

Supplementary Material: Compact description of quantum phase slip junctions

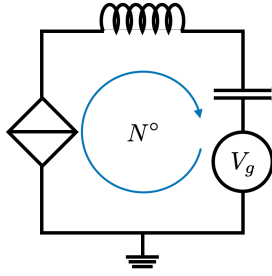
C. Koliofoti and R.-P. Riwar

Peter Grünberg Institute, Theoretical Nanoelectronics, D-52425 Jülich, Germany

SUPPLEMENTARY NOTE I:

Loop-charge quantization for circuits involving one and two QPS junctions

In the main text, we give the loop-charge Hamiltonians of the two circuits of Fig. 5 and Fig. 8. Here we show how they can be derived. In both cases we will use Fig. 3 of Ref. [1] to build the Lagrangian.



Supplementary Figure 1. **Loop-charge treatments of QPS junctions** via a simple circuit coupling a QPS junction with a capacitance and a gate voltage V_g .

The first circuit (main text Fig. 5) consists of a QPS junction, an inductor and a gate voltage with a capacitance in a loop. Note that Fig. 5 in the main text does not explicitly show the loop-charge variables in the circuit scheme, which is why we add here Supplementary Fig. 1 to indicate them. The Lagrangian for this system reads

$$L^\circ = E_S \cos(2\pi N^\circ) + \frac{L}{2} (2e)^2 \dot{N}^{\circ 2} - \frac{(2e)^2}{2C_g} N^{\circ 2} - 2eV_g N^\circ. \quad (1)$$

First we have to find the canonically conjugate momentum to the loop-charge N° , φ° ,

$$\varphi^\circ = \frac{\partial L^\circ}{\partial \dot{N}^\circ} = L(2e)^2 \dot{N}^\circ \quad (2)$$

Thus we can now find the Hamiltonian by means of the Legendre transformation

$$H^\circ = \dot{N}^\circ \varphi^\circ - L^\circ = \frac{1}{2L} \left(\frac{\varphi^\circ}{2e} \right)^2 - E_S \cos(2\pi N^\circ) - \frac{C_g}{2} V_g^2 + \frac{(2e)^2}{2C_g} \left[N^{\circ 2} + \frac{2C_g}{(2e)} V_g N^\circ + \left(\frac{C_g}{(2e)} V_g \right)^2 \right]. \quad (3)$$

By setting $E_L = 1/(4e^2 L)$, $E_C = (2e)^2/2C_g$ and $N_g = C_g V_g/(2e)$ we arrive at the Hamiltonian of Eq. (17).

The same calculation can be done for the circuit in Fig. 8, consisting of two QPS junction, two inductor and a gate voltage with a capacitance. Again we will follow the recipe of Ref. [1] to construct the Lagrangian

$$L^\circ = E_{S1} \cos(2\pi N_1^\circ) + E_{S2} \cos(2\pi N_2^\circ) + \frac{L_1}{2} (2e \dot{N}_1^\circ)^2 + \frac{L_2}{2} (2e \dot{N}_2^\circ)^2 - \frac{(2e)^2}{2C_g} (N_1^\circ - N_2^\circ)^2 - 2eV_g (N_1^\circ - N_2^\circ). \quad (4)$$

The canonically conjugate phase φ_i° will be of the form of φ° in Supplementary Eq. (2) with $N^\circ \rightarrow N_i^\circ$ and $L \rightarrow L_i$. We find the Hamiltonian again through

$$H^\circ = \dot{N}_1^\circ \varphi_1^\circ + \dot{N}_2^\circ \varphi_2^\circ - L^\circ = \sum_{i \in \{1,2\}} \left[\frac{1}{L_i} \left(\frac{\varphi_i^\circ}{2e} \right)^2 - E_{Si} \cos(2\pi N_i^\circ) \right] + \frac{(2e)^2}{2C_g} (N_1^\circ - N_2^\circ + N_g) + \text{const.} \quad (5)$$

This leads us to the Hamiltonian of Eq. (26).

