	0.9281
	± 0.0011	± 0.0011	± 0.001	± 0.0009	± 0.0011	± 0.0012	± 0.0012	± 0.0013

Supplementary Table 2. Settings and results for $d = 3$ guessing game. Wave plates H1 and H2 are used to prepare the input states, and Q1 and H12 perform the corresponding optimal measurements, see the red and purple regions in FIG. 3(b) of the main text. For each strategy, the theoretically predicted results are given as P_{guess} and the corresponding experimental results as $P_{\text{guess}}^{\text{exp}}$.

states ρ_R^x are performed using wave plates Q1 and H12, whose angles, together with the corresponding guessing probabilities, are also given in Supplementary Table 2. The relation between the measurement basis and the angles of wave plates can be found in Eq. (8). Wave plates H5, H6, H8, and H11 are 45° rotated to regulate the directions of the beams to make the photons combine coherently in the right places. The role of the remaining wave plates is discussed in Supplementary Note 3 in relation to the implementation of the three-dimensional Fourier gate.

Supplementary Note 3. Implementation of the Fourier transformation operation

In the $d = 3$ guessing game, the standard basis states $|k\rangle$ and the Fourier basis states

$$|w_k\rangle = U^\dagger |k\rangle, \quad (10)$$

where $k = 0, 1, 2$, constitute mutually unbiased bases and the Fourier transformation matrix is given by

$$U = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w \end{bmatrix}, \quad (11)$$

with $w = e^{2\pi i/3}$.

Here we will show how to experimentally realize the transformation U . The method we use comes from Ref. [3], which gives a universal algorithm to decompose such a multi-mode transformation matrix into a set of transformations $T_{m,n}$ between two modes m, n . Specifically, U will be re-written as a sequentially ordered $T_{m,n}$, $U = D \prod_{(m,n) \in S} T_{m,n}$, where S defines the order and D applied at the end adds an appropriate phase shift in each output mode. In the experiment, $T_{m,n}$ denotes a lossless variable beam splitter taking input modes m and n , with the reflectivity $\cos \theta$ and phase shift ϕ at input m , where $\theta \in [0, \pi/2]$, $\phi \in [0, 2\pi]$:

$$T_{m,n} = \begin{bmatrix} e^{i\phi} \cos \theta & -\sin \theta \\ e^{i\phi} \sin \theta & \cos \theta \end{bmatrix}. \quad (12)$$

Here we have omitted the nonfunctional elements of $T_{m,n}$. This decomposition method is based on the work of Reck *et al.* [4], and robust to the optical losses.

For our three mode transformation matrix U , we obtain the following expression according to the decomposition procedure in [3]:

$$D = T_{1,2} T_{0,1} U T_{0,1}^{-1}, \quad (13)$$

which can be rewritten as $U = T_{0,1}^{-1} T_{1,2}^{-1} D T_{0,1}$. For any matrix $T_{m,n}^{-1}$, one can find a matrix $A_{m,n}$ and a matrix D' so that $T_{m,n}^{-1} D = D' A_{m,n}$, then

$$U = D' A_{0,1} A_{1,2} T_{0,1}, \quad (14)$$

where for our U

$$\begin{aligned} A_{0,1} &= \begin{bmatrix} e^{-\frac{5i\pi}{6}} \cos(\frac{\pi}{4}) & -\sin(\frac{\pi}{4}) & 0 \\ e^{-\frac{5i\pi}{6}} \sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ A_{1,2} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{\frac{2i\pi}{3}} \cos(\frac{54.74\pi}{180}) & -\sin(\frac{54.74\pi}{180}) \\ 0 & e^{\frac{2i\pi}{3}} \sin(\frac{54.74\pi}{180}) & \cos(\frac{54.74\pi}{180}) \end{bmatrix}, \\ T_{0,1} &= \begin{bmatrix} e^{\frac{2i\pi}{3}} \cos(\frac{\pi}{4}) & -\sin(\frac{\pi}{4}) & 0 \\ e^{\frac{2i\pi}{3}} \sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ D' &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{\frac{i\pi}{3}} & 0 \\ 0 & 0 & e^{\frac{2i\pi}{3}} \end{bmatrix}. \end{aligned} \quad (15)$$

