Supplementary Information: Multiphoton Quantum van Cittert–Zernike Theorem

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In this supplementary information we present: (i) the explicit derivation of the van Cittert-Zernike theorem for polarized two-photon fields; (ii) explicit derivation of Eq. (9) of the main manuscript; (iii) the difference between our theory and classical theory of coherence; (iv) each element of the BCP matrix upon propagation to the far field; (v) an example calculation of indistinguishable two-photon states; and (vi) photon statistics of distinguishable combined fields.

Supplementary Note 1. Derivation of the van Cittert-Zernike theorem for polarized two-photon fields

Let us start by considering the second-order coherence matrix for a polarized, quasi-monochromatic field [1]

$$J\left(\boldsymbol{r}_1,\boldsymbol{r}_2,z\right)=\left\langle j\left(\boldsymbol{r}_1,\boldsymbol{r}_2,z\right)\right\rangle=\begin{bmatrix} J_{HH}\left(\boldsymbol{r}_1,\boldsymbol{r}_2,z\right) & J_{HV}\left(\boldsymbol{r}_1,\boldsymbol{r}_2,z\right) \\ J_{VH}\left(\boldsymbol{r}_1,\boldsymbol{r}_2,z\right) & J_{VV}\left(\boldsymbol{r}_1,\boldsymbol{r}_2,z\right)\end{bmatrix}, \tag{1}$$

where z stands for the propagation distance of the field and $\boldsymbol{r}$ is used to specify the position of a point at the transverse plane of observation. The elements of the coherence-polarization matrix [Eq. (1)] are given by

$$J_{\alpha\beta}\left(\boldsymbol{r}_1,\boldsymbol{r}_2,z\right)=\left\langle j_{\alpha\beta}\left(\boldsymbol{r}_1,\boldsymbol{r}_2,z\right)\right\rangle=\left\langle E_\alpha^{(-)}\left(\boldsymbol{r}_1,z;t\right)E_\beta^{(+)}\left(\boldsymbol{r}_2,z;t\right)\right\rangle, \tag{2}$$

with $\alpha,\beta=H,V$. The angle brackets denote time average, whereas the quantities $E_\alpha^{(-)}\left(\boldsymbol{r},z;t\right)$ and $E_\alpha^{(+)}\left(\boldsymbol{r},z;t\right)$ represent the negative- and positive-frequency components of the α-polarized field-operator at the space-time point $(\boldsymbol{r},z;t)$, respectively.

Note that Eq. (1) is valid for classical fields, as well as single-photon sources [1, 2]. However, to include multi-photon effects, higher-order correlation functions are needed. In particular, for light in an arbitrary quantum state, the polarized, two-photon four-point correlation matrix can readily be written as

$$G\left(\boldsymbol{r}_1,\boldsymbol{r}_2;\boldsymbol{r}_3,\boldsymbol{r}_4,z\right)=\left\langle j\left(\boldsymbol{r}_1,\boldsymbol{r}_2,z\right)\otimes j\left(\boldsymbol{r}_3,\boldsymbol{r}_4,z\right)\right\rangle, \tag{3}$$

where $\otimes$ stands for the Kronecker (tensor) product. Note that the elements defined by the matrix in Eq. (3) are given by the four-point correlation matrix [3]

$$G_{\alpha\beta\alpha'\beta'}\left(\boldsymbol{r}_1,\boldsymbol{r}_2;\boldsymbol{r}_3,\boldsymbol{r}_4,z\right)=\left\langle E_\alpha^{(-)}\left(\boldsymbol{r}_1,z;t\right)E_\beta^{(+)}\left(\boldsymbol{r}_2,z;t\right)E_{\alpha'}^{(-)}\left(\boldsymbol{r}_3,z;t\right)E_{\beta'}^{(+)}\left(\boldsymbol{r}_4,z;t\right)\right\rangle, \tag{4}$$

with $\alpha,\beta,\alpha',\beta'=H,V$. Furthermore, we realize that the elements defined by the previous equation follow a propagation formula of the form [4]

$$\begin{aligned}G_{\alpha\beta\alpha'\beta'}\left(\boldsymbol{r}_1,\boldsymbol{r}_2;\boldsymbol{r}_3,\boldsymbol{r}_4,z\right)=\int\int\int\int &G_{\alpha\beta\alpha'\beta'}\left(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2;\boldsymbol{\rho}_3,\boldsymbol{\rho}_4,0\right)K^*\left(\boldsymbol{r}_1,\boldsymbol{\rho}_1,z\right)K\left(\boldsymbol{r}_2,\boldsymbol{\rho}_2,z\right)\\ &\times K^*\left(\boldsymbol{r}_3,\boldsymbol{\rho}_3,z\right)K\left(\boldsymbol{r}_4,\boldsymbol{\rho}_4,z\right)d^2\boldsymbol{\rho}_1 d^2\boldsymbol{\rho}_2 d^2\boldsymbol{\rho}_3 d^2\boldsymbol{\rho}_4,\end{aligned} \tag{5}$$