SUPPLEMENTARY NOTE II:

Loop-charge quantization for the SQUID

In the main text we considered the dc SQUID in terms of loop-charge quantization (Fig. 1c), and pointed out that there is a lingering issue regarding the capacitances in the circuits. We here provide details on the conclusions presented in the main text. Physically, in the SQUID, there only exists a bulk-capacitance C_{tot} . As pointed out in Ref. [2], this capacitance can in principle be split it up into the two self-capacitances C_1 and C_2 of the Josephson junctions, so that $C_1 + C_2 = C_{\text{tot}}$. However, in the most general case, it can happen that one of the two self-capacitance could now be negative (while the sum C_{tot} remains positive). We here show in detail, why this leads to an instability in the Hamiltonian.

The Lagrangian of the circuit in Fig. 1c is

$$\mathcal{L} = \frac{C_1}{2(2e)^2} \dot{\varphi}_1^2 + E_J \cos(\varphi_1) + \frac{C_2}{2(2e)^2} \dot{\varphi}_2^2 + E_J \cos(\varphi_2) + \frac{L}{2} (2e \dot{N}^\circ)^2 + N^\circ (\dot{\varphi}_1 - \dot{\varphi}_2), \quad (6)$$

which is an example of a mixed node-flux/loop-charge approach. The first two lines refer to the node-flux part describing the Josephson effects of each junction. The inductance term $L \left(2e\dot{N}^\circ \right)^2 / 2$ is expressed through the SQUID's loop charge. The last term $N^\circ (\dot{\varphi}_1 - \dot{\varphi}_2)$ is included to satisfy Kirchhoff laws (as explained in Ref. [1]).

Through Legendre transformation we obtain the Hamiltonian

$$H = (N_1 - N^\circ)^2 \frac{(2e)^2}{2C_1} + (N_2 + N^\circ)^2 \frac{(2e)^2}{2C_2} - E_J \cos(\varphi_1) - E_J \cos(\varphi_2) + \frac{(\varphi^\circ)^2}{2L(2e)^2}, \quad (7)$$

where N_i is the canonically conjugated charge to the phase φ_i . Notice that the above Hamiltonian has a continuous symmetry, leading to the conserved Noether current

$$\varphi'^\circ = \varphi^\circ + \varphi_1 - \varphi_2, \quad (8)$$

which can be eliminated. For this purpose, we perform the coordinate transformation $N'_1 = N_1 - N^\circ$ and $N'_2 = N_2 + N^\circ$. Considering the limit of large Josephson energies ($\cos(\varphi) \sim \varphi^2/2$), we arrive at

$$H \approx (N'_1)^2 \frac{(2e)^2}{2C_1} + (N'_2)^2 \frac{(2e)^2}{2C_2} + \frac{(\varphi' - \varphi'_1 + \varphi'_2)^2}{2L(2e)^2} + \frac{E_J}{2} \varphi_1^2 + \frac{E_J}{2} \varphi_2^2, \quad (9)$$

We see that the above Hamiltonian is simply the model of two coupled harmonic oscillators. If one of the capacitances is negative, this is equivalent to one of the oscillators acquiring negative mass. Hence, one can no longer guarantee that all eigenvalues of the Hamiltonian are real and positive. In fact, through a standard Bogoliubov-transformation we find that one eigenvalues of this Hamiltonian becomes imaginary. That is, the eigenfrequencies are

$$\omega_\pm = \frac{1}{\sqrt{8}} \left(C_{\parallel+} \left(\frac{1}{L} + 4e^2 E_J \right) \pm \sqrt{\frac{1}{L^2} C_{\parallel+}^2 + 8e^2 E_J \left(\frac{1}{L} + 2e^2 E_J \right) C_{\parallel-}^2} \right)^{1/2}, \quad (10)$$

with

$$C_{\parallel\pm} = \frac{1}{C_L} \pm \frac{1}{C_R}. \quad (11)$$

In particular $\omega_- \approx \sqrt{C_{\parallel-}}/(2\sqrt{L})$ is imaginary, if, e.g., $C_1 < 0$ (and $1/(e^2 L) \gg E_J, e^2/C_{1,2}$).

