Supplementary Information for "Characterizing Quantum Codes via the Coefficients in Knill-Laflamme Conditions"

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Supplementary Information A: Details of the ((6,2,3)) codes

1. The $|S_i\rangle$ basis

In the six-qubits system $q_1q_2q_3q_4q_5q_6$, we choose a subspace for $q_2q_3q_4q_5q_6$ with basis $|S_i\rangle$:

$$
\begin{aligned}
|S_1\rangle &= \frac{1}{\sqrt{2}}\left(|00001\rangle + |11110\rangle\right),\\
|S_2\rangle &= \frac{1}{\sqrt{2}}\left(|00010\rangle + |11101\rangle\right),\\
|S_3\rangle &= \frac{1}{\sqrt{2}}\left(|00100\rangle + |11011\rangle\right),\\
|S_4\rangle &= \frac{1}{\sqrt{2}}\left(|01000\rangle + |10111\rangle\right),\\
|S_5\rangle &= \frac{1}{\sqrt{2}}\left(|10000\rangle + |01111\rangle\right).
\end{aligned}
$$

Their reduced density matrices (RDM) have clean forms. For example, 1-RDM

$$2\mathrm{Tr}_{q_r q_s q_t q_\mu}\left[|S_i\rangle\langle S_j|\right] = \delta_{ij} I_2$$

2-RDM are

$$4\mathrm{Tr}_{q_r q_s q_t}\left[|S_i\rangle\langle S_i|\right] = I_4 + (-1)^{\delta}\, Z\otimes Z$$

where the sign $\delta \in \{0,1\}$ depends on how which qubits are chosen, for example:

$$
\begin{aligned}
4\mathrm{Tr}_{q_2q_3q_4}\left[|S_1\rangle\langle S_1|\right] &= I_4 - Z\otimes Z,\\
4\mathrm{Tr}_{q_2q_3q_6}\left[|S_1\rangle\langle S_1|\right] &= I_4 + Z\otimes Z.
\end{aligned}
\tag{A1}
$$

Still, 2-RDM but with different basis

$$4\mathrm{Tr}_{q_r q_s q_t}\left[|S_i\rangle\langle S_j|\right] = \left(X\otimes X + Y\otimes Y\right)\delta^{(i,j)},$$

$$4\mathrm{Tr}_{q_2q_3q_4}\left[|S_1\rangle\langle S_2|\right] = \left(X\otimes X + Y\otimes Y\right),$$

$$\mathrm{Tr}_{q_2q_3q_5}\left[\left|S_1\right\rangle\left\langle S_2\right|\right]=0.$$

Our code are designed in the following 10-dimensional subspace

$$\{\left|0\right\rangle,\left|1\right\rangle\}\otimes\{\left|S_1\right\rangle,\left|S_2\right\rangle,\left|S_3\right\rangle,\left|S_4\right\rangle,\left|S_5\right\rangle\}$$

From above, all 2-RDM of pure states in this subspace will be in the form:

$$\mathrm{RDM}\left(q_i,q_j\right)=\frac{1}{4}I_4+\alpha_{ij}\left(XX+YY\right)+\beta_{ij}ZZ.$$

The logical states are defined as

$$\left|x_i\right\rangle=\gamma_i\left|0\right\rangle+\gamma_{i+5}\left|1\right\rangle,\ i=1,2,3,4,5$$

$$\left|y_i\right\rangle=\gamma_{i+5}^*\left|0\right\rangle-\gamma_i^*\left|1\right\rangle,\ i=1,2,3,4,5$$

$$\left|0_L\right\rangle=\sum_{i=1}^{5}\left|x_i\right\rangle\left|S_i\right\rangle,\ \left|1_L\right\rangle=\sum_{i=1}^{5}\left|y_i\right\rangle\left|S_i\right\rangle.$$

It is convenient to introduce the following shorthand notation for the 2-RDMs:

$$\begin{aligned}M_{ij}^{xx}&=\left\langle x_i\right|\left.x_j\right\rangle, \quad M_{ij}^{yx}=\left\langle y_i\right|\left.x_j\right\rangle,\\ M_{ij}^{xy}&=\left\langle x_i\right|\left.y_j\right\rangle, \quad M_{ij}^{yy}=\left\langle y_i\right|\left.y_j\right\rangle.\end{aligned}$$

$$\begin{aligned}M\left(\square_1,\square_2,\square_3\right)=\square_1&\left(\left|00\right\rangle\left\langle 00\right|+\left|11\right\rangle\left\langle 11\right|\right)\\ +\square_2&\left(\left|01\right\rangle\left\langle 01\right|+\left|10\right\rangle\left\langle 10\right|\right)\\ +\square_3&\left(\left|01\right\rangle\left\langle 10\right|+\left|10\right\rangle\left\langle 01\right|\right).\end{aligned}$$

