The Supplementary Material for
The superconductor to metal transition in overdoped cuprates

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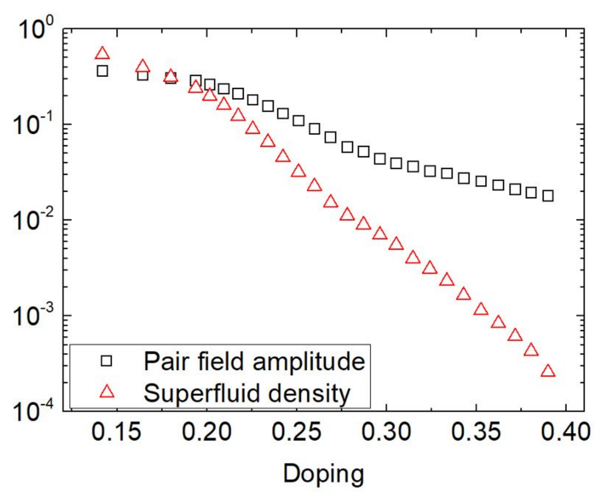
Supplementary Note 1: The zero-temperature pair field amplitude and superfluid density

We employ self-consistent, superconducting, mean-field approximation to study Eq.(1) in the main text. We consider two different forms of $H_{\rm int}$: 1) To trigger d-wave pairing, we use the nearest-neighbor Heisenberg antiferromagnetic exchange interaction $H_{\rm int} = J\sum_{\langle ij\rangle} \boldsymbol{S}_i \cdot \boldsymbol{S}_j$. 2) To trigger s-wave pairing, we use the attractive Hubbard interaction $H_{\rm int} = -U\sum_i c^\dagger_{i\uparrow}c_{i\uparrow}c^\dagger_{i\downarrow}c_{i\downarrow}$. After the mean-field decoupling, we obtain, respectively,

$$\begin{aligned} H^{\rm d}_{\rm MF} &= -\frac{1}{2}\sum_{ij}\Delta_{ij}(c^\dagger_{i\uparrow}c^\dagger_{j\downarrow} - c^\dagger_{i\downarrow}c^\dagger_{j\uparrow}) + h.c. \\ H^{\rm s}_{\rm MF} &= -\sum_i \Delta_i c^\dagger_{i\uparrow}c^\dagger_{i\downarrow} + h.c. \end{aligned} \tag{1}$$

where pair field for d-wave and s-wave are given by: $\Delta_{ij} = J\left\langle c_{j\downarrow}c_{i\uparrow} - c_{j\uparrow}c_{i\downarrow}\right\rangle$ and $\Delta_i = U\left\langle c_{i\downarrow}c_{i\uparrow}\right\rangle$, respectively, which are determined self-consistently.

To study the averaged pair field amplitude and superfluid density in strongly disordered regime, we plot the logarithm of the pair field amplitude and superfluid density as a function of doping concentration, or impurity concentration (see Fig.2 in main text) in Fig. 1.



Supplementary Figure 1. **The logarithmic plot of pair field amplitude and superfluid density as a function of doping concentration for the d-wave pairing**.

In the regime where doping (hence impurity) concentration is high, the pair field amplitude obeys approximately an exponential behavior. This is consistent with the prediction of Ref.[2], suggesting the significant pair field originates from the exponentially rare regions in which the disorder is relatively weak. Compared with pair field amplitude, the superfluid drops considerably more rapidly with increasing doping concentration, or impurity concentration.

Assuming the presence of 90 degree rotation symmetry after impurity averaging, we compute the zero temperature superfluid density by twisting the phase of the pair field in a pre-chosen, say, x-direction. Alternatively, it can be calculated as the current response to a gauge field in $\hat{x}$. Using the standard Kubo formula:

$$\rho_s = -\langle K_x \rangle - \Lambda_{xx}(q_x = 0, q_y \to 0, \omega = 0). \tag{2}$$