with the Fresnel propagation kernel defined by [5]

$$K(\boldsymbol{r},\boldsymbol{\rho},z)=\frac{-i\exp(ikz)}{\lambda z}\exp\left[\frac{ik}{2z}(\boldsymbol{r}-\boldsymbol{\rho})^2\right], \tag{6}$$

where $k=2\pi/\lambda$. Interestingly, in the context of the scalar theory, a spatially incoherent source is characterized by means of a delta-correlated intensity function, which indicates that subfields—making up for the whole source—at any two distinct points across the source plane are uncorrelated [6]. In the same spirit, and following previous authors

[1, 2], we define a partially polarized, spatially incoherent source as one whose four-point correlation matrix elements have the form

$$G_{\alpha\beta\alpha'\beta'}\left(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2;\boldsymbol{\rho}_3,\boldsymbol{\rho}_4,0\right)=\lambda^4 I_{\alpha\beta}\left(\boldsymbol{\rho}_1\right)I_{\alpha'\beta'}\left(\boldsymbol{\rho}_3\right)\left[\delta\left(\boldsymbol{\rho}_2-\boldsymbol{\rho}_1\right)\delta\left(\boldsymbol{\rho}_3-\boldsymbol{\rho}_4\right)+\delta\left(\boldsymbol{\rho}_2-\boldsymbol{\rho}_3\right)\delta\left(\boldsymbol{\rho}_1-\boldsymbol{\rho}_4\right)\right]. \tag{7}$$

Here, $I_{\alpha\beta}\left(\boldsymbol{\rho}\right)$ stands for the intensity, position-dependent, polarized two-photon source function. Note that the sum of delta functions in Eq. (7) is a result of the wavefunction symmetrization due to photon (in)distinguishability [3]. By substituting Eq. (7) into Eq. (5) we can thus obtain

$$\begin{aligned}G_{\alpha\beta\alpha'\beta'}\left(\boldsymbol{r}_1,\boldsymbol{r}_2;\boldsymbol{r}_3,\boldsymbol{r}_4,z\right)=&\frac{\exp\left[\frac{ik}{2z}\left(r_2^2-r_1^2\right)\right]\exp\left[\frac{ik}{2z}\left(r_2^4-r_1^3\right)\right]}{z^4}\iint I_{\alpha\beta}\left(\boldsymbol{\rho}_1\right)I_{\alpha'\beta'}\left(\boldsymbol{\rho}_3\right)\\&\times\exp\left[\frac{-2\pi i}{\lambda z}\boldsymbol{\rho}_1\cdot\left(\boldsymbol{r}_2-\boldsymbol{r}_1\right)\right]\exp\left[\frac{-2\pi i}{\lambda z}\boldsymbol{\rho}_3\cdot\left(\boldsymbol{r}_4-\boldsymbol{r}_3\right)\right]d^2\boldsymbol{\rho}_1 d^2\boldsymbol{\rho}_3,\end{aligned} \tag{8}$$

which represents the extension of the van Cittert-Zernike theorem to two-photon, partially polarized fields.

Supplementary Note 2. Second-order coherence matrix

To show some consequences of the two-photon vectorial van Cittert-Zernike theorem, we now present an example where two-photon correlations are built up during propagation. Let us start from a spatially incoherent and unpolarized source, whose four-point correlation matrix is written, in the $\{|HH\rangle,|HV\rangle,|VH\rangle,|VV\rangle\}$ basis, as

$$\begin{aligned}G_{\text{ini}}\left(x_1,x_2;x_3,x_4,0\right)&=j_{\text{ini}}\left(x_1,x_2,0\right)\otimes j_{\text{ini}}\left(x_3,x_4,0\right)\\&=\lambda^4 I_0^2\left[\delta\left(x_2-x_1\right)\delta\left(x_3-x_4\right)+\delta\left(x_2-x_3\right)\delta\left(x_1-x_4\right)\right]\\&\quad\times\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\end{aligned} \tag{9}$$

where

$$j_{\text{ini}}\left(x_1,x_2,0\right)=\lambda^2 I_0\delta\left(x_2-x_1\right)\begin{bmatrix}1&0\\0&1\end{bmatrix} \tag{10}$$

stands for the two-point correlation matrix for a spatially incoherent and unpolarized photon source [1], and I_0 describes a constant-intensity factor. Note that, for the sake of simplicity, we have restricted ourselves to a one-dimensional case, i.e., we have taken only one element of the transversal vector $\boldsymbol{r}=(x,y)$.

