Supplementary Information for "Finite-temperature critical behaviors in 2D long-range quantum Heisenberg model"

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Supplementary Note 1: SSE QMC update scheme.

The Hamiltonian of the long-range ferromagnetic Heisenberg model discussed in main text can be decomposed as diagonal and off-diagonal operators,

$$\begin{aligned} H_{0,0} &= I \\ H_{1,a(ij)} &= J_{ij}(\frac{1}{4} + S_i^z S_j^z) \\ H_{2,a(ij)} &= \frac{J_{ij}}{2}(S_i^+ S_j^- + S_i^- S_j^+), \end{aligned} \tag{1}$$

where $H - \frac{\sum_{i<j} J_{ij}}{4} = -\sum_{a(i<j)} H_{1,a(ij)} + H_{2,a(ij)}$ and $a(ij)$ is the bond index. Take the eigenstates of σ^z as basis, the non-zero matrix elements in Eq. (1) are

$$\begin{aligned} \langle\uparrow\uparrow| H_{1,a} |\uparrow\uparrow\rangle = \langle\downarrow\downarrow| H_{1,a} |\downarrow\downarrow\rangle &= \frac{J_{ij}}{2} \\ \langle\uparrow\downarrow| H_{2,a} |\downarrow\uparrow\rangle = \langle\downarrow\uparrow| H_{2,a} |\uparrow\downarrow\rangle &= \frac{J_{ij}}{2}. \end{aligned} \tag{2}$$

In the SSE QMC simulation, the loop update scheme is maintained the same with the ferromagnetic Heisenberg model with nearest-neighbor couplings. However, to efficiently carry out the diagonal update scheme, we choose the candidate bonds for inserting diagonal operators with an importance sampling procedure with $P_{\text{choose}} \propto J_{ij}$. The diagonal update scheme is thus

1. If a diagonal operator ($H_{1,a}$) is visited, remove it with probability

$$P_{\text{remove}} = \min\left(\frac{2(M-n+1)}{\beta \sum_{i<j} J_{ij}}, 1\right) \tag{3}$$

2. If an identity operator ($H_{0,a}$) is visited, insert a diagonal operator according to:

   - First choose a candidate bond a to make the insertion with probability

$$P_{\text{choose}} = \frac{J_{ij}}{\sum_{i<j} J_{ij}} \tag{4}$$

   - Then accept the insertion of a diagonal operator at this position with probability

$$P_{\text{accept}} = \min\left(\frac{\beta \sum_{i<j} J_{ij}}{2(M-n)}, 1\right) \tag{5}$$

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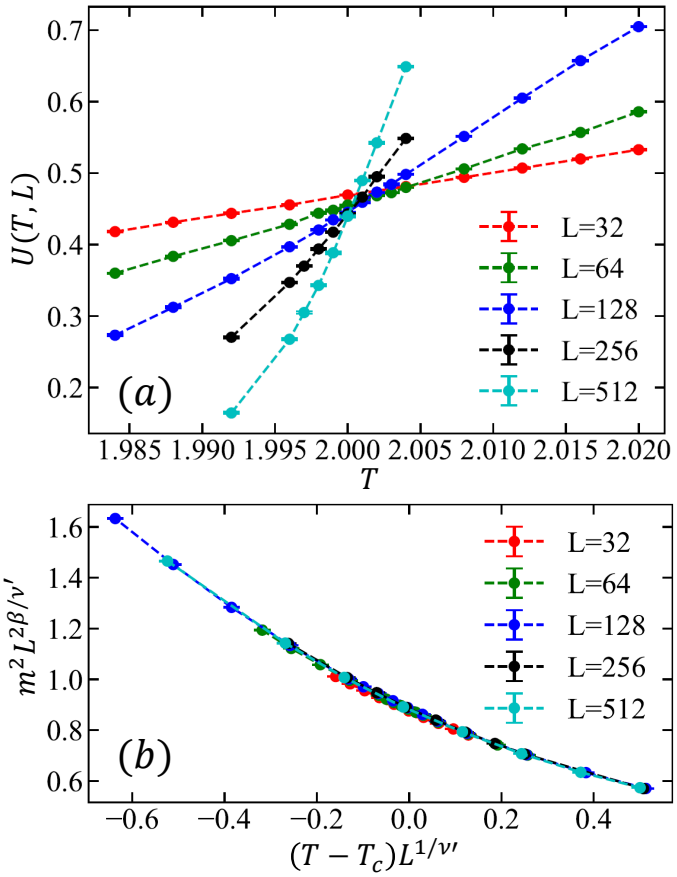


Supplementary Figure 1. **The determination of the critical point and exponents at** $\alpha = 1.8$**.** (a) Binder ratio $U(T, L)$ versus temperature T for different system sizes. (b) Data collapse of the order parameter $\langle m^2 \rangle$ near the critical point T_c. The obtained results are $\nu' = 1.00(5)$ and $\beta = 0.505(8)$. The standard error of the mean (SEM) is used when estimating the errors of the physical quantities.