In the following, we will show how to realize the variable beam splitter in our experiment. As the bottom part of Supplementary Figure 2 shows, an HBD-HWP-HBD structure is adopted to implement the $T_{m,n}$ ($A_{m,n}$). Firstly, an HBD maps the spacial path modes $|m\rangle$ and $|n\rangle$ in the input port into the polarization basis as follows: $|m\rangle \rightarrow |H\rangle$ and $|n\rangle \rightarrow |V\rangle$. Then the second HBD maps the polarization basis into the spacial path modes $|m\rangle$ and $|n\rangle$ again: $|V\rangle \rightarrow |m\rangle$ and $|H\rangle \rightarrow |n\rangle$. Hence, in the basis $\{|H\rangle, |V\rangle\}$, matrix $T_{m,n}$ ($A_{m,n}$) is represented as:

$$T_{m,n}^{\text{pol}} (A_{m,n}^{\text{pol}}) = \begin{bmatrix} e^{i\phi} \sin \theta & \cos \theta \\ e^{i\phi} \cos \theta & -\sin \theta \end{bmatrix}, \quad (16)$$

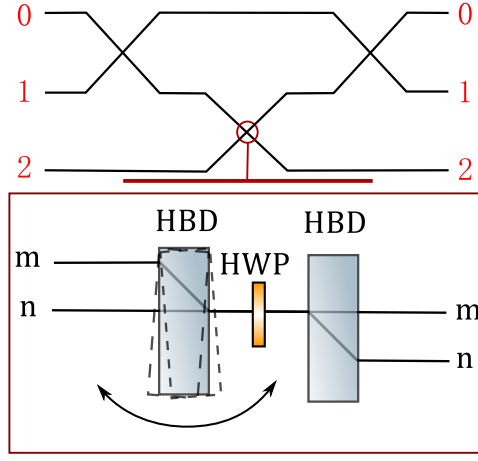
where the superscript "pol" has been used to indicate the operation performed after conversion into the polarization encoding. We now rewrite Eq. (16) such that

$$T_{m,n}^{\text{pol}} (A_{m,n}^{\text{pol}}) = \begin{bmatrix} \sin \theta & \cos \theta \\ \cos \theta & -\sin \theta \end{bmatrix} \begin{bmatrix} e^{i\phi} & 0 \\ 0 & 1 \end{bmatrix}. \quad (17)$$

Note that the first matrix:

$$R^{\text{pol}} = \begin{bmatrix} \sin \theta & \cos \theta \\ \cos \theta & -\sin \theta \end{bmatrix} \quad (18)$$

in the polarization basis can be implemented by a $(\pi/4 - \theta/2)$ rotated HWP. The full $T_{m,n}$ ($A_{m,n}$) is realized by additionally applying phase shift by angle ϕ to the $|H\rangle$ basis state, which is realized by slightly tuning the first



Supplementary Figure 2. The structure diagram of the implementation of the Fourier transformation matrix. In the top part, the lines represent the optical mode, and the crossing between two lines represents a variable beam splitter with a phase shift in one mode. In our experiment, we use two HBDs and an HWP, as shown in the bottom part of the figure, to realize the variable beam splitter, i.e., the crossing in the red circle, where the phase shift is added by slightly tuning the HBD.

HBD. In our experimental setup, the angles of middle-placed wave plates H7, H9 and H10 are 22.5° , 17.6° and 22.5° respectively.

The action of the matrix D' does not affect Alice's measurement outcomes as her measurement is in the eigenbasis of D' . However, it will contribute phase shifts to the post-measurement states of the register R : ρ_R^x . For $x = 0, 1, 2$, the phase shifts between the upper layer and lower layer are 0 , $e^{\frac{i\pi}{3}}$ and $e^{\frac{2i\pi}{3}}$, respectively. In our experimental setup, these phase shifts are added after the second VBD, where the upper layer and lower layer are translated to the polarization modes V and H respectively. Here, an individual wave plate with the rotated angle 0° is inserted in the path x to add the corresponding phase shift, which is not shown in the setup in FIG. 3 of the main text.