To polarize the source, we make use of a linear polarizer. Specifically, we cover the source with a linear polarization grating whose angle between its transmission axis and the x-axis is a linear function of the form $\theta=\pi x/L$, with L being the length of the grating. The four-point correlation matrix after the polarization grating can thus be written as

$$\begin{aligned}G_{\text{out}}\left(x_1,x_2;x_3,x_4,0\right)=&\left[P^\dagger\left(x_1\right)j_{\text{ini}}\left(x_1,x_2,0\right)P\left(x_2\right)\right]\otimes\left[P^\dagger\left(x_3\right)j_{\text{ini}}\left(x_3,x_4,0\right)P\left(x_4\right)\right]\\&\times\text{rect}\left(x_1/L\right)\text{rect}\left(x_2/L\right)\text{rect}\left(x_3/L\right)\text{rect}\left(x_4/L\right)\\&\times\lambda^4 I_0^2\left[\delta\left(x_2-x_1\right)\delta\left(x_3-x_4\right)+\delta\left(x_2-x_3\right)\delta\left(x_1-x_4\right)\right]\end{aligned} \tag{11}$$

where the product of rect $(\cdots)$ functions describe the finite size of the source, and the action of the polarization grating is given by the Jones matrix,

$$P\left(x\right)=\begin{bmatrix}\cos^2\left(\frac{\pi x}{L}\right)&\cos\left(\pi x/p\right)\sin\left(\pi x/L\right)\\\cos\left(\pi x/p\right)\sin\left(\pi x/L\right)&\sin^2\left(\pi x/L\right)\end{bmatrix}. \tag{12}$$

By substituting Eqs. (9)-(12) into Eq. (8), we can obtain the explicit form of the polarized, four-point correlation matrix elements. As an example, we can find that, in the far-field—i.e. the region where the quadratic phase factor in front of the integral of Eq. (8) goes to one—the normalized four-point correlation function for H-polarized photons reads as

$$\begin{aligned}
G_{\mathrm{HHHH}}\left(\nu_1,\nu_2,\nu_3,\nu_4;z\right) = &\left[\mathrm{sinc}\left(\nu_1\right)+\frac{1}{2}\,\mathrm{sinc}\left(\nu_1-1\right)+\frac{1}{2}\,\mathrm{sinc}\left(\nu_1+1\right)\right]\\
&\times\left[\mathrm{sinc}\left(\nu_2\right)+\frac{1}{2}\,\mathrm{sinc}\left(\nu_2-1\right)+\frac{1}{2}\,\mathrm{sinc}\left(\nu_2+1\right)\right]\\
&+\frac{1}{16}\left[\mathrm{sinc}\left(2+\nu_3\right)\left(\mathrm{sinc}\left(\nu_4\right)+2\,\mathrm{sinc}\left(1-\nu_4\right)+\mathrm{sinc}\left(2-\nu_4\right)\right)\right.\\
&+2\,\mathrm{sinc}\left(1+\nu_3\right)\left(3\,\mathrm{sinc}\left(\nu_4\right)+3\,\mathrm{sinc}\left(1-\nu_4\right)+\mathrm{sinc}\left(2-\nu_4\right)+\mathrm{sinc}\left(1+\nu_4\right)\right)\\
&+\mathrm{sinc}\left(2-\nu_3\right)\left(\mathrm{sinc}\left(\nu_4\right)+2\,\mathrm{sinc}\left(1+\nu_4\right)+\mathrm{sinc}\left(2+\nu_4\right)\right)\\
&+2\,\mathrm{sinc}\left(1-\nu_3\right)\left(3\,\mathrm{sinc}\left(\nu_4\right)+\mathrm{sinc}\left(1-\nu_4\right)+3\,\mathrm{sinc}\left(1+\nu_4\right)+\mathrm{sinc}\left(2+\nu_4\right)\right)\\
&\left.+\mathrm{sinc}\left(\nu_3\right)\left(10\,\mathrm{sinc}\left(\nu_4\right)+6\,\mathrm{sinc}\left(1-\nu_4\right)+\mathrm{sinc}\left(2-\nu_4\right)+6\,\mathrm{sinc}\left(1+\nu_4\right)+\mathrm{sinc}\left(2+\nu_4\right)\right)\right]
\end{aligned} \tag{13}$$

with

$$\nu_1 = L\frac{x_2-x_3}{\lambda z};\quad \nu_2 = L\frac{x_4-x_1}{\lambda z};\quad \nu_3 = L\frac{x_2-x_1}{\lambda z};\quad \nu_4 = L\frac{x_4-x_3}{\lambda z}. \tag{14}$$

Finally, by realizing that when monitoring the two-photon correlation function with two detectors, at the observation plane in z, we must set [2]: $x_2 = x_3$ and $x_1 = x_4$, we find that

$$G_{\mathrm{HHHH}}\left(\nu_1,\nu_2,\nu_3,\nu_4;z\right) = G_{\mathrm{HHHH}}\left(0,0,\nu_1,-\nu_1;z\right). \tag{15}$$

We can follow the same procedure as above to obtain the remaining terms of the four-point correlation matrix.