SUPPLEMENTARY NOTE III:

Differing charge definitions and renormalization effects for a Josephson junction array

In the main text, we revisit the low-energy physics of QPS junctions. In this section we illustrate some of the conclusion of the main text, by means of a finite element approach, when the wire is modelled by a discrete chain of Josephson junctions, see Supplementary Fig. 2. We use this model also to explain how different basis choices for the low-energy states of the QPS junction correspond to the measurement or interaction with different charges N and $\tilde{N}$ as hinted at in the main text. This discussion in turn helps illustrating the existence of a renormalization effect for the QPS amplitude E_S .

We decompose the QPS wire into a finite but large number islands indexed by $j = 1, \dots, J$, hosting charges and phases $[\hat{N}_j, \hat{\varphi}_j] = i\delta_{jj'}$. Here the last phase $\varphi := \varphi_J$ is kept arbitrary but constant. The islands may be capacitively coupled (either among themselves or to ground) and exchange Cooper pairs formally via JJs. The ends of the resulting JJ array are going to be phase-biased due to a finite φ and choosing $\varphi_0 = 0$ (contact to ground). The total Hamiltonian is given as

$$H_{\text{JJA}} = T + V, \quad (12)$$

where we cast the entirety of the capacitive energy into the kinetic energy term

$$T = \frac{(2e)^2}{2} \sum_{j,k=1} \hat{N}_j (C^{-1})_{jk} \hat{N}_k, \quad (13)$$

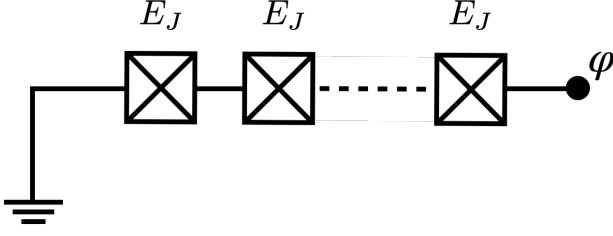
where offset gate charges on each island can be accounted for by shifting all $\hat{N}_j \rightarrow \hat{N}_j + N_{gj}$. All Josephson energies appear in the potential energy term

$$V = -E_J \sum_{j=1}^{J-1} \cos(\hat{\varphi}_j - \hat{\varphi}_{j-1}) - E_J \cos(\varphi - \hat{\varphi}_{J-1}). \quad (14)$$

While this model is obviously valid if the QPS junction is made of a Josephson junction array, we believe that it should retain some validity even in a continuous limit, i.e., for a thin wire.

A. Derive low-energy Hamiltonian in Eq. (7)

As for the influence of the capacitive contribution to the circuit dynamics, we notice the following. With all Josephson energies the same, and at sufficiently low-energies, the potential energy (the Josephson-energy contributions of the JJ array) have well-defined local minima. From this perspective, the capacitances provide a mechanism whereby the system can perform quantum



Supplementary Figure 2. **Josephson junction array.** Note that the individual islands may be in general connected by a highly complex capacitive network. For our considerations the details of the capacitive coupling is irrelevant, which is why we do not explicitly show it here. Also note that if we use the JJ array model to simulate a bulk superconducting wire, then the circuit dynamics will likely be dominated by the capacitances to ground.

jumps between different local minima. These are the quantum phase slips (QPS). The energy associated to the amplitude of the QPSs has already been computed for JJ arrays, see Refs. [3–6]. In the main text, we show the formula provided by Ref. [5] in Eq. (3).

