

Supplementary information for "Unconventional Superconductivity near a Nematic Instability in a Multi-Orbital system"

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SUPPLEMENTARY DISCUSSION I: MODEL HAMILTONIAN

We consider an effective two dimensional 2-orbital model Hamiltonian with two hole pockets, centered at the Γ point of the Brillouin zone(BZ) and two electron pockets, centered at $X = (0, \pi)$ and $Y = (\pi, 0)$ points of the BZ, respectively. The hole pockets and the corresponding hole bands are composed of d_{xz} and d_{yz} orbitals. The X pocket/band is composed of d_{yz} and the Y pocket/band is composed of d_{xz} orbitals respectively. For simplicity, we neglect the d_{xy} orbital contribution to the X and Y electron pockets. We also neglect the spin-orbit coupling on the band dispersion. The kinetic energy of the model Hamiltonian near the Γ , Y and X points are captured by

$$H_{\text{Kin}} = H_{\Gamma} + H_Y + H_X, \quad (1)$$

with

$$H_{\Gamma} = \sum_{\mathbf{k}, \sigma} \psi_{\Gamma, \mathbf{k}, \sigma}^{\dagger} \left[\left(\mu_h - \frac{\mathbf{k}^2}{2m_h} \right) \tau_0 - \frac{b}{2} \mathbf{k}^2 \cos 2\theta_{\mathbf{k}} \tau_3 + \frac{b}{2} \mathbf{k}^2 \sin 2\theta_{\mathbf{k}} \tau_1 \right] \psi_{\Gamma, \mathbf{k}, \sigma}, \quad (2)$$

$$H_Y = \sum_{\mathbf{k}, \sigma} \xi_{Y, \mathbf{k}} f_{1, \mathbf{k}, \sigma}^{\dagger} f_{1, \mathbf{k}, \sigma}, \quad (3)$$

$$H_X = \sum_{\mathbf{k}, \sigma} \xi_{X, \mathbf{k}} f_{2, \mathbf{k}, \sigma}^{\dagger} f_{2, \mathbf{k}, \sigma}. \quad (4)$$

Here, $\psi_{\Gamma, \mathbf{k}, \sigma}^{\dagger} = \left(d_{xz, \mathbf{k}, \sigma}^{\dagger}, d_{yz, \mathbf{k}, \sigma}^{\dagger} \right)$ where $d_{i, \mathbf{k}, \sigma}^{\dagger}$ creates a fermion with the orbital index $i(xz, yz)$, momentum $\mathbf{k}$ (measured from Γ) and spin index $\sigma(\pm 1)$. $\theta_{\mathbf{k}}$ is the polar angle for momentum $\mathbf{k}$, measured from the $\Gamma - X$ -direction in the anti-clockwise direction. τ_i 's are the conventional Pauli matrices with τ_0 being the 2×2 identity matrix. The parameters μ_h, m_h and $b > 0$ are taken from the ARPES data for FeSe at $k_z = \pi$ [1]. Diagonalizing Eq. (2), we find two bands called

outer(h) and inner(H) hole pockets with the dispersion relation $\xi_{hh} = \mu_h - \mathbf{k}^2/2m_h \pm b\mathbf{k}^2$. The fermionic band operators h, H are the linear combination of the orbital operators,

$$h_{\mathbf{k},\sigma} = \cos\theta_{\mathbf{k}} d_{yz,\mathbf{k},\sigma} + \sin\theta_{\mathbf{k}} d_{xz,\mathbf{k},\sigma} \quad (5)$$

$$H_{\mathbf{k},\sigma} = \sin\theta_{\mathbf{k}} d_{yz,\mathbf{k},\sigma} - \cos\theta_{\mathbf{k}} d_{xz,\mathbf{k},\sigma}. \quad (6)$$

