

Supplementary Material: Demonstration of Optimal Non-Projective Measurements of Binary Coherent States with Photon Counting

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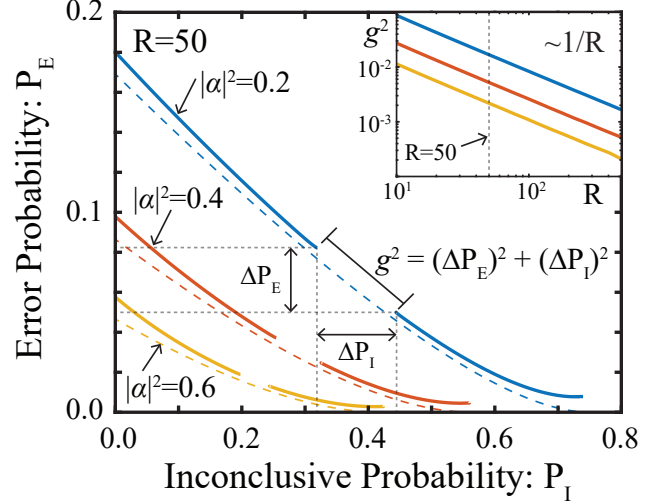
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SUPPLEMENTARY NOTE I: IMPACT OF EXPERIMENTAL IMPERFECTIONS

We investigate the effect of non-ideal experimental parameters on the performance of the optimal inconclusive measurement and the Dolinar receiver. A critical experimental parameter is the ratio $R = \max(|u(t)|^2)/|\alpha|^2$ between the maximum local oscillator (LO) energy $\max(|u(t)|^2)$ and the energy of the input state $|\alpha|^2$ that can be reliably implemented in the experiment. In principle, the optimal inconclusive measurement requires very large values of R (e.g. $R = \infty$ at $t = 0$ for equiprobable states). However, having a finite value of R that can be applied limits our ability to implement the desired optimal displacement $u(t)$ perfectly, which impacts the performance of the strategy. Supplementary Figure 1 shows the effect of a finite value of $R = 50$ on the optimal inconclusive measurement for $|\alpha|^2 = 0.2, 0.4$, and 0.6 . The dashed lines show the performance for when $R = \infty$, for reference. We obtain each data point by solving the evolution equations for P_I and P_E using the bounded optimal waveform $u'(t) = \min[u(t), \sqrt{R}|\alpha|\text{sign}(u(t))]$ with a finite R , and then plotting the obtained values of P_E and P_I . We observe that the main effect of finite R is a gap g in the achievable $P_I - P_E$ curve. We define this gap as $g^2 = (\Delta P_E)^2 + (\Delta P_I)^2$, where the dashed gray lines show ΔP_E and ΔP_I for $|\alpha|^2 = 0.2$. The inset shows g^2 as a function of R for each value of $|\alpha|^2$ on a log-log scale, showing a $1/R$ scaling.

Intrinsic imperfections such as non-ideal interference visibility and non-zero dark counts can produce unexpected photon detections, which cause errors in the discrimination process. Supplementary Figure 2(a-c) shows the performance of the optimal inconclusive measurement for $|\alpha|^2 = 0.2$ for different levels of experimental imperfections (solid colored lines) compared to the ideal case (dashed line). Non-ideal values of (a) the detection efficiency η , (b) the interference visibility ξ , and (c) the dark count rate ν all increase the attainable error probability compared to the ideal case. A reduction in η increases the error probability by approximately the same amount across the entire range of values of P_I . On the other hand, higher dark counts have a larger impact closer to the MESD ($P_I \approx 0$) and USD ($P_E \approx 0$) regimes. Supplementary Figure 2(d-f) shows the performance of the optimal MESD measurement, the Dolinar receiver, for different levels of experimental imperfections (solid colored lines) compared to the ideal case (dashed black