Supplementary Note 3. Classical theory of coherence does not describe multiphoton scattering

In order to emphasize the quantum nature of our main result we present the classical analysis of the same system. We will present this calculation in the form of the classical Beam Coherence Polarization (BCP) matrix and highlight the differences that arise between the two analyses. We begin by presenting the first order BCP-matrix for an incoherent, unpolarized source as

$$J\left(\boldsymbol{r}_1,\boldsymbol{r}_2,z\right) = \left\langle j\left(\boldsymbol{r}_1,\boldsymbol{r}_2,z\right)\right\rangle = \begin{bmatrix} J_{HH}\left(\boldsymbol{r}_1,\boldsymbol{r}_2,z\right) & J_{HV}\left(\boldsymbol{r}_1,\boldsymbol{r}_2,z\right)\\ J_{VH}\left(\boldsymbol{r}_1,\boldsymbol{r}_2,z\right) & J_{VV}\left(\boldsymbol{r}_1,\boldsymbol{r}_2,z\right)\end{bmatrix} = \lambda^2 I_0\delta\left(x_2-x_1\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}. \tag{16}$$

Where in contrast to Eq. (1) we define $J_{\alpha\beta} = \langle E^*_\alpha(\boldsymbol{r}_1,z;t)E_\beta(\boldsymbol{r}_2,z;t)\rangle$. Therefore, the key difference between the classical approach to the system and the quantum approach previously outlined is that we propagate the first order classical correlations directly rather than converting it to the operator form and propagating the operators through space. This creates an immediate change when viewing the initial second order BCP matrix which is now written as

$$\begin{aligned}
G_{\mathrm{ini}}\left(x_1,x_2;x_3,x_4,0\right) &= j_{\mathrm{ini}}\left(x_1,x_2,0\right)\otimes j_{\mathrm{ini}}\left(x_3,x_4,0\right)\\
&= \lambda^4 I_0^2\delta\left(x_2-x_1\right)\delta\left(x_3-x_4\right)\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},
\end{aligned} \tag{17}$$

This removed the $\delta\left(x_2-x_3\right)\delta\left(x_1-x_4\right)$ term from Eq. (9) which is ultimately the quantum term. In other words, a classical system presumes that there is no relation between individual photons regardless of measurements. This effect holds when propagating through the polarization grating allowing the van Cittert-Zernike theorem to be written as

$$G^{(2)}_{jklm}(\mathbf{X},z) = \int dx_1\int dx_2\int dx_3\int dx_4 C_{jklm}(\boldsymbol{x})F(\boldsymbol{x},\mathbf{X},z)\delta\left(x_1-x_2\right)\delta\left(x_3-x_4\right), \tag{18}$$

where the limits of integration for each integral is $-L/2$ to $L/2$, $\boldsymbol{x} = [x_1,x_2,x_3,x_4]$, $C_{jklm}(\boldsymbol{x})$ is the coefficient of the $\mathbf{j}_{jk}\otimes\mathbf{j}_{lm}$ of the second order BCP matrix, and $j,k,l,m\in\{H,V\}$. Furthermore, $F\left(\boldsymbol{x},\mathbf{X},z\right)$ is given as

$$F\left(\boldsymbol{x},\mathbf{X},z\right) = \mathrm{Exp}[\frac{2\pi i}{\lambda z}\left(X_4x_4-X_3x_3+X_2x_2-X_1x_1\right)]. \tag{19}$$

Upon propagation we then have the far-field result as

$$g^{(2)}(\nu) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{20}$$

Which vastly differs from the results presented in Fig. 2 of the main paper. Mainly, we lose all spatial correlation as described before. This results in our $g^{(2)}(0)$ diagonal elements to become 1 while all other elements are zero. This of course, matches the $g^{(2)}(0)$ obtained when modeling thermal light as a classical source with a constant, non-fluctuating intensity, since bunching needs to be modeled by a quantum theory. Furthermore, the lack of off-diagonal correlations erases the quantum degree of polarization we discovered in our paper and instead states that the source is unpolarized, as noted in the first order classical analysis of the system [1]. Overall, this highlights that our calculation produces results incapable of being replicated by a classical analysis.