On the other hand, $f_{1,\mathbf{k}}^\dagger$ and $f_{2,\mathbf{k}}^\dagger$ creates fermion with orbital d_{xz} and d_{yz} and momentum $\mathbf{k}$ near $Y-$ and $X-$ points respectively. The corresponding energy dispersions for the electron pockets are taken of the form

$$\xi_y(\mathbf{k}) = \frac{k_x^2}{2m_x} + \frac{k_y^2}{2m_y} - \mu_e, \quad \xi_x(\mathbf{k}) = \frac{k_x^2}{2m_y} + \frac{k_y^2}{2m_x} - \mu_e, \quad (7)$$

where m_x and m_y are fitting parameters from the ARPES data. In the orbital basis, the interaction part of the Hamiltonian involves 14 distinct C_4 symmetric interactions between the low energy fermions near the Γ, X and Y pockets,

$$\begin{aligned} H_{\text{int}} = & \frac{U_4}{2} \sum \left[d_{xz,\sigma}^\dagger d_{xz,\sigma} d_{xz,\sigma'}^\dagger d_{xz,\sigma'} + d_{yz,\sigma}^\dagger d_{yz,\sigma} d_{yz,\sigma'}^\dagger d_{yz,\sigma'} \right] + \tilde{U}_4 \sum d_{xz,\sigma}^\dagger d_{xz,\sigma} d_{yz,\sigma'}^\dagger d_{yz,\sigma'} + \tilde{\bar{U}}_4 \sum d_{xz,\sigma}^\dagger d_{yz,\sigma} d_{yz,\sigma'}^\dagger d_{xz,\sigma'} \\ & + \frac{\bar{U}_4}{2} \sum \left[d_{xz,\sigma}^\dagger d_{yz,\sigma} d_{xz,\sigma'}^\dagger d_{yz,\sigma'} + d_{yz,\sigma}^\dagger d_{xz,\sigma} d_{yz,\sigma'}^\dagger d_{xz,\sigma'} \right] + U_1 \sum \left[f_{1,\sigma}^\dagger f_{1,\sigma} d_{xz,\sigma'}^\dagger d_{xz,\sigma'} + f_{2,\sigma}^\dagger f_{2,\sigma} d_{yz,\sigma'}^\dagger d_{yz,\sigma'} \right] \\ & + \bar{U}_1 \sum \left[f_{1,\sigma}^\dagger f_{1,\sigma} d_{yz,\sigma'}^\dagger d_{yz,\sigma'} + f_{2,\sigma}^\dagger f_{2,\sigma} d_{xz,\sigma'}^\dagger d_{xz,\sigma'} \right] + U_2 \sum \left[f_{1,\sigma}^\dagger d_{xz,\sigma} d_{xz,\sigma'}^\dagger f_{1,\sigma'} + f_{2,\sigma}^\dagger d_{yz,\sigma} d_{yz,\sigma'}^\dagger f_{2,\sigma'} \right] \\ & + \bar{U}_2 \sum \left[f_{1,\sigma}^\dagger d_{yz,\sigma} d_{yz,\sigma'}^\dagger f_{1,\sigma'} + f_{2,\sigma}^\dagger d_{xz,\sigma} d_{xz,\sigma'}^\dagger f_{2,\sigma'} \right] + \frac{U_3}{2} \sum \left[f_{1,\sigma}^\dagger d_{xz,\sigma} f_{1,\sigma'}^\dagger d_{xz,\sigma'} + f_{2,\sigma}^\dagger d_{yz,\sigma} f_{2,\sigma'}^\dagger d_{yz,\sigma'} \right] \\ & + \frac{\bar{U}_3}{2} \sum \left[f_{1,\sigma}^\dagger d_{yz,\sigma} f_{1,\sigma'}^\dagger d_{yz,\sigma'} + f_{2,\sigma}^\dagger d_{xz,\sigma} f_{2,\sigma'}^\dagger d_{xz,\sigma'} \right] + \frac{U_5}{2} \sum \left[f_{1,\sigma}^\dagger f_{1,\sigma} f_{1,\sigma'}^\dagger f_{1,\sigma'} + f_{2,\sigma}^\dagger f_{2,\sigma} f_{2,\sigma'}^\dagger f_{2,\sigma'} \right] \\ & + \tilde{U}_5 \sum f_{1,\sigma}^\dagger f_{1,\sigma} f_{2,\sigma'}^\dagger f_{2,\sigma'} + \tilde{\bar{U}}_5 \sum f_{1,\sigma}^\dagger f_{2,\sigma} f_{2,\sigma'}^\dagger f_{1,\sigma'} + \frac{\bar{U}_5}{2} \sum \left[f_{1,\sigma}^\dagger f_{2,\sigma} f_{1,\sigma'}^\dagger f_{2,\sigma'} + f_{2,\sigma}^\dagger f_{1,\sigma} f_{2,\sigma'}^\dagger f_{1,\sigma'} \right]. \end{aligned} \quad (8)$$