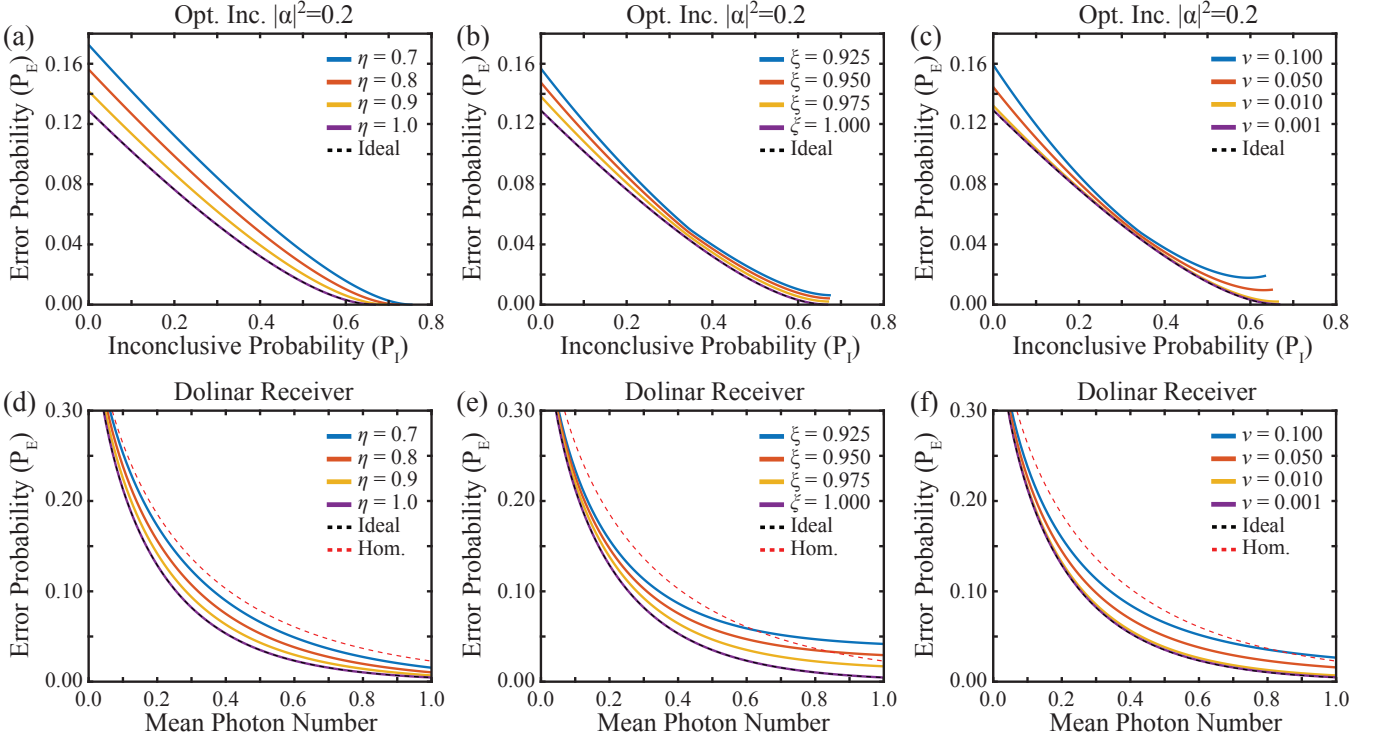


Supplementary Figure 1. Impact of a finite value of $R = \max(|u(t)|^2)/|\alpha|^2$ on the optimal inconclusive measurement. The blue, orange, and yellow solid lines show the simulated results for $|\alpha|^2 = 0.2, 0.4$, and 0.6 , respectively, for a strategy with $R = 50$, and the dashed lines show the results for $R = \infty$. A value of $R < \infty$ produces a gap $g^2 = (\Delta P_E)^2 + (\Delta P_I)^2$ in the $P_I - P_E$ curve. The dashed gray lines show the ranges ΔP_I and ΔP_E for $|\alpha|^2 = 0.2$ that cannot be accessed by the measurement. The inset shows g^2 as a function of R for each value of $|\alpha|^2$ on a log-log scale, showing a $1/R$ scaling.

line) and homodyne limit (dashed red line). We observe that for the Dolinar receiver, non-ideal interference visibility (e) has the largest detrimental effect, significantly increasing the error probability at larger values of $|\alpha|^2$.