Let us now evaluate the phase profile for the energy minima of V . For this purpose, we may consider all phase variables as classical parameters $\hat{\varphi}_j \rightarrow \varphi_j$. We can deploy the minimization scheme used in Refs. [7, 8]. For fixed $\varphi := \varphi_J$, and large J , the minimization for the remaining φ_j (with $j \neq 0, J$) results in

$$\varphi_{j=1\dots J-1}^{(0,f)} = \frac{j}{J} (\varphi + 2\pi f) \mod 2\pi. \quad (15)$$

In order to obtain the energy at the minima, we approximate the potential energy V (Supplementary Eq. (14)) for small phase differences $\varphi_j - \varphi_{j-1} \ll 1$ as

$$V \approx \text{const.} + \frac{E_J}{2} \sum_{j=0}^J (\varphi_j - \varphi_{j-1})^2. \quad (16)$$

Importantly, due to the potential energy being periodic in all phases (and all phases being compact), the minima can be defined up to modulo 2π . This corresponds to the compact representation of the local minima, see Fig. 3b in the main text. Plugging in Supplementary Eq. (15) into the approximated potential energy will yield

$$V \approx \frac{E_J}{2} \frac{1}{J} (\varphi + 2\pi f)^2 \quad (17)$$

The first quadratic term corresponds to the inductive energy appearing in the main text in Eq. (7). The Josephson junction array has an effective array inductance $E_L \equiv E_J/J$. In the continuous limit we find $E_L \rightarrow 1/(4e^2 l L)$, with the array (wire) length $L = \Delta x J$, where Δx is the distance between two junctions. The quantum phase slips between different minima f are in the main text described by the operators $e^{\pm i\hat{S}} \equiv \sum_f |f \pm 1\rangle_{\varphi_J} \langle f|_{\varphi_J}$. The

feature usually neglected is the φ -dependence of the f -basis. Namely, we can think of the minima solutions as wave functions centered around the minima. This gives rise to the φ -dependent QPS operators, see also main text.

B. Local versus non-local charge definition and renormalization of E_S

As we discussed in the main text, neglecting charge quantization in QPS junctions is intimately connected to neglecting the phase dependence of the QPS basis $|f\rangle$. Here we elucidate this point. And we show how the phase dependence leads to a renormalization of E_S .

As already stated above, we can think of the minima given in Supplementary Eq. (15) as the reference points around which the wave function $|f\rangle$ is centered. Since the minima in Supplementary Eq. (15) depend on the total phase difference φ between the two contacts linked by the QPS junction, the basis of $\hat{f}$ is likewise φ -dependent, $|f\rangle_\varphi$. As pointed out in the main text, this phase dependence of the minima can be removed by a unitary transformation, $U(\varphi) = \sum_f |\tilde{f}\rangle \langle f|_\varphi$.

In the space of the individual phases φ_j , this unitary is simply a translation operator,

$$U = \prod_{j=1}^{J-1} e^{-i \frac{j}{J} \varphi \hat{N}_j}. \quad (18)$$

This realization allows us now to formulate the two different charges. Originally, φ is only defined on the rightmost contact. Its conjugate charge, here denoted as N is guaranteed to have integer eigenvalues in the finite element model given by the Hamiltonian in Supplementary Eqs. (13) and (14), because it is assigned to a sharply defined region, see also the arguments in Ref. [8]. The charge emerging from the unitary transformation on the other hand is given by

$$\hat{\tilde{N}} = \hat{N} + iU^\dagger \partial_\varphi U = \hat{N} + \sum_{j=1}^{J-1} \frac{j}{J} \hat{\tilde{N}}_j. \quad (19)$$

