Nonlinear Phase Gates as Airy Transforms of the Wigner Function - Supplementary Material

Darren W. Moore* and Radim Filip†
Department of Optics, Palacký University, 17. listopadu 1192/12, 771 46 Olomouc, Czech Republic

I. ROBUSTNESS OF WIGNER NEGATIVITY FOR THERMAL STATES

The full expression for the Wigner function of the nonlinear phase states (see main text, Methods) allows us to directly consider the effect of initial Gaussian thermal noise on the negativity in the Wigner function induced by the nonlinear phase states. It follows directly from the analytical form of the $W_{\mathcal{G}_A}$ that the negativity must survive arbitrarily large initial thermal noise $\bar{n}$. The Wigner function is composed of a product of positive functions with an Airy function. Since the Airy function is always negative at at least one point, it follows that the Wigner function is always negative at at least one point.

To illustrate this more concretely consider that the cubic phase gate acting on a thermal state (see also Fig. 1) results in the Wigner function

$$W_{\mathrm{th}}(q,p)=\frac{2^{\frac{2}{3}}e^{\frac{4\bar{n}(1+\bar{n})q^2}{1+2\bar{n}}+\frac{(1+2\bar{n})^3+6(1+2\bar{n})\gamma p}{6\gamma^2}}\mathrm{Ai}\left(\frac{1+4\bar{n}(1+\bar{n})+4\gamma(p+\gamma q^2)}{(2\gamma)^{\frac{4}{3}}}\right)}{\sqrt{\pi(1+\bar{n})}|\gamma^{\frac{1}{3}}|}. \tag{1}$$

Due to the parabolic symmetry of this function we can take a cut of the Wigner function at $q=0$, which shows the negative ripples along the p axis and has the form

$$W_{\mathrm{th}}(0,p)=\frac{2^{\frac{2}{3}}e^{\frac{(1+2\bar{n})^3+6(1+2\bar{n})\gamma p}{6\gamma^2}}\mathrm{Ai}\left(\frac{1+4\bar{n}(1+\bar{n})+4\gamma p}{(2\gamma)^{\frac{4}{3}}}\right)}{\sqrt{\pi(1+\bar{n})}|\gamma^{\frac{1}{3}}|}. \tag{2}$$

For $p<0$, the Airy function always provides oscillations between positive and negative values for any finite $\bar{n}$ (see Fig. 1, bottom row). However the exponential term involving p is a decaying function which reduces the amplitude of the oscillations, with the reduction proportional to $\bar{n}$. The Airy oscillations in the negative regions are thus suppressed to zero *from below* and reach zero in the limit $\bar{n}\to\infty$. Similar arguments can be applied to the results of other nonlinear phase gates.

II. NUMERICAL INTEGRATION AS COMPARED WITH AIRY TRANSFORMS

Interpreting the cubic and quartic phase gates as Airy transforms potentially provides some computational advantage, in addition to the broad analytical results mentioned in the main text. Here we illustrate briefly in what form this advantage might take by comparing the direct numerical integration of the Wigner function with the analytical result. In fact, the simplest possible example comes from 'cuts' or 'slices' of the Wigner function and in this case the advantage of the analytical expressions is already visible.

Consider the cubic phase state $U_3|0\rangle$, which has the Wigner function (with $\hbar=1$)

$$W(q,p)=\frac{1}{\pi}\int e^{2ipt}e^{2i\frac{\gamma t^3}{3}}\frac{e^{-\frac{1}{2}(q-t)^2}}{\pi^{\frac{1}{4}}}\frac{e^{-\frac{1}{2}(q+t)^2}}{\pi^{\frac{1}{4}}} \tag{3}$$

$$=\frac{2^{\frac{2}{3}}e^{\frac{1+6\gamma p}{6\gamma^2}}}{\sqrt{\pi}|\gamma|^{\frac{1}{3}}}\mathrm{Ai}\left(\frac{1+4\gamma\mathcal{S}_3(p)}{(2\gamma)^{\frac{4}{3}}}\right). \tag{4}$$

Then the Wigner cut $W(0,p)$ can be calculated both directly and by numerical integration. We do not aim for a detailed numerical analysis, or take advantage of specialised techniques relevant to the kind of integral being calculated. Rather, we straightforwardly compare the times required for the common mathematical software Mathematica using the command NIntegrate on its default settings to evaluate 101 evenly spaced points of the