SUPPLEMENTARY NOTE II: PHASE CORRECTION CALIBRATION

The LO amplitude modulator (AM) also incidentally modulates the LO phase depending on the depth of the amplitude modulation, which is related to the parameter R . The relationship between the applied amplitude modulation and the resulting extra phase modulation is typically linear for low modulation depths. However, the implementation of the optimal inconclusive measurement requires modulation as large as experimentally possible (in principle infinitely large at $t = 0$ for equal prior prob-



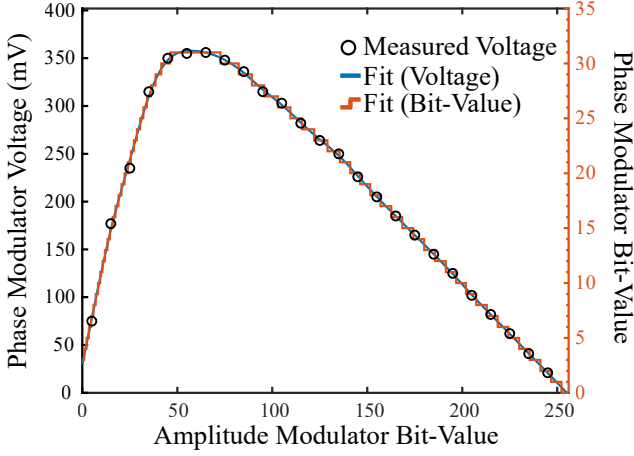
Supplementary Figure 2. Impact of imperfections on the optimal inconclusive measurement (top row) and the optimal MESD measurement, the Dolinar receiver (bottom row). Colored lines show the effect of different levels of detection efficiency η (a,d), interference visibility ξ (b,e), and dark counts ν (c,f) on the attainable performances of the corresponding measurements for $|\alpha|^2 = 0.2$.

abilities) to implement the necessary displacement waveform for a particular strategy. Thus, to correct the phase modulation caused by the AM we apply a phase correction to the phase modulator (PM) in the LO arm, such that any extra phase modulation from the AM will be exactly canceled out once the light passes through both modulators. This correction allows us to experimentally implement the displacement waveform which implements the desired optimal inconclusive measurement. A critical aspect of the strategy for the optimal inconclusive measurement is that the optimal waveform for the LO magnitude has a deterministic shape, i.e. takes on values which implement a predetermined displacement magnitude. Since the LO magnitude is predetermined, the necessary phase correction is predetermined as well. Thus, in our implementation of this strategy in the FPGA, the correction for the unwanted phase shift introduced by the amplitude modulator is included as an extra factor for the phase of the LO in the FPGA. This modification allows us to realize feedforward operations to the PM of the LO in real time for implementing the phase correction of this undesirable effect.

We calibrate this phase correction by measuring the voltage required to correct the phase shift from the AM as a function of the bit-value (proportional to voltage via the DAC) applied to the AM. To measure the necessary amount of phase correction, we use an interferomet-

ric measurement where the Signal and LO fields interfere, while the interferometer is stabilized. We apply two copies of a waveform for phase calibration to the PM for the Signal field back-to-back within a single light pulse. The phase calibration waveform implements the phase pattern $[0, \pi/2, 0, \pi, 0, 3\pi/2, 0]$, where each value has the same time duration [1]. During the first copy, the phase and amplitude modulation of the LO is turned off, and during the second copy we enable the AM of the LO with a particular bit-value.

We calibrate the phase shift induced by the AM of the LO by observing the intensity of the interference pattern after the Signal and LO interfere. Comparing the measured intensity for relative phases $\pi/2$ and $3\pi/2$ allows us to correct the phase shift induced by the AM by adjusting the PM of the LO to achieve the expected interference for these two phases. These two phases should correspond to the same intensity, but the phase error from the AM causes them to be uneven and we simply adjust the phase correction until they are balanced. We repeat this process to obtain the phase correction voltage for different values of amplitude modulation to obtain a phase correction calibration curve. This calibration curve corresponds to the extra voltage we must apply to the LO PM to cancel out the phase error from amplitude modulation in the measurement. During the experimental runs, this phase correction is simply added to the current LO



Supplementary Figure 3. Example correction curve for the phase error imparted by the LO AM. The points show the phase modulator voltage required to correct the incidental phase shift from amplitude modulation. At low modulation depth (large bit-value due to our configuration), the correction is relatively linear as expected. However at large modulation (small bit-value) the needed phase correction becomes highly non-linear. The blue line shows the data fit using a smoothing spline and the orange line shows the corresponding bit-value added to the phase of the LO to correct the phase error.

phase, which depends on the random photon detections in the measurement, and the strength of the correction depends on the value sent to the AM of the LO, which is coded in the FPGA as a look-up table.