Here, we omit the momentum index in the fermionic operators for the simplicity of the notation and " $\sum$ " represents summation over the momenta with momentum conservation. In terms of bare Hubbard-Hund interactions, these interactions are

$$U_1 = U_2 = U_3 = U_4 = U_5 = U, \quad (9)$$

$$\bar{U}_1 = \tilde{U}_4 = \tilde{\bar{U}}_5 = U', \quad (10)$$

$$\bar{U}_2 = \tilde{U}_4 = \tilde{\bar{U}}_5 = J, \quad (11)$$

$$\bar{U}_3 = \bar{U}_4 = \bar{U}_5 = J', \quad (12)$$

with U, U', J, J' are intra-orbital Hubbard repulsion, inter-orbital Hubbard repulsion, inter-orbital Hund exchange and inter-orbital Hund exchange term known as pair hopping term respectively. We convert these interactions in the band basis(for $X-, Y-$ pockets Eq. (8) is already in the band basis) using the Eqs. (5-6) and write the interaction Hamiltonian as

$$H_{\text{int}} = \sum_{i,j \in \{h, X, Y\}} H_{i,j} + H_{\text{other}} \quad (13)$$

Here, $H_{i,j}$ is the interaction between the pockets i and $j \in \{h, Y, X\}$ and takes the form

$$H_{hh} = \frac{1}{2} \sum_{\mathbf{k}, \mathbf{p}, \mathbf{q}} h_{\mathbf{k},\sigma}^\dagger h_{\mathbf{k}+\mathbf{q},\sigma} h_{\mathbf{p},\sigma'}^\dagger h_{\mathbf{p}-\mathbf{q},\sigma'} V_{hh}^{\text{den}}(\mathbf{k}, \mathbf{k} + \mathbf{q}; \mathbf{p}, \mathbf{p} - \mathbf{q}), \quad (14)$$

$$\begin{aligned} H_{h,Y} = & \sum_{\mathbf{k}, \mathbf{p}, \mathbf{q}} f_{1,\mathbf{k},\sigma}^\dagger f_{1,\mathbf{k}+\mathbf{q},\sigma} h_{\mathbf{p},\sigma'}^\dagger h_{\mathbf{p}-\mathbf{q},\sigma'} V_{h,Y}^{\text{den}}(\mathbf{p}, \mathbf{p} - \mathbf{q}) + \sum_{\mathbf{k}, \mathbf{p}, \mathbf{q}} f_{1,\mathbf{k},\sigma}^\dagger h_{\mathbf{k}+\mathbf{q},\sigma} f_{1,\mathbf{p},\sigma'}^\dagger h_{\mathbf{p}-\mathbf{q},\sigma'} V_{h,Y}^{\text{ph}}(\mathbf{k} + \mathbf{q}, \mathbf{p} - \mathbf{q}) \\ & + \sum_{\mathbf{k}, \mathbf{p}, \mathbf{q}} f_{1,\mathbf{k},\sigma}^\dagger h_{\mathbf{k}+\mathbf{q},\sigma} h_{\mathbf{p},\sigma'}^\dagger f_{1,\mathbf{p}-\mathbf{q},\sigma'} V_{h,Y}^{\text{ex}}(\mathbf{k} + \mathbf{q}, \mathbf{p}), \end{aligned} \quad (15)$$

