Supplemental Material: Anomalous Josephson Coupling and High-Harmonics in Non-Centrosymmetric Superconductors with S-wave Spin-Triplet Pairing

Yuri Fukaya

CNR-SPIN, I-84084 Fisciano (Salerno), Italy,

c/o Universitá di Salerno, I-84084 Fisciano (Salerno), Italy and

Department of Applied Physics, Nagoya University, Nagoya 464-8603, Japan

Yukio Tanaka and Keiji Yada

Department of Applied Physics, Nagoya University, Nagoya 464-8603, Japan

Paola Gentile and Mario Cuoco

CNR-SPIN, I-84084 Fisciano (Salerno), Italy,

c/o Universitá di Salerno, I-84084 Fisciano (Salerno), Italy

Supplementary Note 1 : Josephson current

We report about the evolution of the Josephson current as a function of the electronic parameters. We start by considering a configuration with an amplitude mismatch of the chemical potentials between the left (μ_{L}) and right (μ_{R}) side of the junction (Fig. 1). We track the evolution of the first four harmonics of the Josephson current as a function of μ_R at a given value of $\mu_{\mathrm{L}} = 0.0$ (Fig. 1(a) as well of the CPR for two representative cases of μ_{L} ((Fig. 1(b),(c)). For these configurations, corresponding to equal amplitude of the spin-orbit (λ_{SO}) and orbital Rashba couplings on the two sides of the junction, the CPR is marked by a 0-type Josephson coupling. It is more interesting to observe the evolution as a function of the spin-orbit and orbital Rashba interactions assuming a fixed amplitude mismatch of the chemical potentials. In Figs. 2,3 we show the evolution of the first four harmonics and the corresponding CPR as a function of the orbital Rashba coupling (α_{OR}) and the spin-orbit coupling (λ_{SO}). We find that the first harmonic can change sign at a critical value of α_{OR} or λ_{SO} thus signalling a 0- to π-phase transition. In proximity of the transition the CPR is dominated by high harmonics as manifested by the presence of double maximum and asymmetric amplitudes of the maximum and minimum in the current phase relations.

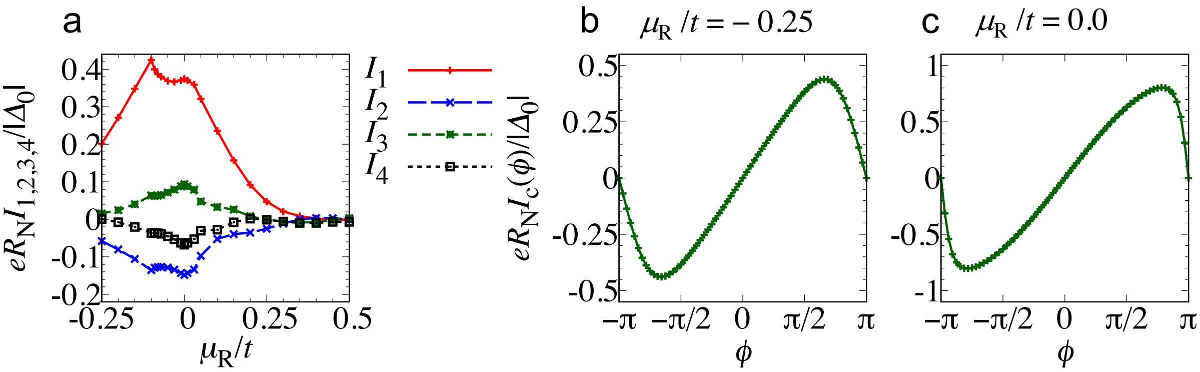
To evaluate the dependence of the critical current on an applied magnetic field H we follow the standard approach for a short junction with length L and width W that is thread by a magnetic flux $\Phi = LWH$. Then, we can expand the current phase relation in all the harmonics and consider that the spatial dependent vector potential associated to the magnetic field modifies the phase difference ϕ across the junction with a factor $2\pi\tilde{\Phi}\tilde{x}$ where $\tilde{\Phi} = \Phi/\Phi_0$, with Φ_0 being the unit of magnetic flux, and $\tilde{x}$ indicating the ratio x/W, i.e. the scaled length with respect to the lateral size of the junction. The general expression of the supercurrent expressed via an expansion in harmonics is given by

$$I(\tilde{\Phi}, \phi) = \sum_n \int_0^w I_n \sin\left[n\phi + 2\pi\tilde{\Phi}\tilde{x}\right] d\tilde{x}\,. \tag{1}$$