We now see very explicitly, that while each local charge within the array, $\hat{N}_j$, has again integer eigenvalues, the transformed charge $\hat{\tilde{N}}$ does not, due to the prefactor $j/J < 1$ obviously being a rational number, which, in the continuum limit $J \rightarrow \infty$ approaches an irrational value. As a matter of fact, the transformed charge $\hat{\tilde{N}}$ can be understood as a “fuzzy” measurement of the charge, with a support falling off linearly along the JJ-array, see Supplementary Fig 3c in the main text. Equivalently, we can regard these two charges as a basis choice on the level of the Josephson junction array Hamiltonian, H_{JJA} , Eq. (12). The quantized charge corresponds to the choice

where φ enters only on the right-most junction in the potential energy term V in Supplementary Eq. (14). The fuzzy charge corresponds to the choice where φ enters in a perfectly distributed way in each junction energy, with a participation ratio $1/J$, that is $\cos(\varphi_j - \varphi_{j-1} + \varphi/J)$. Any other charge, with a different spatial profile, could now be represented on the same footing, with yet another spatial distribution of how φ enters the individual Josephson junction energies in V . Note that while the charge can either be quantized or not, the size of the low-energy Hilbert space does not change. This illustrates the importance of conserving the size of the low-energy Hilbert space, one of our main statements in the main text.

Furthermore, as we argue in the main text, if we shunt the QPS junction with a capacitor, the resulting Hamiltonian describing the whole circuit now depends on the spatial resolution with which the capacitor couples to the charge. That is, given a certain electrostatic situation, we have to make the basis choice inside V (the distribution of the phase φ along the individual junction contributions in V) according to that spatial resolution. In the language of the node flux approach, the phase difference φ is first assumed to be a scalar (the default assumption in H_{JJA} , which acquires a finite kinetic energy due to the capacitor. In a second step, we canonically quantize the charge and phase across the QPS junction. Therefore, in order to describe the effect of a capacitor on the Lagrangian level, we need to account for a time-dependent φ . The appropriate low-energy treatment is achieved by transforming the Hamiltonian H_{JJA} in such a way that the resulting low-energy basis $|f\rangle$ is independent of φ (such that the charge-phase quantization procedure can be performed without complications).

As just discussed, $|f\rangle$ is φ -independent in the maximally fuzzy regime, corresponding to a measurement of the charge $\hat{N}$ as defined in Supplementary Eq. (19). Hence, if the capacitor couples to this maximally fuzzy charge, and we choose H_{JJA} in the basis where the phase drop φ is equivalently distributed along all junction elements, we are already in the basis where $|f\rangle$ does not depend on φ , and no further basis transformation is required. If the capacitor couples to any other charge profile, this is no longer true, and we have to make a basis transformation, which is time-dependent due to $\dot{\varphi} \neq 0$ in the presence of the capacitor. This time-dependent transformation yields the extra term $-i\dot{\varphi}U\partial_\varphi U^\dagger$ in the Hamiltonian of the Josephson junction array. For instance, if the capacitor couples to the rightmost charge, defined as N within this section, then the time-dependent transformation U is exactly as given in Supplementary Eq. (18). As a consequence, the junction array Hamiltonian H_{JJA} receives extra offset charge terms $\sim \frac{2}{J}\dot{\varphi}N_j$, which enter as vector potentials in the kinetic energy term T in Supplementary Eq. (13). Crucially, as we pointed out in the main text, Ref. [5] showed that the resulting quantum phase slip amplitude E_S of the junction array depends

on the offset charges N_{gj} on the individual islands. With our time-dependent unitary transformation, we get an additional, dynamical contribution to the offset charges, $\sim \frac{2}{J}\dot{\varphi}$, hence the resulting $\dot{\varphi}$ -dependence of E_S . The above treatment can be significantly generalized to the case of an arbitrary spatial resolution for the charge with which the capacitor couples across the QPS junction. Hence, we get a generic shift of the offset charges by $\sim \alpha_j \dot{\varphi}$ where α_j represents the spatial profile, as stated in the main text.