Supplementary Note 4. BCP Matrix Elements

Each element of the final BCP matrix upon detection is given as follows:

$$\begin{aligned} g^{(2)}_{\mathrm{HHHH}}(\nu) = \frac{1}{16}&(10\mathrm{sinc}(\nu)^2 + 2(6\mathrm{sinc}(\nu+1) + \mathrm{sinc}(\nu+2) + 6\mathrm{sinc}(1-\nu) + \mathrm{sinc}(2-\nu))\mathrm{sinc}(\nu) \\ &+ 6\mathrm{sinc}(\nu+1)^2 + \mathrm{sinc}(\nu+2)^2 + 6\mathrm{sinc}(1-\nu)^2 + \mathrm{sinc}(2-\nu)^2 \\ &+ 4\mathrm{sinc}(\nu+1)\mathrm{sinc}(\nu+2) + 4(\mathrm{sinc}(\nu+1) + \mathrm{sinc}(2-\nu))\mathrm{sinc}(1-\nu) + 16) \end{aligned} \tag{21}$$

$$\begin{aligned} g^{(2)}_{\mathrm{HHVV}}(\nu) = g^{(2)}_{\mathrm{VVHH}}(\nu) = \frac{1}{16}&(2\mathrm{sinc}(\nu)^2 - 2(\mathrm{sinc}(\nu+2) + \mathrm{sinc}(2-\nu))\mathrm{sinc}(\nu) + 2(\mathrm{sinc}(1-\nu) \\ &- \mathrm{sinc}(\nu+1))^2 + \mathrm{sinc}(\nu+2)^2 + \mathrm{sinc}(2-\nu)^2 + 16) \end{aligned} \tag{22}$$

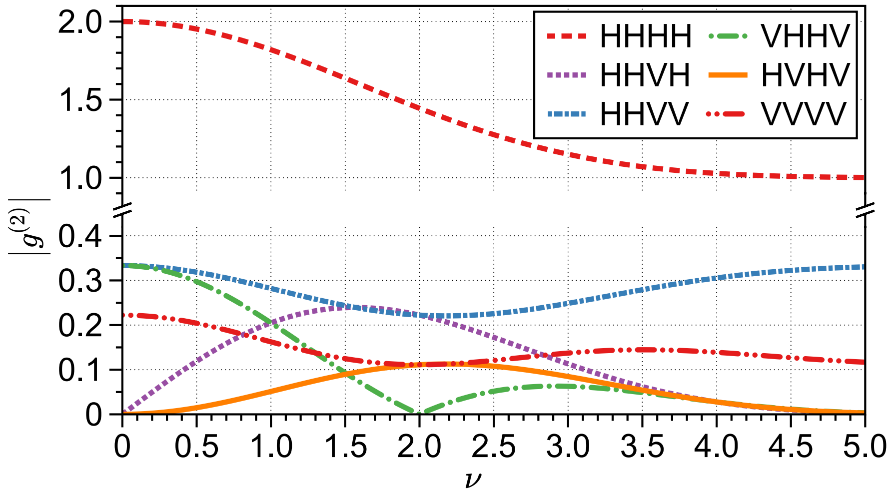
$$\begin{aligned} g^{(2)}_{\mathrm{HHVH}}(\nu) = g^{(2)}_{\mathrm{VHHH}}(\nu) &= -\frac{i}{16}(\mathrm{sinc}(\nu-2)^2 + 2\mathrm{sinc}(1-\nu)\mathrm{sinc}(\nu-2) - \mathrm{sinc}(\nu+2)^2 \\ &- 2\mathrm{sinc}(\nu+2)\mathrm{sinc}(\nu+1) + 2(\mathrm{sinc}(1-\nu) - \mathrm{sinc}(\nu+1))(\mathrm{sinc}(\nu) + \mathrm{sinc}(\nu+1) + \mathrm{sinc}(1-\nu))) \\ &= \left(g^{(2)}_{\mathrm{HHHV}}(\nu)\right)^* = \left(g^{(2)}_{\mathrm{HVHH}}(\nu)\right)^* \end{aligned} \tag{23}$$

$$\begin{aligned} g^{(2)}_{\mathrm{HVHV}}(\nu) = g^{(2)}_{\mathrm{VHVH}}(\nu) = \frac{1}{16}&(2\mathrm{sinc}(\nu)^2 - 2(\mathrm{sinc}(\nu+2) + \mathrm{sinc}(2-\nu))\mathrm{sinc}(\nu) \\ &+ 2(\mathrm{sinc}(1-\nu) - \mathrm{sinc}(\nu+1))^2 + \mathrm{sinc}(\nu+2)^2 + \mathrm{sinc}(2-\nu)^2) \end{aligned} \tag{24}$$

$$\begin{aligned} g^{(2)}_{\mathrm{VHHV}}(\nu) = g^{(2)}_{\mathrm{HVVH}}(\nu) = \frac{1}{16}&(6\mathrm{sinc}(\nu)^2 - 2(\mathrm{sinc}(\nu+2) + \mathrm{sinc}(2-\nu))\mathrm{sinc}(\nu) + 2(\mathrm{sinc}(1-\nu) \\ &- \mathrm{sinc}(\nu+1))^2 - \mathrm{sinc}(\nu+2)^2 - \mathrm{sinc}(2-\nu)^2) \end{aligned} \tag{25}$$