$$\begin{aligned} H_{h,X} = & \sum_{\mathbf{k}, \mathbf{p}, \mathbf{q}} f_{2,\mathbf{k},\sigma}^\dagger f_{2,\mathbf{k}+\mathbf{q},\sigma} h_{\mathbf{p},\sigma'}^\dagger h_{\mathbf{p}-\mathbf{q},\sigma'} V_{h,X}^{\text{den}}(\mathbf{p}, \mathbf{p} - \mathbf{q}) + \sum_{\mathbf{k}, \mathbf{p}, \mathbf{q}} f_{2,\mathbf{k},\sigma}^\dagger h_{\mathbf{k}+\mathbf{q},\sigma} f_{2,\mathbf{p},\sigma'}^\dagger h_{\mathbf{p}-\mathbf{q},\sigma'} V_{h,X}^{\text{ph}}(\mathbf{k} + \mathbf{q}, \mathbf{p} - \mathbf{q}) \\ & + \sum_{\mathbf{k}, \mathbf{p}, \mathbf{q}} f_{2,\mathbf{k},\sigma}^\dagger h_{\mathbf{k}+\mathbf{q},\sigma} h_{\mathbf{p},\sigma'}^\dagger f_{2,\mathbf{p}-\mathbf{q},\sigma'} V_{h,X}^{\text{ex}}(\mathbf{k} + \mathbf{q}, \mathbf{p}). \end{aligned} \quad (16)$$

The interaction within the electron pockets labeled as $H_{X,X}$, $H_{Y,Y}$ and $H_{X,Y}$ are already in the band basis and captured by the last 4 terms of Eq.(8). We group interactions into density-density, exchange and pair-hopping by labeling them "den", "ex" and "ph" respectively. Since the electron pockets are assumed to be mono-orbital, in Eqs.(14-16), interaction strengths are momentum dependent only for the hole pockets in the following way

$$\begin{aligned} V_{hh}^{\text{den}} = & U_4 \left\{ \sin \theta_{\mathbf{k}} \sin \theta_{\mathbf{k}+\mathbf{q}} \sin \theta_{\mathbf{p}} \sin \theta_{\mathbf{p}-\mathbf{q}} + \cos \theta_{\mathbf{k}} \cos \theta_{\mathbf{k}+\mathbf{q}} \cos \theta_{\mathbf{p}} \cos \theta_{\mathbf{p}-\mathbf{q}} \right\} \\ & + \tilde{U}_4 \left\{ \sin \theta_{\mathbf{k}} \sin \theta_{\mathbf{k}+\mathbf{q}} \cos \theta_{\mathbf{p}} \cos \theta_{\mathbf{p}-\mathbf{q}} + \cos \theta_{\mathbf{k}} \cos \theta_{\mathbf{k}+\mathbf{q}} \sin \theta_{\mathbf{p}} \sin \theta_{\mathbf{p}-\mathbf{q}} \right\} \\ & + \tilde{\tilde{U}}_4 \left\{ \sin \theta_{\mathbf{k}} \cos \theta_{\mathbf{k}+\mathbf{q}} \cos \theta_{\mathbf{p}} \sin \theta_{\mathbf{p}-\mathbf{q}} + \cos \theta_{\mathbf{k}} \sin \theta_{\mathbf{k}+\mathbf{q}} \sin \theta_{\mathbf{p}} \cos \theta_{\mathbf{p}-\mathbf{q}} \right\} \\ & + \bar{U}_4 \left\{ \sin \theta_{\mathbf{k}} \cos \theta_{\mathbf{k}+\mathbf{q}} \sin \theta_{\mathbf{p}} \cos \theta_{\mathbf{p}-\mathbf{q}} + \cos \theta_{\mathbf{k}} \sin \theta_{\mathbf{k}+\mathbf{q}} \cos \theta_{\mathbf{p}} \sin \theta_{\mathbf{p}-\mathbf{q}} \right\}, \end{aligned} \quad (17)$$

$$V_{h,Y}^{\text{den}} = U_1 \sin \theta_{\mathbf{p}} \sin \theta_{\mathbf{p}-\mathbf{q}} + \bar{U}_1 \cos \theta_{\mathbf{p}} \cos \theta_{\mathbf{p}-\mathbf{q}}, \quad (18)$$

$$V_{h,X}^{\text{den}} = U_1 \cos \theta_{\mathbf{p}} \cos \theta_{\mathbf{p}-\mathbf{q}} + \bar{U}_1 \sin \theta_{\mathbf{p}} \sin \theta_{\mathbf{p}-\mathbf{q}}, \quad (19)$$