$M\left(\square_1,\square_2,\square_3\right)$ is a 4-by-4 matrix with diagonal elements $\square_1$ and $\square_2$, off-diagonal element $\square_3$. Four important properties about the tensor M will be used later:

$$M_{ii}^{xy}=0 \tag{A2}$$

$$M_{ii}^{xx}=M_{ii}^{yy} \tag{A3}$$

$$M_{ij}^{xx}=\gamma_i^*\gamma_j+\gamma_{i+5}^*\gamma_{j+5}=M_{ji}^{yy} \tag{A4}$$

$$\begin{aligned}M_{ij}^{xy}+M_{ji}^{xy}=&\left(\gamma_i^*\gamma_{j+5}^*-\gamma_{i+5}^*\gamma_j^*\right)\\ &+\left(\gamma_j^*\gamma_{i+5}^*-\gamma_{j+5}^*\gamma_i^*\right)=0.\end{aligned} \tag{A5}$$

Since $\left\langle 0_L\right|\left.0_L\right\rangle=\sum_i\left\langle x_i\right|\left.x_i\right\rangle$, we have

$$\left\langle 0_L\right|\left.0_L\right\rangle=\sum_i M_{ii}^{xx}=\sum_i\gamma_i\gamma_i^*=1.$$

With such a basis chosen, our code can already satisfy most of KL-conditions.

2. The 2-RDMs and KL conditions

a. RDM-(q_2, q_3) We first introduce RDM-(q_2, q_3):

$$\begin{aligned}
\mathrm{Tr}_{(23)^c}\left[\left|0_L\right\rangle\left\langle 0_L\right|\right] &= M\left(\Box_1, \Box_2, \Box_3\right), \\
2\Box_1 &= M_{11}^{xx} + M_{22}^{xx} + M_{33}^{xx}, \\
2\Box_2 &= M_{44}^{xx} + M_{55}^{xx}, \\
2\Box_3 &= M_{45}^{xx} + M_{54}^{xx},
\end{aligned} \tag{A6}$$

$$\begin{aligned}
\mathrm{Tr}_{(23)^c}\left[\left|1_L\right\rangle\left\langle 1_L\right|\right] &= M\left(\Box_1', \Box_2', \Box_3'\right), \\
2\Box_1' &= M_{11}^{yy} + M_{22}^{yy} + M_{33}^{yy}, \\
2\Box_2' &= M_{44}^{yy} + M_{55}^{yy}, \\
2\Box_3' &= M_{45}^{yy} + M_{54}^{yy},
\end{aligned} \tag{A7}$$

$$\begin{aligned}
\mathrm{Tr}_{(23)^c}\left[\left|1_L\right\rangle\left\langle 0_L\right|\right] &= M\left(\Box_1'', \Box_2'', \Box_3''\right), \\
2\Box_1'' &= M_{11}^{yx} + M_{22}^{yx} + M_{33}^{yx}, \\
2\Box_2'' &= M_{44}^{yx} + M_{55}^{yx}, \\
2\Box_3'' &= M_{45}^{yx} + M_{54}^{yx}.
\end{aligned} \tag{A8}$$

- From Eq.(A3), we have $\Box_1 = \Box_1'$ and $\Box_2 = \Box_2'$.
- From Eq.(A4), we have $\Box_3 = \Box_3'$.
- From Eq.(A2), we have $\Box_1'' = \Box_2'' = 0$.
- From Eq.(A5), we have $\Box_3'' = 0$

The $|0_L\rangle\langle 0_L|$ RDM is equal to $|1_L\rangle\langle 1_L|$ RDM and others are zero. It should be emphasized that this is true to all RDM$(q_i, q_j), i \neq 1, j \neq 1$.