We prepare the Fourier basis states $|w_j\rangle$ with $(j = 0, 1, 2)$ defined in Eq. (10) to probe the quality of the operation U . For the perfect gate the photon entering in state $|w_j\rangle$ should be detected in the output mode j . The probabilities $P_{ij}^{\text{F,exp}}$ experimentally observed by obtaining a click in mode i for input state w_j are given in Supplementary Table 3, from which we can see that the average probability of detecting the Fourier basis state in the correct mode is 0.9771 ± 0.0009 , which shows the high quality of our Fourier gate implementation.

Probe state	$ w_0\rangle$	$ w_1\rangle$	$ w_2\rangle$
Output mode 0	0.9722 ± 0.001	0.0176 ± 0.0009	0.0103 ± 0.0007
Output mode 1	0.0183 ± 0.0009	0.9742 ± 0.001	0.0075 ± 0.0006
Output mode 2	0.0095 ± 0.0006	0.0055 ± 0.0005	0.9851 ± 0.0008

Supplementary Table 3. The normalized probabilities $P_{ij}^{\text{F,exp}}$ of the photon detected in mode i for the probe state $|w_j\rangle$, where $i, j = 0, 1, 2$. For each state, we collect about 20,000 photons in total.

Supplementary Note 4. Error analysis for the $d = 2$ guessing game

For the $d = 2$ guessing game, Bob should be able to perfectly guess Alice's measurement result X if he has access to all the quantum information about Alice's measurement basis choice, i.e., the optimal guessing probability is $P_{\text{guess}}^{\text{max}}(\gamma = 1, d = 2) = 1$. However, in our experiment we observe the highest value of $P_{\text{guess}}^{\text{exp}}(\gamma, d = 2) = 0.9953$. In the following, we provide a short numerical justification why the experimentally observed $P_{\text{guess}}^{\text{exp}}(\gamma, d = 2)$ cannot reach 1.

For that purpose we need to recall the two main sources of error in our experiment. The first one relates to the imperfections in the preparation of the state ρ_R . Specifically, using a QWP, an HWP and a PBS we perform a polarization analysis of the control state ρ_R and estimate the highest experimentally achievable value of the coherence

parameter to be $\gamma = 0.9918$.

The second one relates to the imperfections of the interferometer. The VBD transforms the photon's polarization degree of freedom to spacial modes $|u\rangle$ and $|l\rangle$. Ideally, after the photon undergoes the controlled Hadamard transformation $C_U = |0\rangle\langle 0| \otimes \mathbb{I}_B + |1\rangle\langle 1| \otimes H_B$, post-selecting on the measurement outcomes on the system B , we should obtain the following sub-normalized post-measurement states of the register R :

$$\tilde{\rho}_R^x = \begin{bmatrix} [\rho_R]_{00} \langle x | \rho_B | x \rangle & [\rho_R]_{01} \langle x | \rho_B H^\dagger | x \rangle \\ [\rho_R]_{10} \langle x | H \rho_B | x \rangle & [\rho_R]_{11} \langle x | H \rho_B H^\dagger | x \rangle \end{bmatrix}, \quad (19)$$

where $[\rho_R]_{ij}$ represents the matrix elements of the initial state of the register R in the basis $|i\rangle\langle j|$. In our experiment of the $d = 2$ game, the visibility of the interferometer composed of the two VBDs stays about $v = 0.99$, which introduces a dephasing noise for the post-measurement state, such that:

$$\rho_{\text{out}} = \frac{1+v}{2} \rho_{\text{in}} + \frac{1-v}{2} \sigma_z \rho_{\text{in}} \sigma_z. \quad (20)$$

Therefore, the state $\tilde{\rho}_R^x$ becomes

$$\tilde{\rho}_R^{x,\text{deph}} = \begin{bmatrix} [\rho_R]_{00} \langle x | \rho_B | x \rangle & v [\rho_R]_{01} \langle x | \rho_B H^\dagger | x \rangle \\ v [\rho_R]_{10} \langle x | H \rho_B | x \rangle & [\rho_R]_{11} \langle x | H \rho_B H^\dagger | x \rangle \end{bmatrix}. \quad (21)$$

Finally, Bob performs a measurement to distinguish the two possible states $\tilde{\rho}_R^{x,\text{deph}}$. In our experiment the measurements performed and the prepared input state ρ_B are optimized for the ideal case, that is when the states to be distinguished are $\tilde{\rho}_R^x$ for the ideal initial state of R , namely

$$\rho_R(\gamma) = 1/2 \begin{bmatrix} 1 & \gamma \\ \gamma & 1 \end{bmatrix}. \quad (22)$$