$$V_{h,Y}^{\text{ph}} = \frac{U_3}{2} \sin \theta_{\mathbf{k}+\mathbf{q}} \sin \theta_{\mathbf{p}-\mathbf{q}} + \frac{\bar{U}_3}{2} \cos \theta_{\mathbf{k}+\mathbf{q}} \cos \theta_{\mathbf{p}-\mathbf{q}}, \quad (20)$$

$$V_{h,X}^{\text{ph}} = \frac{U_3}{2} \cos \theta_{\mathbf{k}+\mathbf{q}} \cos \theta_{\mathbf{p}-\mathbf{q}} + \frac{\bar{U}_3}{2} \sin \theta_{\mathbf{k}+\mathbf{q}} \sin \theta_{\mathbf{p}-\mathbf{q}}, \quad (21)$$

$$V_{h,Y}^{\text{ex}} = U_2 \sin \theta_{\mathbf{k}+\mathbf{q}} \sin \theta_{\mathbf{p}} + \bar{U}_2 \cos \theta_{\mathbf{k}+\mathbf{q}} \cos \theta_{\mathbf{p}}, \quad (22)$$

$$V_{h,X}^{\text{ex}} = U_2 \cos \theta_{\mathbf{k}+\mathbf{q}} \cos \theta_{\mathbf{p}} + \bar{U}_2 \sin \theta_{\mathbf{k}+\mathbf{q}} \sin \theta_{\mathbf{p}}. \quad (23)$$

On the other hand, all other interactions which involve the inner hole band fermions are contained in H_{other} term. We ignore the effects of this part on the nematic criticality and the superconductivity in the rest of the paper because of the two reasons: (a) unlike the outer hole pocket, inner hole pocket is quite small in size and can even disappear below the Fermi energy in the presence of a small spin-orbit coupling or external strain, and (b) the presence of the inner hole pocket does not change the main results of our paper qualitatively, but complicates the calculation unnecessarily.

SUPPLEMENTARY DISCUSSION II: ORBITAL ORDER INSTABILITY

We define the orbital/nematic order as the density difference between the d_{xz} and d_{yz} fermions. A non-zero value for this order parameter breaks the Z_2 symmetry of the system. In the band language, the order parameter translates to two zero momentum d-wave components: one for the hole pocket

$$\phi_h = \sum_{\mathbf{k}, \sigma} \langle d_{yz, \mathbf{k}, \sigma}^\dagger d_{yz, \mathbf{k}, \sigma} \rangle - \langle d_{xz, \mathbf{k}, \sigma}^\dagger d_{xz, \mathbf{k}, \sigma} \rangle \quad (24)$$

$$= \sum_{\mathbf{k}, \sigma} \langle h_{\mathbf{k}, \sigma}^\dagger h_{\mathbf{k}, \sigma} \rangle \cos 2\theta_{\mathbf{k}} \quad (25)$$