b. RDM-$(q_{i>1}, q_{j>1})$ Without loss of generality, only $|0_L\rangle\langle 0_L|$ is listed below:

$$\begin{aligned}
\mathrm{Tr}_{(24)^c}\left[\left|0_L\right\rangle\left\langle 0_L\right|\right] &= M\left(\Box_1, \Box_2, \Box_3\right), \\
2\Box_1 &= M_{11}^{xx} + M_{22}^{xx} + M_{44}^{xx}, \\
2\Box_2 &= M_{33}^{xx} + M_{55}^{xx}, \\
2\Box_3 &= M_{35}^{xx} + M_{53}^{xx},
\end{aligned}$$

$$\begin{aligned}
\mathrm{Tr}_{(25)^c}\left[\left|0_L\right\rangle\left\langle 0_L\right|\right] &= M\left(\Box_1, \Box_2, \Box_3\right), \\
2\Box_1 &= M_{11}^{xx} + M_{33}^{xx} + M_{44}^{xx}, \\
2\Box_2 &= M_{22}^{xx} + M_{55}^{xx}, \\
2\Box_3 &= M_{25}^{xx} + M_{52}^{xx},
\end{aligned}$$

$$\begin{aligned}
\mathrm{Tr}_{(26)^c}\left[\left|0_L\right\rangle\left\langle 0_L\right|\right] &= M\left(\Box_1, \Box_2, \Box_3\right), \\
2\Box_1 &= M_{22}^{xx} + M_{33}^{xx} + M_{44}^{xx}, \\
2\Box_2 &= M_{11}^{xx} + M_{55}^{xx}, \\
2\Box_3 &= M_{15}^{xx} + M_{51}^{xx},
\end{aligned}$$

$$\begin{aligned}
\mathrm{Tr}_{(34)^c}\left[\left|0_L\right\rangle\left\langle 0_L\right|\right] &= M\left(\square_1, \square_2, \square_3\right), \\
2\square_1 &= M^{xx}_{11} + M^{xx}_{22} + M^{xx}_{55}, \\
2\square_2 &= M^{xx}_{33} + M^{xx}_{44}, \\
2\square_3 &= M^{xx}_{34} + M^{xx}_{43},
\end{aligned}$$

$$\begin{aligned}
\mathrm{Tr}_{(35)^c}\left[\left|0_L\right\rangle\left\langle 0_L\right|\right] &= M\left(\square_1, \square_2, \square_3\right), \\
2\square_1 &= M^{xx}_{11} + M^{xx}_{33} + M^{xx}_{55}, \\
2\square_2 &= M^{xx}_{22} + M^{xx}_{44}, \\
2\square_3 &= M^{xx}_{24} + M^{xx}_{42},
\end{aligned}$$

$$\begin{aligned}
\mathrm{Tr}_{(36)^c}\left[\left|0_L\right\rangle\left\langle 0_L\right|\right] &= M\left(\square_1, \square_2, \square_3\right), \\
2\square_1 &= M^{xx}_{22} + M^{xx}_{33} + M^{xx}_{55}, \\
2\square_2 &= M^{xx}_{11} + M^{xx}_{44}, \\
2\square_3 &= M^{xx}_{14} + M^{xx}_{41},
\end{aligned}$$

$$\begin{aligned}
\mathrm{Tr}_{(45)^c}\left[\left|0_L\right\rangle\left\langle 0_L\right|\right] &= M\left(\square_1, \square_2, \square_3\right), \\
2\square_1 &= M^{xx}_{11} + M^{xx}_{44} + M^{xx}_{55}, \\
2\square_2 &= M^{xx}_{22} + M^{xx}_{33}, \\
2\square_3 &= M^{xx}_{23} + M^{xx}_{32},
\end{aligned}$$

$$\begin{aligned}
\mathrm{Tr}_{(46)^c}\left[\left|0_L\right\rangle\left\langle 0_L\right|\right] &= M\left(\square_1, \square_2, \square_3\right), \\
2\square_1 &= M^{xx}_{22} + M^{xx}_{44} + M^{xx}_{55}, \\
2\square_2 &= M^{xx}_{11} + M^{xx}_{33}, \\
2\square_3 &= M^{xx}_{13} + M^{xx}_{31},
\end{aligned}$$

$$\begin{aligned}
\mathrm{Tr}_{(56)^c}\left[\left|0_L\right\rangle\left\langle 0_L\right|\right] &= M\left(\square_1, \square_2, \square_3\right), \\
2\square_1 &= M^{xx}_{33} + M^{xx}_{44} + M^{xx}_{55}, \\
2\square_2 &= M^{xx}_{11} + M^{xx}_{22}, \\
2\square_3 &= M^{xx}_{12} + M^{xx}_{21}.
\end{aligned} \tag{A9}$$