Supplementary Figure 3 shows an example of the measured phase correction curve. The dots show the measured voltage applied to the PM required to correct the phase error from amplitude modulation. Different points on the x -axis (Amplitude Modulation Bit-Value) correspond to different levels of amplitude modulation depth which implement different displacement magnitudes. The blue line shows a fit to the measured points using a smoothing spline. The orange curve shows the bit-value that needs to be applied to the phase modulator in order to correct the phase error, which appears to be piece-wise constant due to the finite precision in digitization (8-bits) within the FPGA. For example, an amplitude modulation bit-value of 150 will require the FPGA to add a bit-value of 19 to the LO phase to cancel out any phase errors, and an amplitude modulation value of 50 requires a phase correction bit value of 31 be added to the LO phase. This correction ensures that there is no incidental phase shifts caused by amplitude modulation, allowing us to precisely implement the correct displacement waveform needed for a particular optimal inconclusive measurement.

SUPPLEMENTARY NOTE III: EXTENDING TO HIGH DIMENSIONAL INCONCLUSIVE MEASUREMENTS

We show how to construct inconclusive measurements in high dimensions using the combination of hypothesis testing followed by the binary optimal inconclusive measurement. We extend the strategy proposed in Section III to M-PSK coherent state alphabets by again dividing the single measurement into two stages. In the first stage, the receiver implements hypothesis tests until there are only two possible remaining states. There are on the order of 2^M possible detection histories for the hypothesis testing stage, and most of these outcomes will result in an inconclusive result, which yields $P_I^{(1)}$. The binary optimal inconclusive measurement in the second stage can then achieve arbitrarily small error probabilities P_E at the expense of increasing the total inconclusive probability P_I by a relatively small amount given by $P_I^{(2)}$. In contrast, heterodyne detection requires that P_I^{Het} asymptotically approaches $P_I^{\text{Het}} = 1$ in order to reach asymptotically small error probabilities P_E^{Het} (though $P_E^{\text{Het}} = 0$ only at $P_I^{\text{Het}} = 1$). Thus, examining the minimum possible inconclusive probability $P_I^{(1)}$ allows us to examine the scaling in the performance of the proposed hybrid approach as the number of states M increases, since lower error rates can always be achieved by a small increase in the total $P_I = P_I^{(1)} + P_I^{(2)}$ where $P_I < 1$ for $P_E = 0$.

The minimum inconclusive probability $P_I^{(1)}$ is given by the total probability that the hypothesis testing stage results in a detection history where there are three or more possible states remaining. A given state $s_j \in \{s_1 \dots s_M\}$ is eliminated as a possible state if the receiver detects a photon when testing for that state. The entire hypothesis testing stage consists of testing for each possible input state until either only two states remain, or all states have been tested. We denote the detection history of the hypothesis testing stage as $\mathbf{d}_i$ and the set of all detection histories which result in an inconclusive result as $\mathbf{D}_{\text{inc}}$. We then obtain the minimum inconclusive probability $P_I^{(1)}$ as:

$$P_I^{(1)} = \sum_{\mathbf{d}_i \in \mathbf{D}_{\text{inc}}} P(\mathbf{d}_i) \quad (1)$$

$$= \sum_{\mathbf{d}_i \in \mathbf{D}_{\text{inc}}} \sum_{j=1}^M P(\mathbf{d}_i | s^* = s_j) P(s^* = s_j) \quad (2)$$

$$= \sum_{\mathbf{d}_i \in \mathbf{D}_{\text{inc}}} \sum_{j=1}^M \prod_{k=1}^M \mathcal{L}(d_{i,k} | \hat{s} = s_k, s^* = s_j) \times \frac{1}{M} \quad (3)$$

where $P(\mathbf{d}_i)$ is the probability of obtaining detection history $\mathbf{d}_i$ from the hypothesis testing stage, $P(\mathbf{d}_i | s^* = s_j)$ is the probability of obtaining the detection history $\mathbf{d}_i$ given the actual input state s^* is s_j , and $P(s_j) = 1/M$ is the prior probability. Also, $\mathcal{L}(d_{i,k} | \hat{s} = s_k, s^* = s_j)$ is the

photon counting likelihood of detecting $d_{i,k}$ from a single hypothesis test given that the receiver tests the state $\hat{s} = s_k$ and the actual input state is $s^* = s_j$. We note that $\mathcal{L}(1|s_j, s_j) = 0$ and this enforces that some detection histories never actually occur. Given that $P_I^{(1)} \rightarrow 1$ as $M \rightarrow \infty$, we obtain the maximum *conclusive* probability as $P_{\text{conc}}^{\text{max}} = 1 - P_I^{(1)}$.