and another for the electron pockets

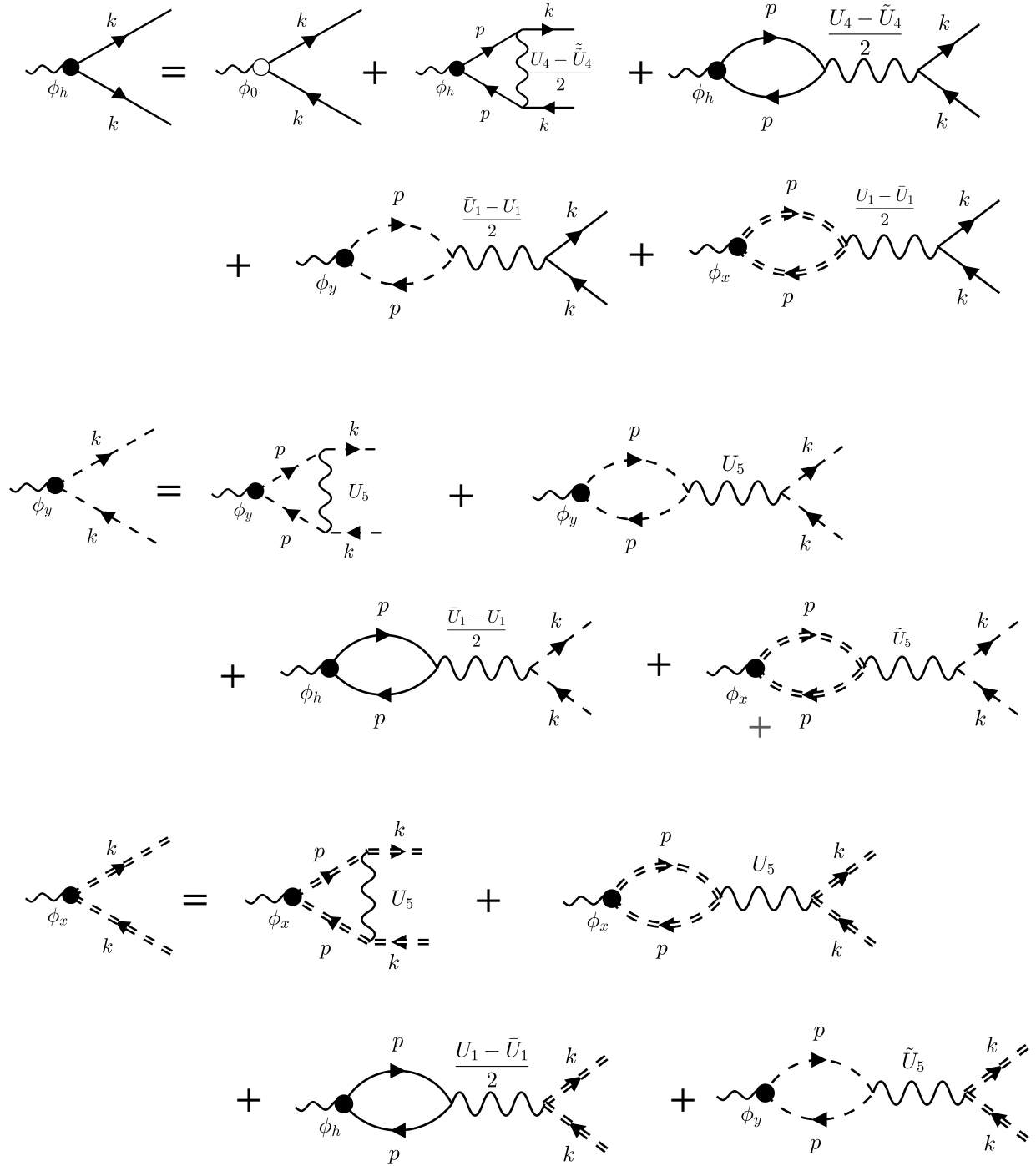
$$\phi_e = \sum_{\mathbf{k}} \langle f_{2, \mathbf{k}, \sigma}^\dagger f_{2, \mathbf{k}, \sigma} \rangle - \langle f_{1, \mathbf{k}, \sigma}^\dagger f_{1, \mathbf{k}, \sigma} \rangle \quad (26)$$

$$= \phi_x - \phi_y. \quad (27)$$

Here, ϕ_x and ϕ_y are the fermionic density in the X - and Y - pockets respectively.

To discuss the nematic instability, we have introduced an infinitesimally small Z_2 symmetry breaking external perturbation to the interaction Hamiltonian in the form of $\phi_0 \cos 2\theta_{\mathbf{k}} h_{\mathbf{k}}^\dagger h_{\mathbf{k}}$ and define nematic susceptibility as $\phi_h = \chi_{\text{nem}} \phi_0$. The onset of the nematic order is signaled by the divergence of its susceptibility, χ_{nem} . We represent the set of self-consistent equations for ϕ_h , ϕ_x and ϕ_y in Fig. 1. They are obtained by adding up the Hartree and Fock diagrams for different bands and turn out to be

$$\begin{aligned} \phi_h \cos 2\theta_{\mathbf{k}} = & \phi_0 \cos 2\theta_{\mathbf{k}} + \phi_h \int_p G_p^h G_p^h \cos 2\theta_{\mathbf{p}} V_{hh}^{\text{den}}(\mathbf{k}, \mathbf{p}; \mathbf{p}, \mathbf{k}) - 2\phi_h \int_p G_p^h G_p^h \cos 2\theta_{\mathbf{p}} V_{hh}^{\text{den}}(\mathbf{k}, \mathbf{k}; \mathbf{p}, \mathbf{p}) \\ & - 2\phi_y \int_p G_p^Y G_p^Y V_{h,Y}^{\text{den}}(\mathbf{k}, \mathbf{k}; \mathbf{p}, \mathbf{p}) - 2\phi_x \int_p G_p^X G_p^X V_{h,X}^{\text{den}}(\mathbf{k}, \mathbf{k}; \mathbf{p}, \mathbf{p}), \end{aligned} \quad (28)$$