c. RDM $(q_1, q_{k>1})$ To satisfy KL-conditions, the RDMs

$$\mathrm{Tr}_{(1\mathrm{k})^c}\left[\left|0_L\right\rangle\left\langle 0_L\right|\right] = \frac{1}{2}\sum_i \left|x_i\right\rangle\left\langle x_i\right| \otimes I_2,$$

$$\mathrm{Tr}_{(1\mathrm{k})^c}\left[\left|1_L\right\rangle\left\langle 1_L\right|\right] = \frac{1}{2}\sum_i \left|y_i\right\rangle\left\langle y_i\right| \otimes I_2$$

$$\mathrm{Tr}_{(1\mathrm{k})^c}\left[\left|1_L\right\rangle\left\langle 0_L\right|\right] = \frac{1}{2}\sum_i \left|y_i\right\rangle\left\langle x_i\right| \otimes I_2$$

must obey

$$\sum_i |x_i\rangle \langle x_i| = \sum_i |y_i\rangle \langle y_i| \,, \ \sum_i |y_i\rangle \langle x_i| = 0.$$

From above, we can derive

$$\gamma = \left[\ a+ib \ \ c+id \ \right] \in \mathbb{C}^{10}, \ a,b,c,d \in \mathbb{R}^5,$$

$$\begin{aligned} a\cdot a = b\cdot b = c\cdot c = d\cdot d = \frac{1}{4}, \\ a\cdot b = a\cdot c = a\cdot d = b\cdot c = b\cdot d = c\cdot d = 0. \end{aligned}$$

$$\sum_i |x_i\rangle \langle x_i| = \frac{1}{2} I_2$$

Thus, vector (a,b,c,d) make four columns of orthogonal group $O(5)$.

3. Expression of λ^*

Let e being the vector orthogonal to a,b,c,d

$$e\cdot e = \frac{1}{4}, e\cdot a = e\cdot b = e\cdot c = e\cdot d = 0, e\in \mathbb{R}^5$$

Then we can prove

$$M^{xx}_{ii} = \frac{1}{4} - e_i^2, M^{xx}_{ij} + M^{xx}_{ji} = -2e_ie_j.$$

Proof. Let's take $i=1$ for example. Denote the unnormalized orthogonal matrix composed from (a,b,c,d,e) as:

$$\begin{aligned} A &= \left[\ a \ \ b \ \ c \ \ d \ \ e\ \right] \\ &= \begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 \\ a_4 & b_4 & c_4 & d_4 & e_4 \\ a_5 & b_5 & c_5 & d_5 & e_5 \end{bmatrix} \end{aligned} \tag{A10}$$

$$\rightarrow AA^T = A^TA = \frac{1}{4}I.$$

which means each column (row) are orthogonal to each other. Then

$$M^{xx}_{11} = a_1^2 + b_1^2 + c_1^2 + d_1^2 = \frac{1}{4} - e_1^2,$$

and similarly

$$\begin{aligned} M^{xx}_{12} + M^{xx}_{21} &= a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2 \\ &= (a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2 + e_1e_2) \\ &\quad - e_1e_2 \\ &= -e_1e_2. \end{aligned} \tag{A11}$$

□

All nonzero components of signature vector can be written as

$$\begin{aligned}\lambda_{X_iX_j} &= \lambda_{Y_iY_j} = -2e_{7-i}e_{7-j},\\ \lambda_{Z_iZ_j} &= 2e_{7-i}^2 + 2e_{7-j}^2, \quad i,j \in \{2,3,4,5,6\}\,.\end{aligned} \tag{A12}$$

For example

$$\lambda_{X_2X_3} = \lambda_{Y_2Y_3} = -2e_4e_5, \lambda_{Z_2Z_3} = 2e_4^2 + 2e_5^2.$$

Then its square of length is

$$\lambda^{*2} = \frac{1}{2} + 8\sum_i e_i^4$$

Proof. Two lemmas:

$$\sum_i \sum_{j\neq i} e_i^2 e_j^2 = \left(\sum_i e_i^2\right)^2 - \sum_i e_i^4 = \frac{1}{16} - \sum_i e_i^4$$

$$\sum_i \sum_{j\neq i} e_i^4 = \sum_i \sum_j e_i^4 - \sum_i e_i^4 = 4\sum_i e_i^4$$