For such ideal $d = 2$ game the optimal input state ρ_B for all γ is the state $|\psi\rangle_B \propto |0\rangle + |-\rangle$ and for $\gamma = 0.9918$, the optimal measurement $\{M_0, M_1\}$ is given by:

$$M_0 = \begin{bmatrix} 0.8550 & 0.3521 \\ 0.3521 & 0.1450 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 0.1450 & -0.3521 \\ -0.3521 & 0.8550 \end{bmatrix}. \quad (23)$$

This measurement is then applied to the actual state $\tilde{\rho}_R^{x,\text{deph}}$, where the actual initial state of the register R prepared in the experiment is:

$$\rho_R^{\text{exp}} = \begin{bmatrix} 0.5124 & 0.4955 - 0.0157i \\ 0.4955 + 0.0157i & 0.4876 \end{bmatrix}. \quad (24)$$

The predicted detection probability P_{ij} in output D_{ij} is then given by

$$P_{ij} = \text{Tr} \left(M_i \tilde{\rho}_R^{x=j,\text{deph}} \right), \quad (25)$$

where $i, j = 0, 1$. In this way, we calculate the probabilities in outputs D_{00} , D_{01} , D_{10} and D_{11} to be 0.5064, 0.0024, 0.0023 and 0.4889 respectively, and the estimated guessing probability $P_{\text{guess}}^{\text{est}} = P_{00} + P_{11} = 0.9953$ agrees with the experimentally obtained probability $P_{\text{guess}}^{\text{exp}} = 0.9953 \pm 0.0003$.

Moreover, by comparing the individual predicted outcomes with the values we obtained in the experiment shown in FIG. 4(a) of the main text, we can see that the probabilities in outputs D_{00} and D_{11} are consistent with the experimental values, and there is only a slight bias between the probabilities in outputs D_{01} and D_{10} . Therefore, this noise model works well and the errors for our $d = 2$ guessing game are mainly coming from two imperfections, namely the preparation of the register state R , and the imperfect interference between the two layers.

Supplementary Note 5. Error analysis for the $d = 3$ guessing game

For the $d = 3$ guessing game, we experimentally test the best known strategy and achieve a guessing probability $P_{\text{guess}}^{\text{exp}}(\gamma = 0.9918, d = 3) = 0.9611 \pm 0.001$, which is indicated as data "3" in FIG. 5 of the main text and Supplementary Table 2. In the following, we will justify this value by performing a detailed analysis of the experimental errors.

Besides the two main sources of error in the $d = 2$ guessing game, i.e., the state preparation error for ρ_R and the dephasing error between the layers $|u\rangle$ and $|l\rangle$, for the $d = 3$ case the error occurring in the Fourier transform also needs to be included. Therefore, let us firstly discuss the main factors limiting the performance of our Fourier gate.

Let us now return to the setup used to test the quality of the Fourier gate discussed in Supplementary note 3 and now depicted in Supplementary Figure 3. As described in Supplementary note 3, we implement the Fourier gate using a series of lossless variable beam splitters, which are denoted by the crossings between two modes in Supplementary Figure 3. The crossing A and C, B and D, and C and E constitute three M-Z (Mach-Zehnder) type interferometers, respectively. In our experiment of the $d = 3$ game, the typical visibilities of all the interferometers are higher than $v = 0.98$.

We consider here the noise model described in [5], where the imperfect visibility in the interferometer can be modeled through an additional fictitious mode F which carries information about other degrees of freedom than the photon path. For perfect interference, F is in the $|0\rangle_F$ state and is uncorrelated from the path information. However, imperfect interference can be seen as leakage of information into F , that is other degrees of freedom than photon path are then no longer the same for all the modes. It is the lack of access to F which results in the effective decoherence of the qudit encoded in the photon path.