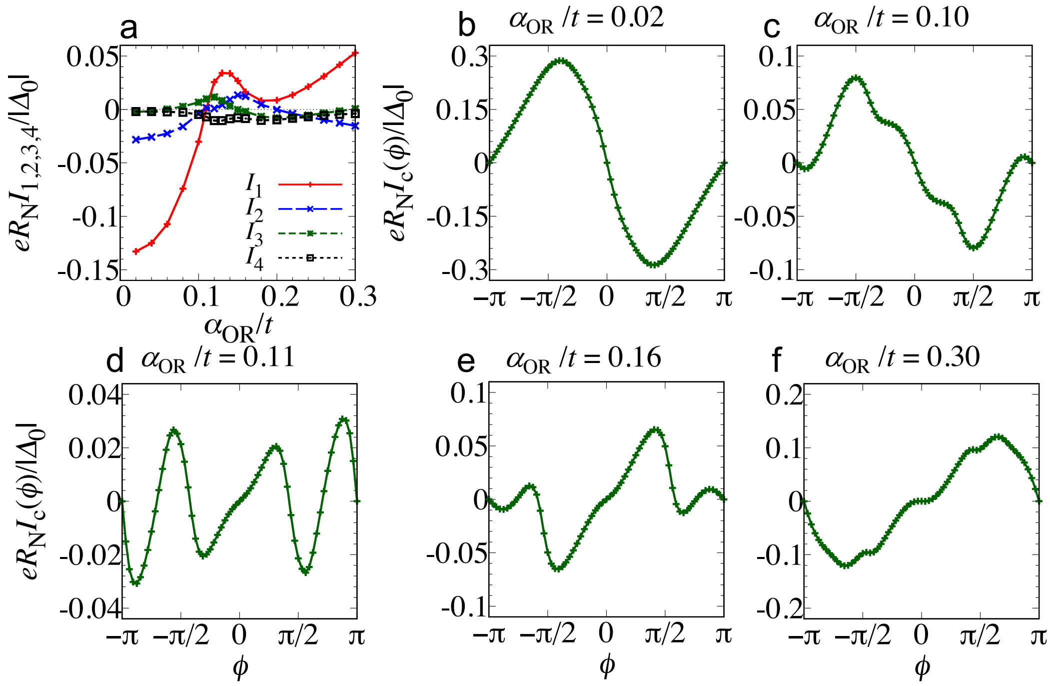
The resulting critical current at a given applied magnetic flux is obtained by maximizing $I(\tilde{\Phi}, \phi)$

with respect to ϕ.

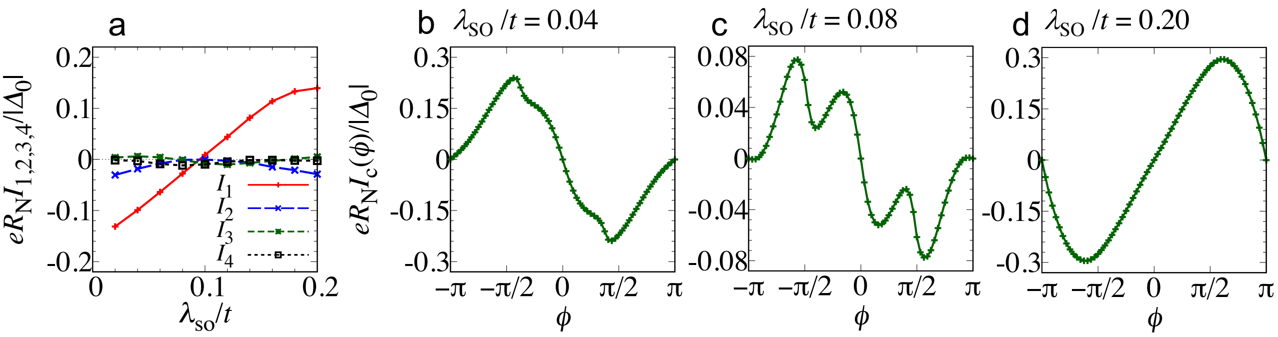
[1] Umerski, A. Closed-form solutions to surface green's functions. *Phys. Rev. B* **55**, 5266–5275 (1997). URL `https://link.aps.org/doi/10.1103/PhysRevB.55.5266`.



Supplementary Figure 1. **Harmonics of the Josephson current and current-phase relation for different chemical potential.** (a) Amplitude of the first four harmonics of the Josephson current $I_{n=1,2,3,4}$ as a function of the chemical potential on the right-side superconductor μ_R at $(\lambda_{SOL}/t, \alpha_{ORL}/t, \mu_L/t) = (0.10, 0.20, 0.00)$, $(\lambda_{SOR}/t, \alpha_{ORR}/t) = (0.10, 0.20)$ and $T/T_c = 0.10$. (b,c) Corresponding current phase relations in (a) at (b) $\mu_R/t = -0.25$ and (c) $\mu_R/t = 0.0$. We showed the current phase relation at $\mu_R/t = 0.35$ in Fig. 3 (e) of the main part.



Supplementary Figure 2. **Harmonics of the Josephson current and current-phase relation for different values of the orbital Rashba interaction.** (a) Amplitude of the first four harmonics of normalized Josephson current, $I_{n=1,2,3,4}$, as a function of $\alpha_{\rm OR} = \alpha_{\rm ORL} = \alpha_{\rm ORR}$. (b,c,d,e,f) Corresponding current phase relations in (a) at (b) $\alpha_{\rm OR}/t = 0.02$, (c) $\alpha_{\rm OR}/t = 0.10$, (d) $\alpha_{\rm OR}/t = 0.11$,(e) $\alpha_{\rm OR}/t = 0.16$, and (f) $\alpha_{\rm OR}/t = 0.30$. We set the parameters as $(\lambda_{\rm SOL}/t, \mu_{\rm L}/t) = (0.10, 0.00)$, $(\lambda_{\rm SOR}/t, \mu_{\rm R}/t) = (0.10, 0.35)$, and $T/T_{\rm c} = 0.025$.



Supplementary Figure 3. **Harmonics of the Josephson current and current-phase relation for different values of the atomic spin-orbit coupling.** (a) Components of normalized Josephson current $I_{n=1,2,3,4}$ as a function of $\lambda_{\rm SO} = \lambda_{\rm SOL} = \lambda_{\rm SOR}$. (b,c,d) Corresponding current phase relations in (a) at (b) $\lambda_{\rm SO}/t = 0.04$, (c) $\lambda_{\rm SO}/t = 0.08$, and (d) $\lambda_{\rm SO}/t = 0.20$. We set the parameters as $(\alpha_{\rm ORL}/t, \mu_{\rm L}/t) = (0.20, 0.00)$, $(\alpha_{\rm ORR}/t, \mu_{\rm R}/t) = (0.20, 0.35)$, and $T/T_{\rm c} = 0.025$.