Using lemmas above,

$$\begin{aligned}\lambda^{*2} &= \sum_i \sum_{j>i} 4e_i^2e_j^2 + 4e_i^2e_j^2 + 4\left(e_i^2+e_j^2\right)^2\\ &= \sum_i \sum_{j\neq i} 4e_i^2e_j^2 + 2\left(e_i^2+e_j^2\right)^2\\ &= \sum_i \sum_{j\neq i} 8e_i^2e_j^2 + 2e_i^4 + 2e_j^4\\ &= \frac{1}{2} + 8\sum_i e_i^4\end{aligned}$$

□

4. Structure of SO(4) symmetry

The Lie algebra $\mathfrak{so}(4)$ comprises all 4×4 skew-symmetric matrices. A general skew-symmetric matrix $X \in \mathfrak{so}(4)$ satisfies $X^T = -X$. Such matrices have zeros on the diagonal and contain six independent off-diagonal elements. These non-zero elements correspond to infinitesimal rotations in the six independent planes of four-dimensional space ($x_i - x_j$ plane for $i \neq j$). Each generator can be represented by a matrix E_{ij}, where the (i,j)-th entry is $+1$, the (j,i)-th entry is -1, and all other entries are zero.

To obtain the Lie group SO(4), we exponentiate the Lie algebra elements. Specifically, for any skew-symmetric matrix $X \in \mathfrak{so}(4)$, the corresponding element in SO(4) can be obtained as:

$$\mathrm{SO}(4) = \left\{e^X \mid X \in \mathfrak{so}(4)\right\}$$

Now choose the set

$$K_1 = E_{12} + E_{34}, K_2 = E_{12} - E_{34},$$

$$K_3 = E_{23} + E_{14}, K_4 = E_{23} - E_{14},$$

$$K_5 = E_{13} + E_{24}, K_6 = E_{13} - E_{24}.$$

Notice that

$$\exp(\theta K_1) = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

$$\exp(\theta K_2) = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$\exp(\theta K_3) = \begin{pmatrix} \cos\theta & 0 & 0 & \sin\theta \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ -\sin\theta & 0 & 0 & \cos\theta \end{pmatrix}$$

$$\exp(\theta K_4) = \begin{pmatrix} \cos\theta & 0 & 0 & -\sin\theta \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ \sin\theta & 0 & 0 & \cos\theta \end{pmatrix}$$

$$\exp(\theta K_5) = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$\exp(\theta K_6) = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & \cos\theta & 0 & -\sin\theta \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & \sin\theta & 0 & \cos\theta \end{pmatrix}$$

And

$$\exp(-i\theta X) = \cos\theta I - i\sin\theta X = \begin{bmatrix} \cos\theta & -i\sin\theta \\ -i\sin\theta & \cos\theta \end{bmatrix}$$

$$\exp(-i\theta Y) = \cos\theta I - i\sin\theta Y = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\exp(-i\theta Z) = \cos\theta I - i\sin\theta Z = \begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix}.$$

Write

$$|x_i\rangle = (a_i + ib_i)\,|0\rangle + (c_i + id_i)\,|1\rangle = \begin{bmatrix} a_i + ib_i \\ c_i + id_i \end{bmatrix}.$$

$$|y_i\rangle = (c_i - id_i)\,|0\rangle - (a_i - ib_i)\,|1\rangle = \begin{bmatrix} c_i - id_i \\ -a_i + ib_i \end{bmatrix}.$$

We now compare

$$\exp\theta K_i$$

with the representation of

- $\exp i\theta X_1, \exp i\theta Y_1, \exp i\theta Z_1$
- $\exp i\theta X_L, \exp i\theta Y_L, \exp i\theta Z_L$

as SO(4) rotations

a. $\exp(-i\theta X_1)$

$$\exp(-i\theta X_1)|x_i\rangle = \begin{bmatrix} \cos\theta & -i\sin\theta \\ -i\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_i + ib_i \\ c_i + id_i \end{bmatrix}$$

$$= \begin{bmatrix} (\cos\theta a_i + \sin\theta d_i) + i(\cos\theta b_i - \sin\theta c_i) \\ (\sin\theta b_i + \cos\theta c_i) + i(-\sin\theta a_i + \cos\theta d_i) \end{bmatrix}$$

As a SO(4) rotation, this is

$$\begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & 0 & -\sin\theta \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ \sin\theta & 0 & 0 & \cos\theta \end{bmatrix},$$

which corresponds to $\exp(\theta K_4)$.