As an example, the minimum inconclusive probability $P_I^{(1)}$ for $M = 3$ as in Sec. III of the main text is given by the probability of detecting zero photons in each hypothesis test, i.e. obtaining $\mathbf{d}_0 = \{0, 0, 0\}$. The histories $\mathbf{d}_1 = \{0, 0, 1\}$, $\mathbf{d}_2 = \{0, 1, 0\}$, and $\mathbf{d}_4 = \{1, 0, 0\}$ result in performing the binary optimal inconclusive measurement on the remaining two states, and the other detection histories never occur. Thus, $\mathbf{D}_{\text{inc}} = \{\mathbf{d}_0\}$ and we obtain:

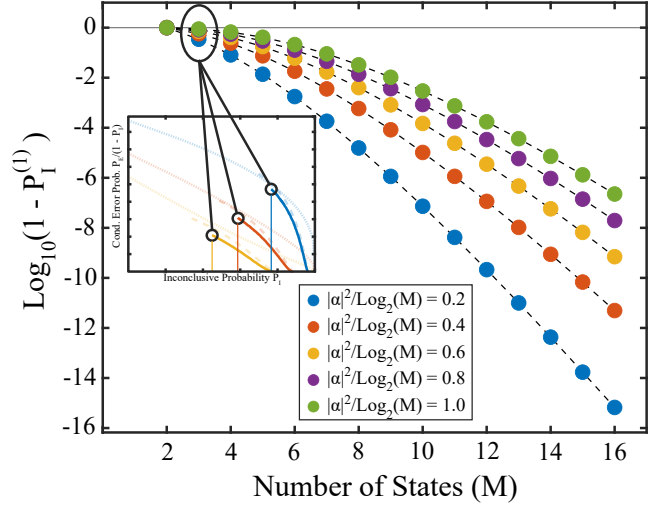
$$\begin{aligned} P_I^{(1)} &= P(\mathbf{d}_0) \\ &= \sum_{j=1}^3 \prod_{k=1}^3 \mathcal{L}(0|s_j, s_k)/3 \\ &= (1) \left(e^{-2 \frac{f_3 |\alpha|^2}{3}} [1 - \cos(\frac{2\pi}{3})] \right) \left(e^{-2 \frac{f_3 |\alpha|^2}{3}} [1 - \cos(\frac{4\pi}{3})] \right). \end{aligned} \quad (4)$$

Which yields $P_I^{(1)} \approx \{0.766, 0.587, 0.449\}$ for $|\alpha|^2 = \{0.2, 0.4, 0.6\}$ respectively with $f_3 = 2/3$ as in Sec. III of the main text.

Supplementary Figure 4 shows the scaling of the maximum conclusive probability $\log_{10}(P_{\text{conc}}^{\text{max}}) = \log_{10}(1 - P_I^{(1)})$ as the number of states M in the alphabet increases. The blue, orange, yellow, purple, and green points show the results for a mean photon number per bit of $|\alpha|^2/\log_2(M) = 0.2, 0.4, 0.6, 0.8$, and 1.0 . The dashed lines are to guide the eye. We note that here the fraction f of total energy $|\alpha|^2$ used to perform the hypothesis testing changes for each alphabet as $f_M = (M - 1)/M$ such that here $M = 3$ (inset) corresponds to the results shown in Supplementary Fig. (3) in the main text. We find that the quantity $\log_{10}(1 - P_I^{(1)})$ roughly scales proportionally to M^2 for all total mean photon numbers, suggesting that $P_{\text{conc}}^{\text{max}} \propto 10^{c_\alpha \times M^2}$ where the coefficient $c_\alpha < 0$ is specific to a particular value of $|\alpha|^2$. We note that this scaling may change with a different choice of f_M . Furthermore, for $M \geq 4$, a Bayesian approach to hypothesis testing may provide better performance as opposed to randomly choosing between the remaining untested states, such as in Refs. [1–3].