Supplementary Figure 1. **Nematic order instability diagram:** Set of ladder equations for the nematic order vertex for hole pocket(ϕ_h), X-pocket(ϕ_x) and Y- pocket(ϕ_y). Here we label only the d-wave component of the interactions. The solid line, dashed line and double dashed line represents Green's function for the hole pocket, X- pocket and Y- pocket fermion. The curvy line represents interaction between fermions. Black dot represent the nematic vertex while the void dot represent the bare vertex for the hole pocket only. We keep both the external fermion momentum $\mathbf{k}$ same.

$$\phi_y = \phi_y \int_p G_p^Y G_p^Y U_5 - 2\phi_y \int_p G_p^Y G_p^Y U_5 - 2\phi_h \int_p G_p^h G_p^h \cos 2\theta \mathbf{p} V_{h,Y}^{\text{den}}(\mathbf{k}, \mathbf{k}; \mathbf{p}, \mathbf{p}) - 2\phi_x \int_p G_p^X G_p^X \tilde{U}_5, \quad (29)$$

$$\phi_x = \phi_x \int_p G_p^X G_p^X U_5 - 2\phi_x \int_p G_p^Y G_p^Y U_5 - 2\phi_h \int_p G_p^h G_p^h \cos 2\theta \mathbf{p} V_{h,X}^{\text{den}}(\mathbf{k}, \mathbf{k}; \mathbf{p}, \mathbf{p}) - 2\phi_y \int_p G_p^Y G_p^Y \tilde{U}_5. \quad (30)$$

Here $G^i(\mathbf{p}, \omega) = 1/i\omega - \xi_i(\mathbf{p})$ is the Green's function for the pocket i with 4- momentum $p = (\mathbf{p}, \omega)$. $\int_p$ stands for $T \sum_{\Omega_n} \int \frac{d^2 \mathbf{p}}{(2\pi)^2}$, where Ω_n is the fermionic Matsubara frequency ($\Omega_n = (2n+1)\pi T$), $\mathbf{p}$ is the lattice momentum and T is the temperature. Because orbital order is a zero momentum order, we only keep the low momentum transfer interaction like density interaction and ignore large momentum transfer interaction like exchange interaction in Eqs.(28-30). Since nematic orders ϕ_h and ϕ_e are d-wave order, only the d-wave component of the interactions(proportional to $\cos 2\theta$ terms) will contribute in Eqs. (28-30). Using Eqs. (17-23), one finds the s - and d - wave components of the interactions

$$V_{hh}^{\text{den}}(\mathbf{k}, \mathbf{p}; \mathbf{p}, \mathbf{k}) = \frac{U_4 + \tilde{U}_4}{2} + \frac{U_4 - \tilde{U}_4}{2} \cos 2\theta_{\mathbf{k}} \cos 2\theta_{\mathbf{p}} + \frac{\tilde{U}_4 + \bar{U}_4}{2} \sin 2\theta_{\mathbf{k}} \sin 2\theta_{\mathbf{p}}, \quad (31)$$

$$V_{hh}^{\text{den}}(\mathbf{k}, \mathbf{k}; \mathbf{p}, \mathbf{p}) = \frac{U_4 + \tilde{U}_4}{2} + \frac{U_4 - \tilde{U}_4}{2} \cos 2\theta_{\mathbf{k}} \cos 2\theta_{\mathbf{p}} + \frac{\tilde{U}_4 + \bar{U}_4}{2} \sin 2\theta_{\mathbf{k}} \sin 2\theta_{\mathbf{p}}, \quad (32)$$

$$V_{h,Y}^{\text{den}}(\mathbf{k}, \mathbf{k}; \mathbf{p}, \mathbf{p}) = \frac{U_1 + \bar{U}_1}{2} - \frac{U_1 - \bar{U}_1}{2} \cos 2\theta_{\mathbf{p}}, \quad (33)$$

$$V_{h,X}^{\text{den}}(\mathbf{k}, \mathbf{k}; \mathbf{p}, \mathbf{p}) = \frac{U_1 + \bar{U}_1}{2} + \frac{U_1 - \bar{U}_1}{2} \cos 2\theta_{\mathbf{p}}. \quad (34)$$