As an example let us examine first the imperfect interference between modes 0 and 1, which occurs e.g. on the crossing C in Supplementary Figure 3. According to the dephasing model in [5], the mode mismatch between the interfering modes will lead to a correlated rotation of mode F . In other words, we can consider the input light modes traveling through fictitious beam splitters acting on mode F . These beam splitters split the input mode $|0\rangle_F$ into two orthogonal modes, where the transmitted part remains in state $|0\rangle_F$ while the reflected part is in the mode $|1\rangle_F$ for the signal being in mode $|0\rangle_B$ and it is in mode $|2\rangle_F$ for the signal being in mode $|1\rangle_B$. Therefore, the input state undergoes the following unitary transformation before the real beam splitter acting on system B:

$$U_{BF} = |0\rangle\langle 0|_B \otimes (\sqrt{v}|0\rangle\langle 0|_F + \sqrt{1-v}|1\rangle\langle 0|_F - \sqrt{1-v}|0\rangle\langle 1|_F + \sqrt{v}|1\rangle\langle 1|_F + |2\rangle\langle 2|_F) \\ + |1\rangle\langle 1|_B \otimes (\sqrt{v}|0\rangle\langle 0|_F + \sqrt{1-v}|2\rangle\langle 0|_F - \sqrt{1-v}|0\rangle\langle 2|_F + \sqrt{v}|2\rangle\langle 2|_F + |1\rangle\langle 1|_F) + |2\rangle\langle 2|_B \otimes \mathbb{I}_F, \quad (26)$$

where v is the measured interferometric visibility in our $d = 3$ guessing game. Since we do not have access to the register F , the state of the register B before the real beam splitter can be described as:

$$\rho_{\text{out}} = \text{Tr}_F[U_{BF}\rho_B \otimes |0\rangle\langle 0|_F U_{BF}^\dagger] = \sum_{i=0}^2 \langle i|_F U_{BF} |0\rangle_F \rho_B (\langle i|_F U_{BF} |0\rangle_F)^\dagger. \quad (27)$$

The resulting channel $\mathcal{K}_{0,1}$ describing the noise arising from the imperfect interference between modes 0 and 1 can be then written in the Kraus representation as follows:

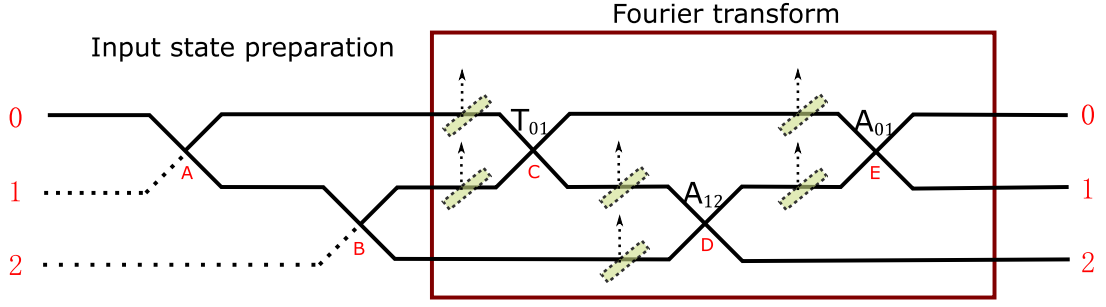
$$\mathcal{K}_{0,1}(\rho) = \sum_{i=0}^2 K_{i0,1} \rho K_{i0,1}^\dagger, \\ K_{00,1} = \langle 0|_F U_{BF} |0\rangle_F = \sqrt{v}(|0\rangle\langle 0| + |1\rangle\langle 1|)_B + |2\rangle\langle 2|_B, \\ K_{10,1} = \langle 1|_F U_{BF} |0\rangle_F = \sqrt{1-v}|0\rangle\langle 0|_B, \\ K_{20,1} = \langle 2|_F U_{BF} |0\rangle_F = \sqrt{1-v}|1\rangle\langle 1|_B. \quad (28)$$

Such a channel rescales the coherences between modes 0 and 1 by v and all the coherences with mode 2 by $\sqrt{v}$.

Similarly, we can model the noise occurring when interfering modes 1 and 2 by the analogous channel $\mathcal{K}_{1,2}$. Hence, for the Fourier gate we implemented, the probability to detect a photon in output i when inputting state $|w_j\rangle$ is given by

$$P_{ij}^{\text{F,deph}} = \langle i| \mathcal{D}' \circ \mathcal{A}_{0,1} \circ \mathcal{K}_{0,1} \circ \mathcal{A}_{1,2} \circ \mathcal{K}_{1,2} \circ \mathcal{T}_{0,1} \circ \mathcal{K}_{0,1} (|w_j\rangle \langle w_j|) |i\rangle, \quad (29)$$

where $i, j = 0, 1, 2$, and we define the channels corresponding to the operations A , T and D' defined in Eq. (15) as: $\mathcal{A}(\rho) = A\rho A^\dagger$ and analogously for $\mathcal{T}$ and $\mathcal{D}'$. The obtained predicted values are shown in Supplementary Table 4.