* darren.moore@upol.cz
† filip@optics.upol.cz

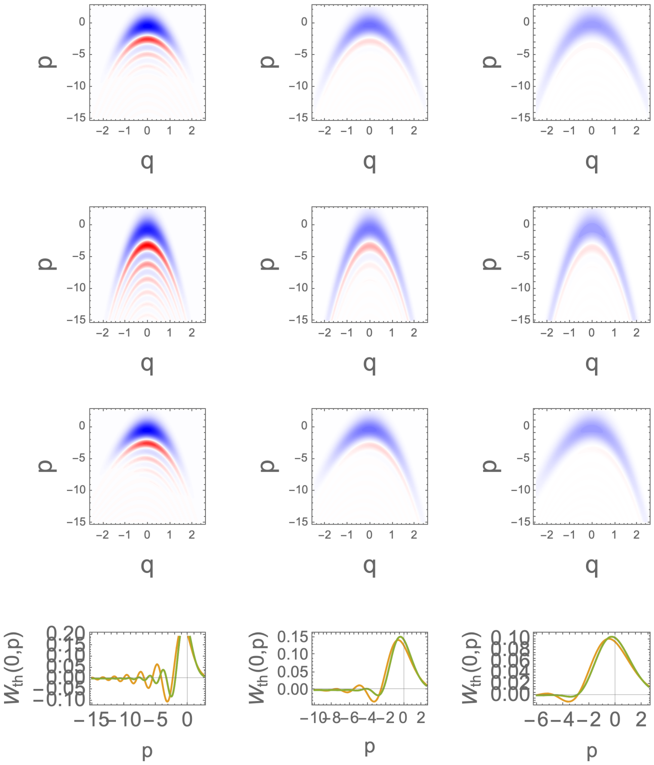

FIG. 1. The Wigner functions for the cubic phase gate acting on various thermal states. Horizontally we set $\bar{n} = 0, 0.5, 1$ and verically $\gamma_3 = 2$ (top), $\gamma_3 = 4$ (middle) and the quartic-bounded cubic gate (bottom) with $\gamma_3 = 2$ and $\gamma_4 = 0.2$. The nonclassical fluctuations are suppressed by the thermal noise, while the state spreads out in phase space, as seen by the extension of the parabolic 'arms'. Below these we show the Wigner cuts at $q = 0$ with blue, yellow and green following the vertical ordering.

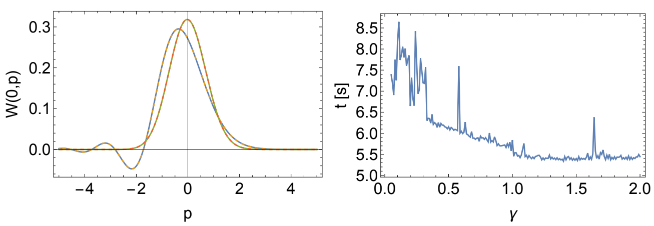

FIG. 2. The Wigner cut $W(0,p)$ for the cubic phase state with $\gamma = 0.05, 1$. The accuracy with respect to the analytical calculation is very good, although some values for $\gamma = 0.05$ were erroneously reported as complex-valued (not shown). Curves from numerical integration are dashed. The lower values of γ appear to require more time to estimate numerically.

Wigner cut in the range $-5 \le p \le 5$. The time taken is estimated using AbsoluteTiming. Strikingly, the time is greater for low values of γ, as demonstrated in Fig. 2.

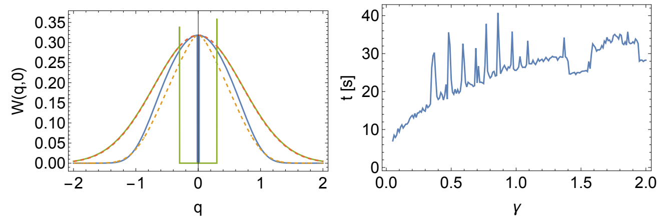

FIG. 3. The Wigner cut $W(q,0)$ for the quartic phase state with $\gamma = 0.05, 1$. The accuracy with respect to the analytical calculation very good, although again some values for $\gamma = 0.05$ were erroneously reported as complex-valued (not shown). Curves from numerical integration are dashed. The higher values of γ appear to require more time to estimate numerically in contrast with the cubic state. In this case, the plotting software fails for the analytical expression as some numbers are too small to be represented numerically.