SUPPLEMENTARY NOTE IV: OPTIMAL INCONCLUSIVE MEASUREMENT SIMULATION DETAILS

We simulate the optimal inconclusive measurement using two methods, one based on evolving equations derived in Ref. [4] and the other based on Monte-Carlo sampling. Both methods allow for inclusion of experimental



Supplementary Figure 4. Scaling of the maximum conclusive probability for inconclusive measurements based on hypothesis testing for high dimensional M-PSK alphabets. Colored points show the maximum conclusive probability $P_{\text{conc}}^{\text{max}} = 1 - P_I^{(1)}$, where a larger $P_{\text{conc}}^{\text{max}}$ is better, for different numbers of states in the PSK alphabet. The value of $P_{\text{conc}}^{\text{max}}$ increases as the mean photon number per bit $|\alpha|^2/\log_2(M)$ increases and $\log_{10}(1 - P_I^{(1)})$ scales approximately proportional to M^2 .

imperfections such as detection efficiency η , interference visibility ξ , dark count rate ν , finite maximum displacement power R , and finite digitization precision from the digital-to-analog conversion within the experiment. A particular optimal inconclusive measurement is fixed by first setting the input energy $|\alpha|^2$, desired inconclusive probability P_I , and prior probabilities of the two states $\{p, 1 - p\}$. These parameters allow for calculation of the switching time $t_1 = t_1(|\alpha|^2, P_I, p)$ as well as the value of $v = v(|\alpha|^2, P_I, p)$, which determines the phase of the displacement field at the beginning of the second temporal mode [4].

Algorithm 1 shows the pseudo-code for the first simulation method where we directly obtain the probability of correct detection $P_C(t)$, error probability $P_E(t)$, and inconclusive probability $P_I(t)$ as a function of time elapsed over the course of the measurement. This method is based on evolving these probabilities in time using Eq. (9) and Eq. (25) from Ref. [4]. Lines 1 through 10 allow for obtaining the magnitude of the experimental displacement waveform including experimental imperfections. Line 8 enforces the maximum displacement power R as discussed above and Line 9-10 enforces the finite precision (8-bits for our implementation) when performing digital-to-analog conversion from the FPGA. Lines 14-21 evolve $P_C(t)$, $P_E(t)$, and $P_I(t)$ according to Eq. (9) of Ref. [4] to simulate the first temporal mode of the measurement. Lines 23-31 determine the value of the probabilities at the start of the second temporal mode based on $|\alpha|^2$, P_I , and p (See Ref. [4] for details). Lines

Algorithm 1 Optimal Inconclusive Simulation: Equations

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1: Fix:  $|\alpha|^2$ ,  $P_I$ , and  $p$   $\triangleright$  Energy, inconclusive probability, and prior
   probability of more likely input state
2: Fix:  $\eta$ ,  $\xi$ ,  $\nu$ ,  $R$   $\triangleright$  Experimental Imperfections
3:
4:  $t_1 \leftarrow t_1(|\alpha|^2, P_I, p)$   $\triangleright$  Eq. 13 in Ref. [4]
5:  $v \leftarrow v(|\alpha|^2, P_I, p)$   $\triangleright$  Eq. 18 in Ref. [4]
6:
7: Get  $|u(t)|$  from Eq. (1)
8:  $|u_{\text{exp}}(t)| = \min(\sqrt{R}, |u(t)|)$ 
9: Transform  $|u_{\text{exp}}(t)|$  to nearest bit-value
10: Transform  $|u_{\text{exp}}(t)|$  back to amplitude
11:
12: Initial:  $P_C(0) = p$ ,  $P_I(0) = 0$ ,  $t = 0$ 
13:  $P_E(0) = 1 - P_C(0) - P_I(0)$ 
14: while  $0 \leq t < t_1$  do  $\triangleright$  Evolve for first temporal mode
15:    $|\beta| \leftarrow |u_{\text{exp}}(t)|$ 
16:    $\langle \hat{n} \rangle_{\pm} \leftarrow \eta(|\alpha|^2 + |\beta|^2 \pm 2\xi|\beta||\alpha|) + \nu$ 
17:    $P_C(t + \delta t) \leftarrow P_C(t)(1 - \delta t \langle \hat{n} \rangle_-) + (1 - P_C(t))\delta t \langle \hat{n} \rangle_+$ 
18:    $P_E(t + \delta t) \leftarrow 1 - P_C(t + \delta t)$ 
19:    $P_I(t + \delta t) \leftarrow 1 - P_C(t + \delta t) - P_E(t + \delta t)$ 
20:    $t \leftarrow t + \delta t$ 
21: end while  $\triangleright$  End of first temporal mode
22:
23: if  $v \leq 0.5$  then
24:    $P_C(t_1) \leftarrow 0$ 
25:    $P_I(t_1) \leftarrow 1$ 
26:    $P_E(t_1) \leftarrow 1 - P_C(t_1) - P_I(t_1)$ 
27: else if  $v > 0.5$  then
28:    $P_C(t_1) \leftarrow P_C(t_1)$ 
29:    $P_I(t_1) \leftarrow P_I(t_1)$ 
30:    $P_E(t_1) \leftarrow 1 - P_C(t_1) - P_I(t_1)$ 
31: end if
32:
33:  $p' \leftarrow \frac{1}{2}(1 - \sqrt{1 - 4p(1-p)e^{-4t_1|\alpha|^2}})$   $\triangleright$  Helstrom Bound for
   states  $|\sqrt{t_1}\alpha\rangle$ 
34: while  $t_1 \leq t < 1$  do  $\triangleright$  Evolve for second temporal mode
35:    $|\beta| \leftarrow |u_{\text{exp}}(t)|$ 
36:    $\langle \hat{n} \rangle_{\pm} \leftarrow \eta(|\alpha|^2 + |\beta|^2 \pm 2\xi|\beta||\alpha|) + \nu$ 
37:    $P_C(t + \delta t) \leftarrow P_C(t)(1 - \delta t \langle \hat{n} \rangle_-) + (p' - P_C(t))\delta t \langle \hat{n} \rangle_+$ 
38:    $P_E(t + \delta t) \leftarrow P_E(t)(1 - \delta t \langle \hat{n} \rangle_+) + (1 - p' - P_E(t))\delta t \langle \hat{n} \rangle_-$ 
39:    $P_I(t + \delta t) \leftarrow 1 - P_C(t + \delta t) - P_E(t + \delta t)$ 
40:    $t \leftarrow t + \delta t$ 
41: end while  $\triangleright$  End of second temporal mode