SUPPLEMENTARY NOTE IV: Solution for two QPS junctions

Here, we explain how to calculate the energy spectrum given in Fig. 9 for the system consisting of two QPS-junctions connected in series with a gate voltage in between (Fig. 8), with the Hamiltonian given in the main text in Eq. (20). In principle the compactness condition (that is, the periodic constraints in Eqs. (9) and (10), for circuits including one or more QPS junctions) are sufficient to compute the correct energy eigenspectra. The (superfluous) larger Hilbert space is carried around for the full calculation of the eigenvectors and eigenenergies of the Schrödinger equation, and can be reduced by imposing the periodic constraints. However, our new treatment also allows to calculate right from the start with the reduced Hilbert space, allowing for significant reduction of the computational effort.

For a generic regime, the Hamiltonian in Eq. (20) can be readily solved numerically by discretizing φ -space and imposing the boundary condition Eq. (21) on the discrete lattice. However, there is a particularly simple regime, for $E_{Si} > E_{Li}$ ($i = 1, 2$) where we can approximate the effect of the two QPS junctions as effective Josephson junction with a Josephson energy given as

$$E_{J,\text{eff}} \approx 16 \sqrt[4]{\frac{E_L E_S^3}{2}} e^{-\sqrt{\frac{8E_S}{\pi^2 E_L}}} . \quad (20)$$

Thus we find a new effective Hamiltonian of the form

$$H_{\text{eff}} = -E_{J,\text{eff}1} \cos(\hat{\varphi}) - E_{J,\text{eff}2} \cos(\hat{\varphi} - \phi_{\text{ext}}) + E_C \left(\hat{N} + N_g \right)^2 , \quad (21)$$

where $E_{J,\text{eff},1,2}$ are given by Eq. (20), by inserting the QPS parameters E_{S1}, E_{S2} and E_{L1}, E_{L2} , respectively. This Hamiltonian is straightforwardly mapped onto a regular charge qubit Hamiltonian, whose solutions are known, via the Mathieu functions [9]. These solutions give rise to the energy spectrum for the compact node-flux approach, see third panel in Fig. 9. The corresponding standard node-flux and loop-charge results (first and second panels) can be obtained by replacing N_g , respectively, ϕ_{ext} by Bloch vectors k , and plotting the resulting energy band widths.

-
- [1] J. Ulrich and F. Hassler, Dual approach to circuit quantization using loop charges, *Phys. Rev. B* **94**, 094505 (2016).
- [2] R.-P. Riwar and D. P. DiVincenzo, Circuit quantization with time-dependent magnetic fields for realistic geometries, *npj Quantum Information* **8**, 36 (2022).
- [3] K. A. Matveev, A. I. Larkin, and L. I. Glazman, Persistent current in superconducting nanorings, *Phys. Rev. Lett.* **89**, 096802 (2002).
- [4] V. Gurarie and A. M. Tsvelik, A superconductor-insulator transition in a one-dimensional array of josephson junctions, *Journal of Low Temperature Physics* **135**, 245 (2004).
- [5] V. E. Manucharyan, N. A. Masluk, A. Kamal, J. Koch, L. I. Glazman, and M. H. Devoret, Evidence for coherent quantum phase slips across a josephson junction array, *Phys. Rev. B* **85**, 024521 (2012).
- [6] M. Houzet and L. I. Glazman, Microwave spectroscopy of a weakly pinned charge density wave in a superinductor, *Phys. Rev. Lett.* **122**, 237701 (2019).
- [7] G. Catelani, R. J. Schoelkopf, M. H. Devoret, and L. I. Glazman, Relaxation and frequency shifts induced by quasiparticles in superconducting qubits, *Phys. Rev. B* **84**, 064517 (2011).
- [8] R.-P. Riwar, Charge quantization and detector resolution, *SciPost Physics* **10**, 093 (2021).
- [9] A. Cottet, Implementation of a quantum bit in a superconducting circuit (2002).