$$\begin{aligned} g^{(2)}_{\mathrm{HVVV}}(\nu) = g^{(2)}_{\mathrm{VVHV}}(\nu) &= \frac{i}{16}(2\mathrm{sinc}(\nu+1)^2 - 2\mathrm{sinc}(\nu)\mathrm{sinc}(\nu+1) - 2\mathrm{sinc}(\nu+2)\mathrm{sinc}(\nu+1) + \mathrm{sinc}(\nu+2)^2 \\ &- 2\mathrm{sinc}(1-\nu)^2 - \mathrm{sinc}(2-\nu)^2 + 2\mathrm{sinc}(\nu)\mathrm{sinc}(1-\nu) + 2\mathrm{sinc}(1-\nu)\mathrm{sinc}(2-\nu) \\ &= \left(g^{(2)}_{\mathrm{VHVV}}(\nu)\right)^* = \left(g^{(2)}_{\mathrm{VVVH}}(\nu)\right)^* \end{aligned} \tag{26}$$

$$\begin{aligned} g^{(2)}_{\mathrm{VVVV}}(\nu) = \frac{1}{16}&(10\mathrm{sinc}(\nu)^2 + 2(-6\mathrm{sinc}(\nu+1) + \mathrm{sinc}(\nu+2) - 6\mathrm{sinc}(1-\nu) + \mathrm{sinc}(2-\nu))\mathrm{sinc}(\nu) \\ &+ 6\mathrm{sinc}(\nu+1)^2 + \mathrm{sinc}(\nu+2)^2 + 6\mathrm{sinc}(1-\nu)^2 + \mathrm{sinc}(2-\nu)^2 \\ &- 4\mathrm{sinc}(\nu+1)\mathrm{sinc}(\nu+2) + 4(\mathrm{sinc}(\nu+1) - \mathrm{sinc}(2-\nu))\mathrm{sinc}(1-\nu) + 16) \end{aligned} \tag{27}$$



Supplementary Figure 1. The behavior of various post-selected measurements for a system acted upon by a purely horizontal beam. Here, all non-purely horizontal measurements exhibit sub-Poissonian-like statistics. This is due to the vertical field is created purely by a Stern-Gerlach like mechanism through the repeated projective measurements by the polarization grating and the post-selection of our measurement.

Supplementary Note 5. Example Calculation: Indistinguishable Two-Photon States

In order to exemplify the characteristics of our system, we consider a two-photon indistinguishable state. Without loss of generality, we will restrict to indistinguishable horizontally-polarized photons. Here the beam coherence polarization matrix $j_{\text{ini}}\left(x_1, x_2, 0\right)$ can be written as

$$j_{\text{ini}}\left(x_1, x_2, 0\right) = \lambda^2 I_0 \delta\left(x_2 - x_1\right) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{28}$$

We can then propagate the system using Eqs. (5) and (11) and then normalize to find the finalized system. The final values of the system are in Supplementary Figure 1. The vertical field and all correlations associated with it are created by the repeated projective measurements of the polarization grating and the post-selected measurements. This allows for a sub-Poissonian-like fields to be created through the use of only linear operations. Furthermore, two measurements will produce fields with spatial anti-bunching-like, the HVHV and HHVH measurement. It is important to note that this result holds for a multiphoton single-mode field as well.

The BCP matrix elements for the system are as follows:

$$g^{(2)}_{\text{HHHH}}(\nu) = \frac{1}{36}\left(6\text{sinc}\left(\frac{\nu}{2}\right) + 4\text{sinc}\left(\frac{\nu}{2}+\right) + \text{sinc}\left(\frac{\nu}{2}+2\right) + 4\text{sinc}\left(-\frac{\nu}{2}\right) + \text{sinc}\left(2-\frac{\nu}{2}\right)\right)^2 + 1 \tag{29}$$

$$g^{(2)}_{\text{HHVV}}(\nu) = g^{(2)}_{\text{VVHH}}(\nu) = \frac{1}{3} - \frac{1}{36}\left(-2\text{sinc}\left(\frac{\nu}{2}+\right) - \text{sinc}\left(\frac{\nu}{2}+2\right) + 2\text{sinc}\left(-\frac{\nu}{2}\right) + \text{sinc}\left(2-\frac{\nu}{2}\right)\right)^2 \tag{30}$$

$$\begin{aligned} g^{(2)}_{\text{HHVH}}(\nu) = g^{(2)}_{\text{VHHH}}(\nu) &= \frac{1}{36} i \left(-2\text{sinc}\left(\frac{\nu}{2}+\right) - \text{sinc}\left(\frac{\nu}{2}+2\right) + 2\text{sinc}\left(-\frac{\nu}{2}\right) + \text{sinc}\left(2-\frac{\nu}{2}\right)\right) \times \\ &\left(6\text{sinc}\left(\frac{\nu}{2}\right) + 4\text{sinc}\left(\frac{\nu}{2}+\right) + \text{sinc}\left(\frac{\nu}{2}+2\right) + 4\text{sinc}\left(-\frac{\nu}{2}\right) + \text{sinc}\left(2-\frac{\nu}{2}\right)\right) \\ &= \left(g^{(2)}_{\text{HHHV}}(\nu)\right)^* = \left(g^{(2)}_{\text{HVHH}}(\nu)\right)^* \end{aligned} \tag{31}$$