The quartic phase state $U_4 |0\rangle$ has the Wigner function

$$W(q,p) = \frac{1}{\pi}\int e^{2ipt} e^{2iq\gamma t^3} \frac{e^{-\frac{1}{2}(q-t)^2}}{\pi^{\frac{1}{4}}} \frac{e^{-\frac{1}{2}(q+t)^2}}{\pi^{\frac{1}{4}}} \tag{5}$$

$$= \frac{2^{\frac{2}{3}} e^{\frac{1+6\gamma pq}{6(q\gamma)^2}}}{\sqrt{\pi}|q\gamma|^{\frac{1}{3}}} \mathrm{Ai}\left(\frac{1+4q\gamma\mathcal{S}_4(p)}{(2q\gamma)^{\frac{4}{3}}}\right). \tag{6}$$

Consider the Wigner cut $W(q,0) = \frac{2^{\frac{2}{3}} e^{\frac{1}{6(q\gamma)^2}}}{\sqrt{\pi}|q\gamma|^{\frac{1}{3}}} \mathrm{Ai}\left(\frac{1+4\gamma^2 q^4}{(2q\gamma)^{\frac{4}{3}}}\right)$. In fact, this immediately poses numerical problems for Mathematica, due to the denominator when $q \to 0$. While Mathematica evaluates $\lim_{q\to 0} W(q,0) = \frac{1}{\pi}$, the plotting software fails to capture this (Fig. 3), due to some expressions involving numbers that are too small to be represented numerically. More precisely, the argument of the Airy function can be large even near the phase space origin, and the evaluation of the function at such values is a small number. The numerical integration also seems to take significantly longer than the cubic state, however it does seem to be able to avoid some of the problem of small numbers.

We can also take note of the accuracy differences between the analytical calculations and numerical integration. In Fig. 4 we show Wigner cuts for the cubic and quartic phase states from the analytical calculation and numerical calculation. The accuracy of the numerical integration is high for these examples, but the time taken to produce the plots is much higher although certainly not prohibitive.

III. PROBABILITY DISTRIBUTIONS OF THE CANONICAL VARIABLES

In the Heisenberg picture it is easy to see that the variable $\hat{q}$ is invariant under the action of the cubic phase gate, with the obvious consequence that if the initial probability distribution of $\hat{q}$ is known then it remains

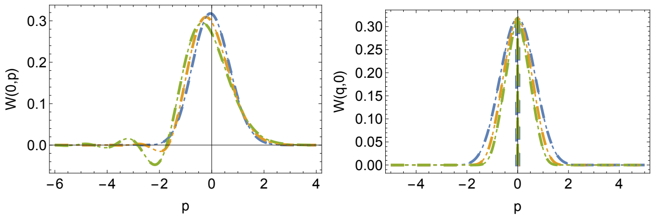

FIG. 4. The Wigner cut $W(0,p)$ for the cubic phase state (left) and $W(q,0)$ for the quartic phase state with $\gamma = 0.1$ (blue), $\gamma = 0.5$ (yellow) and $\gamma = 1$ (green). The thick dashed lines are from analytical expressions and the thin dashed lines are from numerical integration. The accuracy is high for these examples, although analytical expressions are much faster to plot, for example the Wigner cut plot for the cubic phase state using AbsoluteTiming takes approximately 0.03 seconds while the numerically integrated cut using NIntegrate takes approximately 30 seconds.

known after the gate application. The same is not true of the distribution associated with $\hat{p}$ which becomes drastically non-Gaussian; straightforwardly, the calculation of the probability amplitude for $\hat{p}$ for the ideal cubic phase state involves the integral

$$\langle p|e^{i\frac{\gamma}{3}\hat{q}^3}|0\rangle_p = \int \langle p|x\rangle\, e^{i\frac{\gamma}{3}x^3}\, \langle x|0\rangle_p\, dx \tag{7}$$

$$= \frac{1}{2\pi}\int e^{ixp} e^{i\frac{\gamma}{3}x^3} dx \tag{8}$$

$$= \frac{\gamma^{-\frac{1}{3}}}{\pi}\int_0^\infty e^{i\gamma^{-\frac{1}{3}}px} e^{ix^3} dx \tag{9}$$

$$= \gamma^{-\frac{1}{3}}\mathrm{Ai}(\gamma^{-\frac{1}{3}}p)\,, \tag{10}$$

where we have rescaled the integration variable $x \to \gamma^{-\frac{1}{3}}x$.