b. $\exp(-i\theta Y_1)$

$$\exp(-i\theta Y_1)|x_i\rangle = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_i + ib_i \\ c_i + id_i \end{bmatrix}$$

$$= \begin{bmatrix} (\cos\theta a_i - \sin\theta c_i) + i(\cos\theta b_i - \sin\theta d_i) \\ (\sin\theta a_i + \cos\theta c_i) + i(\sin\theta b_i + \cos\theta d_i) \end{bmatrix}$$

As a SO(4) rotation, this is

$$\begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{bmatrix},$$

which corresponds to $\exp(\theta K_5)$.

c. $\exp(-i\theta Z_1)$

$$\exp(-i\theta Z_1)|x_i\rangle =$$

$$\begin{bmatrix} \cos\theta - i\sin\theta & 0 \\ 0 & \cos\theta + i\sin\theta \end{bmatrix} \begin{bmatrix} a_i + ib_i \\ c_i + id_i \end{bmatrix}$$

$$= \begin{bmatrix} (\cos\theta a_i + \sin\theta b_i) + i(-\sin\theta a_i + \cos\theta b_i) \\ (\cos\theta c_i - \sin\theta d_i +) + i(\sin\theta c_i + \cos\theta d_i) \end{bmatrix}$$

As a SO(4) rotation, this is

$$\begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix},$$

which corresponds to $\exp(-\theta K_2)$.

d. $\exp(-i\theta X_L)$

$$\exp(-i\theta X_L)|0_L\rangle = \cos\theta|0_L\rangle - i\sin\theta|1_L\rangle$$

$$\to \cos\theta\begin{bmatrix}a_i + ib_i\\ c_i + id_i\end{bmatrix} - i\sin\theta\begin{bmatrix}c_i - id_i\\ -a_i + ib_i\end{bmatrix}$$

$$= \begin{bmatrix}(\cos\theta a_i - \sin\theta d_i) + i(\cos\theta b_i - \sin\theta c_i)\\ (\sin\theta b_i + \cos\theta c_i) + i(\sin\theta a_i + \cos\theta d_i)\end{bmatrix}$$

As a SO(4) rotation, this is

$$\begin{bmatrix} a & b & c & d \end{bmatrix}\begin{bmatrix}\cos\theta & 0 & 0 & \sin\theta\\ 0 & \cos\theta & \sin\theta & 0\\ 0 & -\sin\theta & \cos\theta & 0\\ -\sin\theta & 0 & 0 & \cos\theta\end{bmatrix},$$

which corresponds to $\exp(\theta K_3)$.

e. $\exp(-i\theta Y_L)$

$$\exp(-i\theta Y_L)|0_L\rangle = \cos\theta|0_L\rangle + \sin\theta|1_L\rangle$$

$$\to \cos\theta\begin{bmatrix}a_i + ib_i\\ c_i + id_i\end{bmatrix} + \sin\theta\begin{bmatrix}c_i - id_i\\ -a_i + ib_i\end{bmatrix}$$

$$= \begin{bmatrix}(\cos\theta a_i + \sin\theta c_i) + i(\cos\theta b_i - \sin\theta d_i)\\ (-\sin\theta a_i + \cos\theta c_i) + i(\sin\theta b_i + \cos\theta d_i)\end{bmatrix}$$

As a SO(4) rotation, this is

$$\begin{bmatrix} a & b & c & d \end{bmatrix}\begin{bmatrix}\cos\theta & 0 & -\sin\theta & 0\\ 0 & \cos\theta & 0 & \sin\theta\\ \sin\theta & 0 & \cos\theta & 0\\ 0 & -\sin\theta & 0 & \cos\theta\end{bmatrix},$$

which corresponds to $\exp(-\theta K_6)$.

f. $\exp(-i\theta Z_L)$

$$\exp(-i\theta Z_L)|0_L\rangle = (\cos\theta - i\sin\theta)|0_L\rangle$$

$$\to (\cos\theta - i\sin\theta)\begin{bmatrix}a_i + ib_i\\ c_i + id_i\end{bmatrix}$$

$$= \begin{bmatrix}(\cos\theta a_i + \sin\theta b_i) + i(-\sin\theta a_i + \cos\theta b_i)\\ (\cos\theta c_i + \sin\theta d_i +) + i(-\sin\theta c_i + \cos\theta d_i)\end{bmatrix}$$

As a SO(4) rotation, this is

$$\begin{bmatrix} a & b & c & d \end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta & 0 & 0\\ \sin\theta & \cos\theta & 0 & 0\\ 0 & 0 & \cos\theta & -\sin\theta\\ 0 & 0 & \sin\theta & \cos\theta\end{bmatrix},$$

which corresponds to $\exp(-\theta K_1)$.