Supplementary Figure 3. Illustration of the scenario used to estimate the quality of the Fourier transform. A horizontal line represents a spacial mode, and a crossing between two lines represents a variable beam splitter with a phase shift in one mode. In our noise model, the dephasing errors can be modeled by adding the fictitious beam splitters before the crossings.

Probe state	$ w_0\rangle$	$ w_1\rangle$	$ w_2\rangle$
Output mode 0	0.9742	0.0170	0.0089
Output mode 1	0.0170	0.9742	0.0089
Output mode 2	0.0089	0.0089	0.9823

Supplementary Table 4. Predicted probabilities $P_{ij}^{\text{F,deph}}$ of obtaining a click in mode i for each input state $|w_j\rangle$ according to our noise model.

By comparing Supplementary Table 3 with Supplementary Table 4 we see that the corresponding probability distributions agree well which verifies that our analytical model provides a good description of the noise processes occurring in our experimental implementation of the three-dimensional Fourier gate.

In the implementation of our game only one of the two layers undergoes the Fourier operation. Let us then consider the corresponding noise model which includes the register R and applies the noise to the state of register B depending on the state of register R . That is the channel $\mathcal{K}$ acts on part of the system B correlated with the state $|1\rangle$ of the register R . The noise is then generated by the following unitary transformation acting on the extended space including register F :

$$U_{RBF} = |0\rangle\langle 0|_R \otimes \mathbb{I}_B \otimes \mathbb{I}_F + |1\rangle\langle 1|_R \otimes U_{BF}. \quad (30)$$

We can now calculate the Kraus operators of the channel $\mathcal{N}$ on the larger space RB to get:

$$\begin{aligned} N_0 &= \langle 0|_F U_{RBF} |0\rangle_F = |0\rangle\langle 0|_R \otimes \mathbb{I}_B + |1\rangle\langle 1|_R \otimes K_{0,B}, \\ N_1 &= \langle 1|_F U_{RBF} |0\rangle_F = |1\rangle\langle 1|_R \otimes K_{1,B}, \\ N_2 &= \langle 2|_F U_{RBF} |0\rangle_F = |1\rangle\langle 1|_R \otimes K_{2,B}. \end{aligned} \quad (31)$$

Note that we have omitted here the second level subscripts because the relation between Kraus operators $\{N_0, N_1, N_2\}$ and $\{K_0, K_1, K_2\}$ assumes the same form independently of which modes interfere.

Let us now consider a simple scenario in which we start with a product state $\rho_R \otimes \rho_B$ with ρ_R given in Eq. (1). We then apply a single round of the channel $\mathcal{N}$ followed by the measurement of the system B . The sub-normalized state on R conditioned on the outcome x would then be:

$$\tilde{\rho}_R^x = \frac{1}{2} \begin{bmatrix} \langle x|\rho_B|x\rangle & \gamma \langle x|\rho_B K_0^\dagger|x\rangle \\ \gamma \langle x|K_0\rho_B|x\rangle & \langle x|\mathcal{K}(\rho_B)|x\rangle \end{bmatrix}. \quad (32)$$

Including all the noisy operations in the lower layer and the imperfections in the preparation of the initial state ρ_R , the actual final state conditioned on the outcome x can be written as:

$$\tilde{\rho}_R^x = \begin{bmatrix} [\rho_R]_{00} \langle x|\rho_B|x\rangle & [\rho_R]_{01} \langle x|\rho_B (D' A_{0,1} K_{0,1} A_{1,2} K_{0,1,2} T_{0,1} K_{0,1})^\dagger |x\rangle \\ [\rho_R]_{10} \langle x|D' A_{0,1} K_{0,1} A_{1,2} K_{0,1,2} T_{0,1} K_{0,1} \rho_B |x\rangle & [\rho_R]_{11} \langle x|D' \circ \mathcal{A}_{0,1} \circ \mathcal{K}_{0,1} \circ \mathcal{A}_{1,2} \circ \mathcal{K}_{1,2} \circ \mathcal{T}_{0,1} \circ \mathcal{K}_{0,1}(\rho_B) |x\rangle \end{bmatrix}, \quad (33)$$