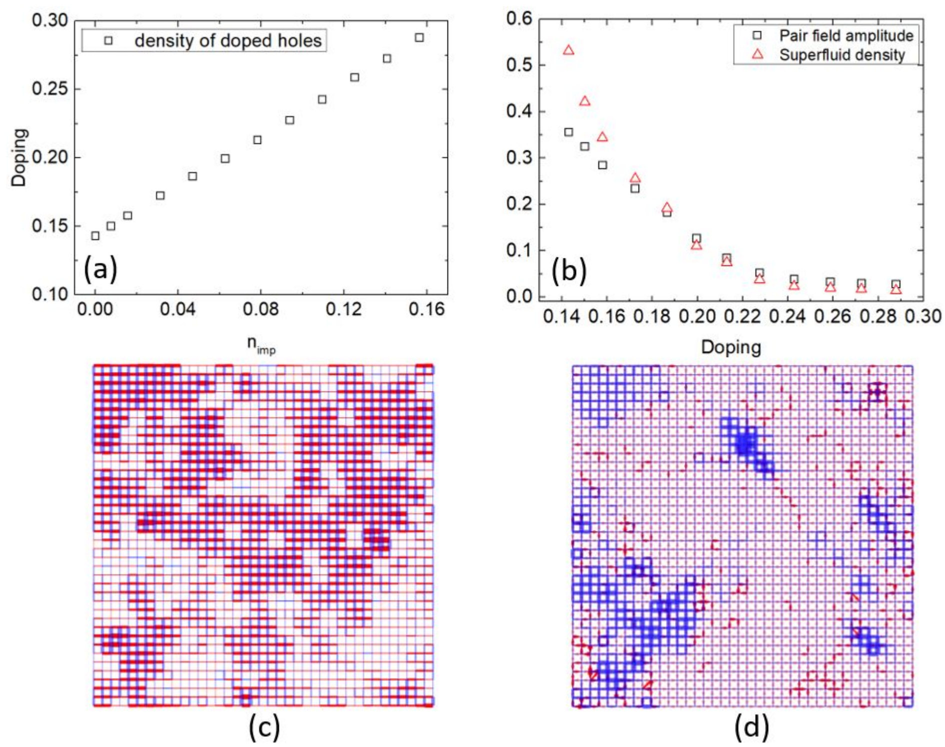
Here

$$K_x = \frac{1}{N}\sum_{i,\sigma}\left(-t_1 c^\dagger_{i\sigma}c_{i+x\sigma} - t_2 c^\dagger_{i\sigma}c_{i+x+y\sigma} - t_2 c^\dagger_{i\sigma}c_{i+x-y\sigma} + h.c.\right),$$

with t_1 being the hopping integral between nearest-neighbor sites, and t_2 is the hopping integral between second-neighbor sites, $\Lambda_{xx}(\boldsymbol{q},\omega)$ is the current-current correlator defined as follow:

$$\Lambda_{xx}(\boldsymbol{q},\omega) = \frac{i}{N}\int_0^\infty dt e^{i\omega t}\left\langle [J_x(t,\boldsymbol{q}), J_x(0,-\boldsymbol{q})]\right\rangle. \tag{3}$$

In the equation above N is the number of lattice sites, $J_x(\boldsymbol{q}) = \sum_i J_x(\boldsymbol{r}_i)e^{i\boldsymbol{q}\cdot\boldsymbol{r}_i}$ is the Fourier transform of the current operator along x-direction.