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33-40 evolve the probabilities through the second temporal mode to give the final values of the error, correct, and inconclusive probabilities.

Algorithm 2 shows the pseudo-code for the second simulation method based on Monte-Carlo sampling. Instead of the probabilities, this method simulates the measurement directly, i.e. as it occurs in the lab based on random sampling from the photon counting distribution. Lines 1-5 fix the particular optimal inconclusive measurement as well as the switching time and value of v . Lines 7-50 repeats the simulation of a single measurement (indexed by i) M_{MC} number of times. During a single measurement, Lines 8-12 randomly choose the sign of the true input state $s_i = \pm 1$, corresponding to $|\pm \alpha\rangle$. Lines 14-17 obtain the displacement magnitude waveform including experimental imperfections. Lines 20-28 simulate the first temporal mode. Line 21 determines the displacement amplitude based on the current total number of photons detected N . Line 22 calculates the mean photon number $\langle \hat{n} \rangle$ of the displaced input state detected by the single photon detector including experimental imperfec-

tions. Line 23 samples a single detection from a Poisson distribution with mean $\langle \hat{n} \rangle \delta t$ where $\delta t = 1/1024$ is the duration of a single time step (i.e. we discretize the measurement into 1024 discrete time steps). Line 24 enforces on/off detection since for each time step the detection result can only be either vacuum or not-vacuum. Line 27 updates the provisional hypothesis for the input state conditioned on the total number of photons detected thus far. The first temporal mode continues until the time t_1 is reached. Lines 30-34 determine the displacement phase for the start of the second temporal mode.