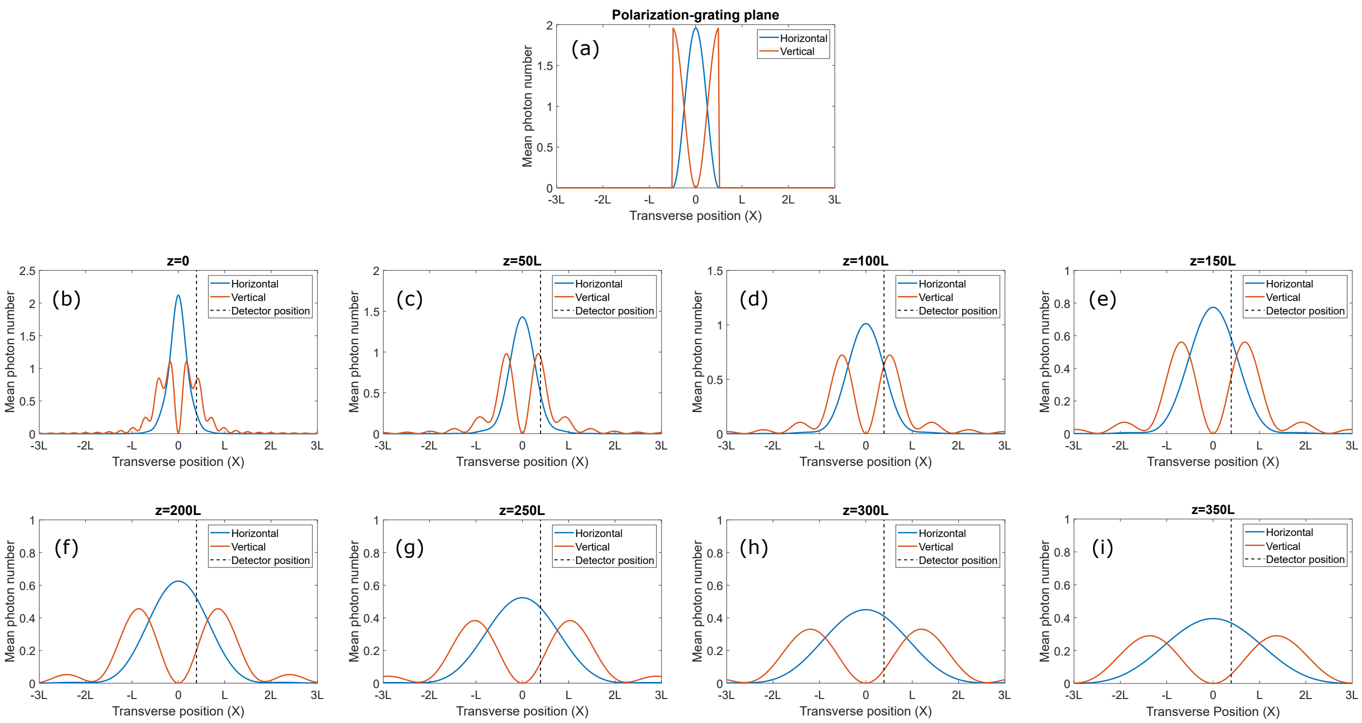
$$g^{(2)}_{\text{HVHV}}(\nu) = g^{(2)}_{\text{VHVH}}(\nu) = -\frac{1}{36}\left(-2\text{sinc}\left(\frac{\nu}{2}+\right) - \text{sinc}\left(\frac{\nu}{2}+2\right) + 2\text{sinc}\left(-\frac{\nu}{2}\right) + \text{sinc}\left(2-\frac{\nu}{2}\right)\right)^2 \tag{32}$$

$$g^{(2)}_{\mathrm{VHHV}}(\nu) = g^{(2)}_{\mathrm{HVVH}}(\nu) = \frac{1}{36}\left(2\mathrm{sinc}\left(\frac{\nu}{2}\right) - \mathrm{sinc}\left(\frac{\nu}{2}+2\right) - \mathrm{sinc}\left(2-\frac{\nu}{2}\right)\right)\times \\ \left(6\mathrm{sinc}\left(\frac{\nu}{2}\right) + 4\mathrm{sinc}\left(\frac{\nu}{2}+\right) + \mathrm{sinc}\left(\frac{\nu}{2}+2\right) + 4\mathrm{sinc}\left(-\frac{\nu}{2}\right) + \mathrm{sinc}\left(2-\frac{\nu}{2}\right)\right) \tag{33}$$