Supplementary Information B: Steps to solve for the ((7,2,3)) codes

We provide the details for solving the following system of equations:

$$c_0^2 + c_1^2 + c_2^2 + c_3^2 + c_4^2 = 1 \tag{E1}$$

$$7c_0^2 + 3c_1^2 - c_2^2 - c_3^2 - 5c_4^2 = 0 \tag{E2}$$

$$\begin{aligned} &2\sqrt{7}\, c_0 c_4 + 2\sqrt{3}\, c_1 c_2 + 4\sqrt{3}\, c_1 c_3 \\ &\quad +4\sqrt{3}\, c_1 c_4 + 4c_2 c_3 + 3c_3^2 = 0 \end{aligned} \tag{E3}$$

$$\begin{aligned} &2\sqrt{7}\, c_0 c_4 + 2\sqrt{3}\, c_1 c_2 + 4\sqrt{3}\, c_1 c_3 \\ &\quad -4\sqrt{3}\, c_1 c_4 - 4c_2 c_3 - 3c_3^2 = 0 \end{aligned} \tag{E4}$$

Our goal is to eliminate c_0, c_2, and c_3 to obtain an equation in terms of c_1 and c_4. To do this, first subtract Eq.(E4) from Eq.(E3):

$$\begin{aligned} &\Big[2\sqrt{7}\, c_0 c_4 + 2\sqrt{3}\, c_1 c_2 + 4\sqrt{3}\, c_1 c_3 \\ &\quad +4\sqrt{3}\, c_1 c_4 + 4c_2 c_3 + 3c_3^2\Big] \\ &- \Big[2\sqrt{7}\, c_0 c_4 + 2\sqrt{3}\, c_1 c_2 + 4\sqrt{3}\, c_1 c_3 \\ &\quad -4\sqrt{3}\, c_1 c_4 - 4c_2 c_3 - 3c_3^2\Big] = 0 \end{aligned}$$

Simplify:

$$\begin{aligned} &8\sqrt{3}\, c_1 c_4 + 8c_2 c_3 + 6c_3^2 = 0 \\ \Rightarrow\quad &4\sqrt{3}\, c_1 c_4 + 4c_2 c_3 + 3c_3^2 = 0 \end{aligned} \tag{E5}$$

Adding Equations (E3) and (E4):

$$\begin{aligned} &\Big[2\sqrt{7}\, c_0 c_4 + 2\sqrt{3}\, c_1 c_2 + 4\sqrt{3}\, c_1 c_3 \\ &+ 4\sqrt{3}\, c_1 c_4 + 4c_2 c_3 + 3c_3^2\Big] \\ &+ \Big[2\sqrt{7}\, c_0 c_4 + 2\sqrt{3}\, c_1 c_2 + 4\sqrt{3}\, c_1 c_3 \\ &- 4\sqrt{3}\, c_1 c_4 - 4c_2 c_3 - 3c_3^2\Big] = 0 \end{aligned}$$

Simplify:

$$\begin{aligned} &4\sqrt{7}\, c_0 c_4 + 4\sqrt{3}\, c_1 c_2 + 8\sqrt{3}\, c_1 c_3 = 0 \\ \Rightarrow\quad &\sqrt{7}\, c_0 c_4 + \sqrt{3}\, c_1 c_2 + 2\sqrt{3}\, c_1 c_3 = 0 \end{aligned} \tag{E6}$$

From Eq.(E1):

$$c_0^2 = 1 - c_1^2 - c_2^2 - c_3^2 - c_4^2 \tag{E1a}$$

Substitute c_0^2 into Eq.(E2):

$$\begin{aligned} 7(1 - c_1^2 - c_2^2 - c_3^2 - c_4^2) + 3c_1^2 - c_2^2 - c_3^2 - 5c_4^2 = 0 \\ 7 - 7c_1^2 - 7c_2^2 - 7c_3^2 - 7c_4^2 + 3c_1^2 - c_2^2 - c_3^2 - 5c_4^2 = 0 \end{aligned}$$