Algorithm 2 Optimal Inconclusive Simulation: Monte-Carlo

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1: Fix:  $|\alpha|^2$ ,  $P_I$ , and  $p$   $\triangleright$  Energy, inconclusive probability, and prior
   probability of more likely input state
2: Fix:  $\eta$ ,  $\xi$ ,  $\nu$ ,  $R$   $\triangleright$  Experimental Imperfections
3:
4:  $t_1 \leftarrow t_1(|\alpha|^2, P_I, p)$   $\triangleright$  Eq. 13 in Ref. [4]
5:  $v \leftarrow v(|\alpha|^2, P_I, p)$   $\triangleright$  Eq. 18 in Ref. [4]
6: Fix:  $M_{\text{MC}}$   $\triangleright$  Number of Monte-Carlo samples
7: for  $i \leftarrow 1$  to  $M_{\text{MC}}$  do
8:   if  $\text{rand} > p$  then  $\triangleright$  Randomly choose actual input state
9:      $s_i = -1$ ,
10:   else
11:      $s_i = 1$ 
12:   end if
13:
14:   Get  $|u(t)|$  from Eq. (1)
15:    $|u_{\text{exp}}(t)| = \min(\sqrt{R}, |u(t)|)$ 
16:   Transform  $|u_{\text{exp}}(t)|$  to nearest bit-value
17:   Transform  $|u_{\text{exp}}(t)|$  back to amplitude
18:
19:   Initial:  $N = 0$ ,  $t = 0$ ,  $\hat{s}_i(0) = 1$   $\triangleright$  Assume  $p \geq 0.5$ 
20:   while  $0 \leq t < t_1$  do  $\triangleright$  Evolve for first temporal mode
21:      $\beta \leftarrow (-1)^N |u_{\text{exp}}(t)|$ 
22:      $\langle \hat{n} \rangle \leftarrow \eta(|\alpha|^2 + |\beta|^2 - 2\xi\beta|\alpha|s_i) + \nu$ 
23:      $d \sim \text{Poisson}(n|\langle \hat{n} \rangle \delta t)$   $\triangleright$  Vacuum/photon detection
24:     if  $d > 1$ , set  $d = 1$ 
25:      $N \leftarrow N + d$   $\triangleright$  Total number of detected photons
26:      $t \leftarrow t + \delta t$ 
27:      $\hat{s}_i(t) = (-1)^N$   $\triangleright$  Current hypothesis for input
28:   end while  $\triangleright$  End of first temporal mode
29:
30:   if  $v \leq 0.5$  then
31:      $N_0 = 1$ 
32:   else if  $v > 0.5$  then
33:      $N_0 = 0$ 
34:   end if
35:
36:    $N \leftarrow N_0$ 
37:   while  $t_1 \leq t < 1$  do  $\triangleright$  Evolve for second temporal mode
38:      $\beta \leftarrow \hat{s}_i(t_1) \times (-1)^N |u_{\text{exp}}(t)|$ 
39:      $\langle \hat{n} \rangle \leftarrow \eta(|\alpha|^2 + |\beta|^2 - 2\xi\beta|\alpha|s_i) + \nu$ 
40:      $d \sim \text{Poisson}(n|\langle \hat{n} \rangle \delta t)$   $\triangleright$  Vacuum/photon detection
41:     if  $d > 1$ , set  $d = 1$ 
42:      $N \leftarrow N + d$   $\triangleright$  Total number of detected photons
43:      $t \leftarrow t + \delta t$ 
44:     if  $\text{mod}(N, 2) = 0$  then
45:        $\hat{s}_i(t) = \hat{s}_i(t_1)$   $\triangleright$  Hypothesis:  $|\pm \alpha\rangle$ 
46:     else
47:        $\hat{s}_i(t) = 0$   $\triangleright$  Hypothesis: Inconclusive
48:     end if
49:   end while  $\triangleright$  End of second temporal mode
50: end for
51:  $P_C(0) \leftarrow p$ ,  $P_I(0) \leftarrow 0$ ,  $P_E(0) \leftarrow 1 - p$ 
52:  $t \leftarrow 0$   $\triangleright$  Reset time
53: while  $0 \leq t < 1$  do  $\triangleright$  Calculate sampled probabilities
54:    $t \leftarrow t + \delta t$ 
55:    $P_C(t) = \sum_i (\hat{s}_i(t) == s_i) / M_{\text{MC}}$   $\triangleright$  Correct
56:    $P_I(t) = \sum_i (\hat{s}_i(t) == 0) / M_{\text{MC}}$   $\triangleright$  Inconclusive
57:    $P_E(t) = 1 - P_C(t) - P_I(t)$   $\triangleright$  Error
58: end while

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Lines 37-50 simulate the second temporal mode in a similar fashion to the first mode, except now the receiver is attempting to discriminate between the hypothesis after the first temporal mode $\hat{s}_i(t_1)$ and the inconclusive outcome denoted by $\hat{s}_i(t) = 0$. Lines 38-41 set the displacement field and sample the photon detection for the time step. Lines 44-48 update the provisional hypothesis

$\hat{s}_i(t)$, which now switches between the hypothesis after the first mode and the inconclusive outcome. This behavior is in contrast to the first temporal mode where the provisional hypothesis switches between the two possible input states. Lines 53-58 calculate the measured values of the correct, error, and inconclusive probabilities based on the value of the hypothesis $\hat{s}_i(t)$ and the true value s_i across all M_{MC} Monte-Carlo samples.

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