$$\begin{aligned} g^{(2)}_{\mathrm{HVVV}}(\nu) = g^{(2)}_{\mathrm{VVHV}}(\nu) &= \frac{1}{36}i\left(2\mathrm{sinc}\left(\frac{\nu}{2}\right) - \mathrm{sinc}\left(\frac{\nu}{2}+2\right) - \mathrm{sinc}\left(2-\frac{\nu}{2}\right)\right)\times \\ &\left(-2\mathrm{sinc}\left(\frac{\nu}{2}+\right) - \mathrm{sinc}\left(\frac{\nu}{2}+2\right) + 2\mathrm{sinc}\left(-\frac{\nu}{2}\right) + \mathrm{sinc}\left(2-\frac{\nu}{2}\right)\right) \\ &= \left(g^{(2)}_{\mathrm{VHVV}}(\nu)\right)^{*} = \left(g^{(2)}_{\mathrm{VVVH}}(\nu)\right)^{*} \end{aligned} \tag{34}$$

$$g^{(2)}_{\mathrm{VVVV}} = \frac{1}{36}\left(\left(2\mathrm{sinc}\left(\frac{\nu}{2}\right) - \mathrm{sinc}\left(\frac{\nu}{2}+2\right) - \mathrm{sinc}\left(2-\frac{\nu}{2}\right)\right)^2 + 4\right) \tag{35}$$

Supplementary Note 6. Photon Statistics of Distinguishable (Polarized) Combined Fields



Supplementary Figure 2. Propagation of distinguishable multiphoton fields. (a) Spatial distribution of horizontally (blue line) and vertically (red line) polarized components of a thermal field at the polarization-grating plane. Each polarization component propagates in free-space following the Fresnel diffraction formula (see Eq. (6)), thus showing a different spatial field distribution at different positions along the propagation axis: (b) $z = 0$, (c) $z = 50L$, (d) $z = 100L$, (e) $z = 150L$, (f) $z = 200L$, (g) $z = 250L$, (h) $z = 300L$, (i) $z = 350L$. The photon distribution at each z is obtained by placing a photon-number-resolving detector at $X = 0.4L$, depicted by the black-dashed line.

To describe the photon statistics evolution of the field that emerges from the polarization grating, we first note that once the unpolarized field traverses the polarization grating, two different field distributions are created, one that bears a horizontal polarization and one that is vertically polarized (see Supplementary Figure 2). We can then describe the propagation of these fields into the far-field by making use of Fresnel propagation (see Eq. (6)). As one might expect, the spatial distribution of both independent and distinguishable fields (vertically- and horizontally-polarized) change upon propagation. This creates different intensity (mean photon number) patterns in the transverse planes located at different positions (z) along the propagation axis, see Supplementary Figure 2(b)-(i). Note that we have labeled as $z = 0$ the field distribution immediately after the polarization grating.

To obtain the combined photon distribution, we first place a photon-number-resolving detector at an arbitrary position along the transverse plane. In the example shown in the main text, we selected a position $X = 0.4L$ (depicted by the black-dashed line in Supplementary Figure 2), with L being the length of the polarization grating. Following the theory of coherence of Glauber and Sudarshan [7, 8]. We can thus find that the photon distribution of the combined field, at a position (X, z), is given by

$$p^{(X,z)}(n) = \sum_{m=0}^{n} p_H^{(X,z)}(n-m) p_V^{(X,z)}(m), \tag{36}$$

where $p_{H,V}^{(X,z)}$ stands for the photon distribution of the horizontally- and vertically-polarized fields, respectively. Because both fields are thermal, we can write their corresponding photon distribution as

$$p_H^{(X,z)}(n) = \frac{\overline{n}_H(X,z)^n}{(\overline{n}_H(X,z)+1)^{n+1}}, \tag{37}$$

$$p_V^{(X,z)}(n) = \frac{\overline{n}_V(X,z)^n}{(\overline{n}_V(X,z)+1)^{n+1}}, \tag{38}$$

with $\overline{n}_H(X,z)$ and $\overline{n}_V(X,z)$ being the mean photon number (extracted from the field distributions shown in Supplementary Figure 2) of the horizontal (H) and vertical (V) fields, at position (X,z), respectively. Finally, by substituting Eqs. (37) and (38) into Eq. (36), with $X = 0.4L$, we find the photon distributions shown in Fig. 3 of the main manuscript. Note that we can use Eq. (36) to evaluate the second-order quantum coherence, i.e. $g^{(2)}(\tau = 0)$, at each point (X,z), by realizing that $\langle n(X,z)\rangle = \sum_n^{\infty} n p^{(X,z)}(n)$, and $\langle n^2(X,z)\rangle = \sum_n^{\infty} n^2 p^{(X,z)}(n)$.

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