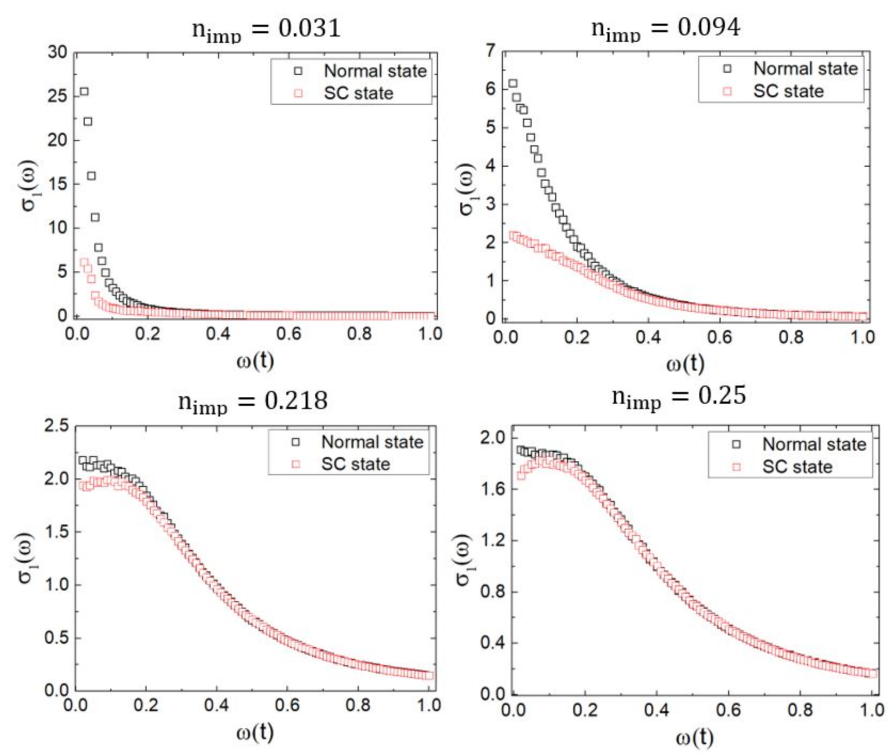
Supplementary Figure 2. **The results of the zero-temperature averaged pair field amplitude, superfluid density, and spontaneous current loops, for disorder strength** $w = 3$. (a) The pair field amplitude and superfluid density as a function of doping level. (b) Real-space distribution of the pair field for impurity concentration $n_{\rm imp} = 0.078$: the thickness of the symbols represents the magnitude and the color(red positive, blue negative) represents the sign. (c) Real-space distribution of the spontaneous current loops and the pair field amplitude for impurity concentration $n_{\rm imp} = 0.125$: the blue colored bonds represent the pair field amplitude, and the red arrows represent the spontaneous generated super-current.

Supplementary Note 2: The pair field amplitude and superfluid density for disorder strength $w = 3$

In the main text, we report the averaged zero-temperature pair field amplitude and superfluid density for a fixed disorder strength $w = 1$. In this section, we present the results for a different value of disorder strength $w = 3$, to demonstrate that the main qualitative results do not depend on the disorder strength. We choose the uniform chemical potential $\mu = -1.08$ such that the density of doped holes is $p = 0.145$, when the system is disorder free. After fixing μ, doping level increases monotonically with impurity concentration $n_{\rm imp}$. The dependence of the density of doped holes, p, on the impurity concentration $n_{\rm imp}$ is shown in Fig. 2(a). We calculate the averaged zero-temperature pair field amplitude and superfluid density as a function of doping level, as shown in Fig. 2(b). Like the case of $w = 1$, the impurity concentration is large, the superconducting pair field amplitude becomes highly heterogeneous (shown in Fig. 2(c)). In addition, like the case of $w = 1$, spontaneous current loops emerge when the disorder is strong (see Fig. 2(d)). These findings are consistent with the results when disorder strength is $w = 1$.

Supplementary Note 3: The band structures of the"flat band" and "steep band" models

In the main text, we present the mean-field results of the averaged zero temperature pair field amplitude and superfluid density for the "flat" and "steep" band structures (see Fig. 6). The parameters used to produce these band structures shown are given as follows: i) For the "flat band": $t_1 = 1$, $t_2 = -0.05$, $t_3 = 0.2$ and $\mu = -0.778$. ii) For the "steep band": $t_1 = 1$, $t_2 = -0.4$, $t_3 = 0.68$ and $\mu = -1.263$. Here t_1, t_2 and t_3 are the hopping integrals between nearest-neighbor sites, second-neighbor sites and third-neighbor sites, respectively, and μ is uniform chemical potential. Under the choice of these parameters, the doping level (23%) and the overall band width (3.875) are the same for the two different band structures. Note the results reported in Fig. 6 (b) and (c) are obtained under a fixed doping level. This is achieved with increasing impurity concentration, by fine tuning the value of uniform chemical potential.