where $[\rho_R]_{ij}$ represents the matrix elements of the initial state of R , in our experiment given in Eq. (24), in the basis $|i\rangle\langle j|$. Furthermore, recall from Eq. (28) that $\mathcal{K}_{0,1}(\rho)$ denotes a channel that rescales the coherences between modes 0 and 1 by v and all the coherences with mode 2 by $\sqrt{v}$. The action of $\mathcal{K}_{1,2}(\rho)$ is analogous. Also recall that $K_{0,1}$ is the K_0 Kraus operator which is a diagonal matrix with $\sqrt{v}$ in the first two diagonal entries and 1 in the third one. The structure of $K_{0,2}$ is analogous.

Finally, we also need to include the dephasing noise between the two layers by rescaling the two off-diagonal entries by a factor v after the channel:

$$\tilde{\rho}_R^{x,\text{deph}} = \frac{1+v}{2}\tilde{\rho}_R^x + \frac{1-v}{2}\sigma_z\tilde{\rho}_R^x\sigma_z, \quad (34)$$

similarly to Eq. (20).

For data "3" in FIG. 5 of the main text, we test the best known strategy for the ideal register state $\rho_R(\gamma = 0.9918)$, and the optimal input state is given by $|\psi\rangle_B = a_1|0\rangle + a_2|1\rangle + a_3|2\rangle$ with the coefficients $a_1 = 0.0938 + 0.5786i$, $a_2 = 0.0109 - 0.1218i$ and $a_3 = 0.8009$. Now we can predict the detection probability P_{ij} in output D_{ij} as

$$P_{ij} = \text{Tr} \left(M_i \tilde{\rho}_R^{x=j,\text{deph}} \right), \quad (35)$$

where $i, j = 0, 1, 2$ and $\{M_0, M_1, M_2\}$ with

$$M_0 = \begin{bmatrix} 0.5003 & 0.2027 + 0.4571i \\ 0.2027 - 0.4571i & 0.4997 \end{bmatrix}, \quad M_1 = \mathbf{0}, \quad M_2 = \begin{bmatrix} 0.4997 & -0.2027 - 0.4571i \\ -0.2027 + 0.4571i & 0.5003 \end{bmatrix} \quad (36)$$

is the projective POVM measurement performed by Bob to guess Alice's outcome x .

Here we need to mention that for the optimal strategy in the $d = 3$ game, the projective measurements are sufficient, i.e., we only aim to distinguish the two dominant outcomes of the three outcomes on system B . In our scenario, the measurements M_0 and M_2 allow us to distinguish the states ρ_R^0 and ρ_R^2 , respectively. Then we can estimate the guessing probability $P_{\text{guess}} = P_{00} + P_{22} = 0.9554$ for $v = 0.98$. With the experimentally observed value $P_{\text{guess}}^{\text{exp}}(\gamma = 0.9918, d = 3) = 0.9611 \pm 0.001$ and taking into account the fact that the actual visibilities can be slightly higher than $v = 0.98$, we can see that the proposed model provides a good description of the noise processes occurring in the experiment. Specifically, due to a large number of interferometers for the $d = 3$ game, we see that the imperfect visibility has a significant impact on the observed guessing probability.

Moreover, when considering the other data points shown in FIG. 5 of the main text, we also need to include an additional error source. In our experiment, the phases of the interferometers are calibrated to prepare the input state for the data "3" to implement the best known strategy, and then other strategies are probed by varying the angles of H1 and H2. Since the surface of the wave plate is not absolutely smooth, the phase of the interferometer will undergo small change while rotating the wave plate. This has a significant effect, especially for H2, for which the photons in the two arms of the interferometer pass through two different places. Then the surface irregularity of the wave plate introduces a relative phase in the prepared state, see FIG. 3 of the main text. That is also the reason why data points "6" and "7" have larger deviations from the corresponding theoretical values. Therefore, besides the error sources we mentioned above, the error in the preparation of ρ_B should also be included for data points other than data "3".

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