The cubic phase gate can be interpreted as applying an Airy transform to the momentum wavefunction. While the probability could also be calculated as the marginal of the associated Wigner distribution, the Airy transform method can bypass the potentially troublesome integral of an Airy function.

For pure states, the momentum space wavefunction $\phi(p) = \frac{1}{\sqrt{2\pi\hbar}}\int e^{\frac{-ixp}{\hbar}}\psi(x)dx$ is just the Fourier transform of the position wavefunction $\psi(x)$. For an arbitrary initial state $|\psi\rangle$ we have that the effect of the cubic phase gate U_3 on the position wavefunction is to add a cubic phase, $\langle x|U_3|\psi\rangle = e^{-i\frac{\gamma x^3}{3\hbar}}\psi(x)$. Then the momentum wavefunction is

$$\langle p|U_3|\psi\rangle = \mathcal{F}\{e^{-i\frac{\gamma x^3}{3\hbar}}\psi(x)\} \tag{11}$$

$$= \mathcal{F}\{e^{-i\frac{\gamma x^3}{3\hbar}}\} * \mathcal{F}\{\psi(x)\} \tag{12}$$

$$= \mathrm{Ai}(p;\alpha) * \phi(p)\,, \tag{13}$$

where we used the convolution theorem and the Fourier pair for the Airy function. What remains is to specify α. If we examine the momentum wavefunction in detail we find

$$\langle p|U_3|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int e^{-i\left(\frac{\gamma x^3}{3\hbar}+\frac{xp}{\hbar}\right)}\psi(x)dx \tag{14}$$

$$= \frac{1}{2\pi\hbar}\int e^{-i\left(\frac{\gamma x^3}{3\hbar}+\frac{xp}{\hbar}\right)}\int e^{i\frac{xt}{\hbar}}\phi(t)dxdt \tag{15}$$

$$= \frac{1}{2\pi\hbar}|\hbar\gamma^{-1}|^{\frac{1}{3}}\int e^{i\left(\frac{x^3}{3}-(\hbar^2\gamma)^{\frac{1}{3}}x(t-p)\right)}\phi(t)dxdt\,, \tag{16}$$

where we used the Fourier pairing between position and momentum wavefunctions and rescaled the integration variable $x = -\left(\hbar\gamma^{-1}\right)^{\frac{1}{3}}$. It follows that we may identify $\alpha = -(\hbar^2\gamma)^{-\frac{1}{3}}$. The effect of the cubic phase gate on the momentum wavefunction therefore is to convolve it with an Airy function, which is the definition of the Airy transform of $\phi(p)$. The probability distribution is then the modulus squared of this new wavefunction. Equivalently, the momentum probability distribution is a product of Airy transforms

$$\mathcal{P}(p) = \langle p|U_3|\psi\rangle\,\langle\psi|U_3^\dagger|p\rangle \tag{17}$$

$$= \mathrm{Ai}(p) * \phi(p)\cdot\mathrm{Ai}(p) * \phi^*(p)\,. \tag{18}$$

In the case that the state is not pure there are no wavefunctions and instead the probability distribution is given by expectation values of the density matrix with the projectors of the canonical variables. That is, $\mathcal{P}(p) = \mathrm{Tr}\left(|p\rangle\langle p|\,U_3\rho U_3^\dagger\right) = \langle p|U_3\rho U_3^\dagger|p\rangle$. One may decompose the density matrix as a convex sum of pure states $\rho = \sum_i p_i\,|\psi_i\rangle\,\langle\psi_i|$. Then the probability distribution is just a convex sum of convolutions of the respective momentum wavefunctions with Airy functions

$$\mathcal{P}(p) = \sum_i p_i \mathrm{Ai}(p) * \phi_i(p)\cdot\mathrm{Ai}(p) * \phi_i^*(p)\,. \tag{19}$$

This method does not generalise to the quartic phase gate as there is no cancellation of the fourth order integration variable as in the Wigner function.