Simplify:

$$-4c_1^2 - 8c_2^2 - 8c_3^2 - 12c_4^2 + 7 = 0 \tag{E7}$$

Divide both sides by -1:

$$4c_1^2 + 8c_2^2 + 8c_3^2 + 12c_4^2 = 7 \tag{E7a}$$

Divide both sides by 4:

$$c_1^2 + 2c_2^2 + 2c_3^2 + 3c_4^2 = \frac{7}{4} \tag{E8}$$

From Eq.(E6):

$$\sqrt{7}\, c_0 c_4 = -\sqrt{3}\, c_1 c_2 - 2\sqrt{3}\, c_1 c_3 \tag{E9}$$

Square both sides of Eq.(E9):

$$\begin{aligned}
(\sqrt{7}\, c_0 c_4)^2 &= \left(-\sqrt{3}\, c_1 c_2 - 2\sqrt{3}\, c_1 c_3\right)^2 \\
7c_0^2 c_4^2 &= 3c_1^2 \left(c_2 + 2c_3\right)^2 \\
\Rightarrow \quad c_0^2 c_4^2 &= \frac{3}{7} c_1^2 \left(c_2 + 2c_3\right)^2
\end{aligned} \tag{E10}$$

From Eq.(E1a), we have:

$$c_0^2 = 1 - c_1^2 - c_2^2 - c_3^2 - c_4^2 \tag{E1a}$$

Therefore:

$$c_0^2 c_4^2 = (1 - c_1^2 - c_2^2 - c_3^2 - c_4^2) c_4^2 \tag{E11}$$

Set Eq.(E10) equal to Eq.(E11):

$$(1 - c_1^2 - c_2^2 - c_3^2 - c_4^2) c_4^2 = \frac{3}{7} c_1^2 \left(c_2 + 2c_3\right)^2 \tag{E12}$$

From Eq.(E8):

$$c_2^2 + c_3^2 = \frac{1}{2}\left(\frac{7}{4} - c_1^2 - 3c_4^2\right) \tag{E13}$$

Simplify:

$$c_2^2 + c_3^2 = \frac{7}{8} - \frac{1}{2} c_1^2 - \frac{3}{2} c_4^2 \tag{E14}$$

First, expand $(c_2 + 2c_3)^2$:

$$(c_2 + 2c_3)^2 = c_2^2 + 4c_2 c_3 + 4c_3^2 \tag{E15}$$

From Eq.(E5), rearranged:

$$c_2 c_3 = -\sqrt{3}\, c_1 c_4 - \frac{3}{4} c_3^2 \tag{E16}$$

Substitute $c_2 c_3$ into Eq.(E15):

$$\begin{aligned}
(c_2 + 2c_3)^2 &= c_2^2 + 4\left(-\sqrt{3}\, c_1 c_4 - \frac{3}{4} c_3^2\right) + 4c_3^2 \\
&= c_2^2 - 4\sqrt{3}\, c_1 c_4 - 3c_3^2 + 4c_3^2 \\
&= c_2^2 - 4\sqrt{3}\, c_1 c_4 + c_3^2
\end{aligned} \tag{E17}$$

Now, using $c_2^2 + c_3^2$ from Eq.(E14):

$$(c_2 + 2c_3)^2 = \frac{7}{8} - \frac{1}{2} c_1^2 - \frac{3}{2} c_4^2 - 4\sqrt{3}\, c_1 c_4 \tag{E14}$$

Thus, from Eq.(E12), we have

$$\begin{aligned}
&\left(1 - c_1^2 - \left(\frac{7}{8} - \frac{1}{2} c_1^2 - \frac{3}{2} c_4^2\right) - c_4^2\right) c_4^2 \\
&= \frac{3}{7} c_1^2 \left(\frac{7}{8} - \frac{1}{2} c_1^2 - \frac{3}{2} c_4^2 - 4\sqrt{3}\, c_1 c_4\right)
\end{aligned} \tag{E18}$$

Simplifying we have

$$\begin{aligned}
&28c_4^4 + \left(7 + 8c_1^2\right) c_4^2 + 96\sqrt{3} c_1^3 c_4 + \left(12c_1^4 - 21c_1^2\right) = 0 \\
\Leftrightarrow\ &\left(c_4 + \sqrt{3} c_1\right) \Big(28c_4^3 - 28\sqrt{3} c_1 c_4^2 + \left(92c_1^2 + 7\right) c_4 \\
&\quad + \sqrt{3}\left(4c_1^3 - 7c_1\right)\Big) = 0
\end{aligned} \tag{E19}$$

This then gives $c_4 = -\sqrt{3} c_1$ as a solution.