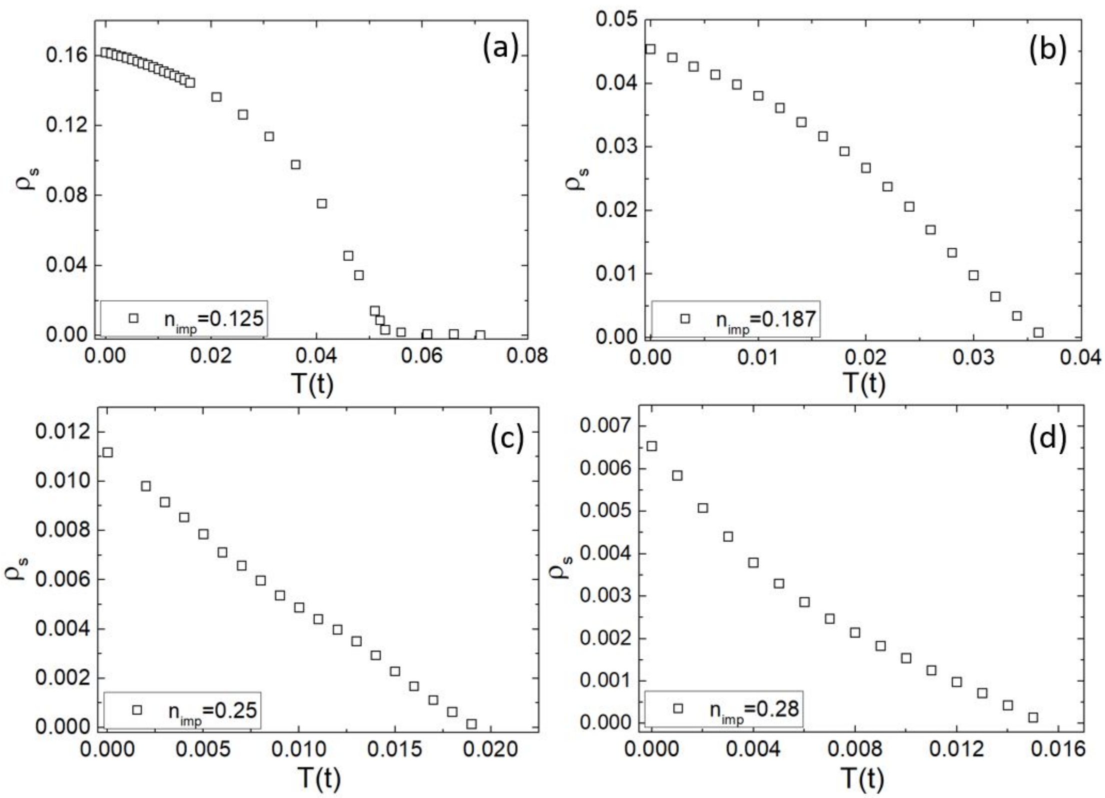
Supplementary Figure 3. **Frequency dependence of the real part of optical conductivity at zero temperature for different impurity concentrations**. The normal state results are obtained by setting the pair field to zero.

Supplementary Note 4: The optical conductivity

In this section, we present the zero-temperature optical conductivity for different impurity concentrations. The optical conductivity is computed from the current-current correlator defined as Eq. (3):

$$\sigma(\omega) = \frac{1}{i\omega}\Lambda_{xx}(\boldsymbol{q}=0,\omega) \tag{4}$$

The frequency dependence of the real-part of $\sigma(\omega)$ is shown in Fig. 3 for several impurity concentrations. The normal state results are obtained by setting the pair field to zero. The result clearly shows that when the impurity concentration is high, a large fraction of the normal-state Drude weight is uncondensed. The result is consistent with physical picture that we have superconducting islands embedded in a normal metal matrix. However, when impurity concentration is high, our optical conductivity results show a broad peak at finite frequency. The peak position is consistent with the averaged superconducting gap. This feature is not observed in the experimental measurement[1]. A possible origin of this discrepancy is that the experimental measurements have not yet reached the frequency regime where the averaged gap will be manifested. Another possibility is the fact that these peaks are very broad, making it difficult to observe experimentally. In any case a more thorough analysis of this discrepancy is left in future studies.



Supplementary Figure 4. **The temperature dependence of superfluid density for different impurity concentrations**. We use the unit in which the Boltzmann constant $k_B = 1$. This results only include the quasiparticle effects. The thermal phase fluctuations are omitted.

Supplementary Note 5: The superfluid density at non-zero temperature

In this section, we present the results of superfluid density at non-zero temperature for four different impurity concentrations. Because the present calculation neglects the suppression of superfluid density by thermal phase fluctuations, it can only give reasonable results at low temperature. The temperature dependences of superfluid density for four different impurity concentrations are shown in Fig. 4. For sufficiently low temperature, we find the superfluid density drops approximately linearly with temperature. This behavior is consistent with the experimental results for LSCO[3]. However, in the same experiment, the linear decrease of the superfluid density with temperature is obeyed all the way up to T_c. Clearly, this is not something we can explain at present.

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