IV. WIGNER FUNCTION OF THE CPE STATE

In Ref [1] the Wigner function of the CPE state is calculated explicitly. Here we show that we recover this result using the Airy transform formalism. The state is defined as

$$|z,\gamma,\theta\rangle = U_3 U_{BS}(\theta)\left[S(z)\otimes S^\dagger(z)\right]|0\rangle^{\otimes 2}\,, \tag{20}$$

where $U_{BS}(\theta) = e^{i\frac{\theta}{\hbar}(\hat{q}_1\hat{p}_2-\hat{q}_2\hat{p}_1)}$ is a beamsplitter operation and $S(z)$ is the squeezing operator (see Wigner Function of the Cubic and Quartic Phase States in the main text). Select $\theta = \frac{\pi\hbar}{4}$ and $z = \ln r > 0$ to recover the state from the reference, although for simplicity here both squeezing operations are assumed to have equal strength. The Gaussian operators induce symplectic transformations on the phase space variables represented by the symplectic matrices:

$$S_{S(r)} = \mathrm{diag}\left(r, \frac{1}{r}\right) \tag{21}$$

$$S_{U_{BS}(\frac{\pi}{4})} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} . \tag{22}$$

Therefore the effect of the squeezing and beamsplitter operations can be calculated by substitution of these transformations into the Wigner function of the product of two harmonic oscillator ground states. The result is the entangled Gaussian two-mode squeezed state with Wigner function

$$W_{\mathrm{TMSS}}(\mathbf{q}, \mathbf{p}) = \frac{e^{-\frac{(q_1-q_2)^2+(p_1+p_2)^2+((p_1-p_2)^2+(q_1+q_2)^2)r^4}{2\hbar r^2}}}{(\pi\hbar)^2} . \tag{23}$$

To apply the cubic phase gate, we need only perform the Airy transform with respect to the variable p_1. Much of the above expression factors out of the Airy transform, and we are left with

$$W_{\mathrm{CPE}}(\mathbf{q}, \mathbf{p}) = \sqrt{\pi}\frac{e^{-\frac{p_2^2+(q_1-q_2)^2+(p_2^2+(q_1+q_2)^2)r^4}{2\hbar r^2}}}{(\pi\hbar)^2} e^{-\frac{(1-r^4)^2p_2^2}{2\hbar r^2(1+r^4)}} \mathcal{A}_{\frac{\hbar}{2}\left(\frac{2\gamma}{\hbar}\right)^{\frac{1}{3}}} \left[\frac{e^{-\left(\sqrt{\frac{1+r^4}{2\hbar r^2}}p_1 + \frac{p_2(1-r^4)}{\sqrt{2\hbar r^2(1+r^4)}}\right)^2}}{\sqrt{\pi}} \right] (\mathcal{S}_3(p_1)) . \tag{24}$$

This is an Airy transform of a Gaussian function, where we have completed the square for p_1. The scaling and translation rules for Airy transforms state that if $\phi_\alpha(x)$ is the Airy transform of $f(x)$ then $\phi_{\alpha k}(kx)$ is the Airy transform of $f(kx)$ and $\phi_\alpha(x+s)$ is the Airy transform of $f(x+s)$. Therefore we can use the Airy transform of the Gaussian function and the translation rule with $s = \frac{p_2(1-r^4)}{\sqrt{2\hbar r^2(1+r^4)}}$ followed by the scaling rule with $k = \sqrt{\frac{1+r^4}{2\hbar r^2}}$ and to get the required Airy transform.

The resulting Wigner function is

$$W_{\mathrm{CPE}}(\mathbf{q}, \mathbf{p}) = \frac{2^{\frac{7}{6}} r e^{\frac{4r^6}{3(1+r^4)^3\gamma^2\hbar}}}{\pi^{\frac{3}{2}}\hbar^{\frac{13}{6}}\sqrt{1+r^4}\gamma^{\frac{1}{3}}} e^{\frac{2r^2(p_1+p_2+(p_1-p_2)r^4)}{(1+r^4)^2\gamma\hbar}} e^{-\frac{4r^4p_2^2+((q_2-q_1+(q_1+q_2)r^4)^2)}{2r^2(1+r^4)\hbar}} \mathrm{Ai}\left(\frac{2^{\frac{2}{3}}r^4}{(1+r^4)^2\gamma^{\frac{4}{3}}\hbar^{\frac{2}{3}}} + \frac{2^{\frac{2}{3}}(p_1+p_2+(p_1-p_2)r^4))}{(1+r^4)\gamma^{\frac{1}{3}}\hbar^{\frac{2}{3}}} + \frac{2^{\frac{2}{3}}\gamma^{\frac{2}{3}}q_1^2}{\hbar^{\frac{2}{3}}}\right) . \tag{25}$$

[1] Peter McConnell, Oussama Houhou, Matteo Brunelli, and Alessandro Ferraro. Unconditional Wigner-negative mechanical entanglement with linear-and-quadratic optomechanical interactions, February 2023. arXiv:2302.03702 [quant-ph].