To generate a set of random bond index according to the probability defined in Eq. (4), we use the naive Walker's method [1] with complexity of $O(N^2)$. Despite there is optimization of this method which reduces the complexity to $O(N)$ [2], we find for the system size we simulate the original method is sufficient and easy to implement.
The above procedure ensures that diagonal operators with higher matrix elements have higher probability to be inserted and compared with randomly choosing candidate bonds this strategy certainly has better efficiency.

Supplementary Note 2: Kac normalization and phase diagram at $\alpha < 2$**.**

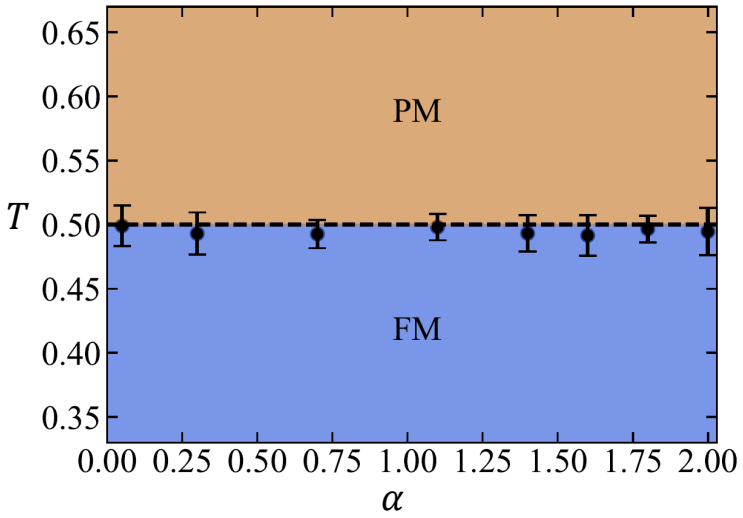
With a LR Hamiltonian defined as

$$H = -\sum_{i<j} J_{ij} \vec{S}_i \vec{S}_j. \tag{6}$$

The ground state is a fully ferromagnetic state with all spins aligned in the same direction. We have shown in the main text that the system undergoes a continuous phase transition at $\alpha \in (2, 4)$. However, when $\alpha \leq 2$ the system is no longer extensive and it is still unclear whether such a system still hold a phase transition point. To make the system extensive, a Kac normalization factor can be added to the Hamiltonian and the normalized Hamiltonian is

$$H = -\frac{N-1}{\sum_{i<j} J_{ij}} \sum_{i<j} J_{ij} \vec{S}_i \vec{S}_j. \tag{7}$$

In this case, the energy density is $\langle \uparrow \cdots \uparrow\uparrow | H/N | \uparrow\uparrow \cdots \uparrow \rangle = -(N-1)/N$ which will be a constant for any value of α. For the Hamiltonian defined in Eq. (7), we perform the QMC simulations and find that there is a continuous phase transition for $\alpha = 1.8$ and the critical exponents also satisfies the prediction of mean-field theory, as shown in



Supplementary Figure 2. **Phase diagram of the 2D LR ferromagnetic Heisenberg model with Kac-normalized Hamiltonian**. The system also undergoes a continuous phase transition from FM to PM as temperature is increased. The black dots are the critical points determined from QMC simulations. The standard error of the mean (SEM) is used when estimating the errors of the physical quantities.

Supplementary Figure **1**. The similar phenomena has also been observed in 1D LR quantum Ising models [3], where at $\alpha = 0.05$ which is below the system dimension, the critical exponents are still consistent with mean-field predictions.

In addition, we also examine this system at other values of α and we find that the phase transition point T_c remains the same for all the $\alpha \leq 2$ we consider, as indicated in Supplementary Figure **2**. There is an intuitive understanding for this finding: the Kac normalization [3, 4] makes the energy scale to be the same for all $\alpha \leq 2$ which somehow suppresses the effect of different decaying exponent α and the phase transition temperature T_c is thus scaled to be the same for all α. This point should be further examined by more robust analysis.

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