Combining all these terms for Eqs. (28-30), we get

$$\phi_h = \phi_0 - \phi_h U_h^d \Pi_h^d - \phi_e U_{he} \Pi_e, \quad (35)$$

$$\phi_e = -\phi_e U_e^d \Pi_e - 2\phi_h U_{he} \Pi_h^d. \quad (36)$$

Here, $U_h^d = \frac{U_4 - 2\tilde{U}_4 + \tilde{U}_4}{2}$ and $U_e^d = U_5 - 2\tilde{U}_5$ are the effective d-wave intra-pockets interactions for hole and electron pockets respectively. $U_{he} = (U_1 - \bar{U}_1)$ is the effective d-wave inter-pocket interaction for the nematic order. We define the effective polarization bubbles with vertex factor $\cos 2\theta$ for hole pocket and 1 for electron pockets as

$$\Pi_h^d = - \int_p G_p^h G_p^h \cos^2 2\theta_{\mathbf{p}}, \quad (37)$$

$$\Pi_e = - \int_p G_p^e G_p^e \quad (38)$$

respectively. Within our convention, the polarization bubbles are positive $\Pi_h^d, \Pi_e > 0$. Eq. (36) is found from two other equations, $\phi_y = -\phi_y U_5 \Pi_e + \phi_h U_{he} \Pi_h^d - 2\phi_x \tilde{U}_5 \Pi_e$ and $\phi_x = -\phi_x U_5 \Pi_e - \phi_h U_{he} \Pi_h^d - 2\phi_y \tilde{U}_5 \Pi_e$. Combining Eqs. (35) and (36)), we find the expression for χ_{nem} ,

$$\chi_{\text{nem}} = \frac{1}{(1 + U_h^d \Pi_h^d)(1 + U_e^d \Pi_e) - 2U_{he}^2 \Pi_h^d \Pi_e} = \frac{1}{Z}. \quad (39)$$

As stated before the onset of the orbital order is set when $Z = 0$. To see what kind of orbital order (measured by the relative sign between ϕ_h and ϕ_e) is produced at the transition point, we need to solve Eqs. (35) and (36)) without the source term ϕ_0 . At the bare interaction level(9-12), we find no solution exists for $J > U/5$ while only sign preserving solution(called d^{++} with $\text{sgn}(\phi_h) = \text{sgn}(\phi_e)$) exists when $J < U/5$. To see this we need to include the effect of the large momentum transfer interaction($U_2, \tilde{U}_2, \tilde{U}_5$) as at the bare level they are equal in magnitude to the low momentum transfer density interaction and can't be ignored. The calculation is straightforward [2] and makes both the effective intra-pocket and inter-pocket interactions comparable: $U_h^d = U_{he} = U_e^d/2 = (5J - U)/2$. A possibility of another solution called $d^\pm$ orbital order with $\text{sgn}(\phi_h) = -\text{sgn}(\phi_e)$ can exist if we consider U_h^d, U_e^d and U_{he} differ from the bare values such that $U_h^d U_e^d \neq 2U_{he}^2$ and $U_{he} > 0$. This can happen as a consequence of the renormalization coming from the high energy fermions. In experiments, $d^\pm$ orbital order is found [3].

SUPPLEMENTARY DISCUSSION III: BARE PAIRING INTERACTION

In the superconducting channel, the bare pairing interaction from Eq. (8) reads

$$H_{\text{pair}} = \frac{V_0^h}{2} h_{\mathbf{k},\uparrow}^\dagger h_{-\mathbf{k},\downarrow}^\dagger h_{-\mathbf{p},\downarrow} h_{\mathbf{p},\uparrow} + \frac{V_0^e}{2} \sum_{i=1}^2 f_{i,\mathbf{k},\uparrow}^\dagger f_{i,-\mathbf{k},\downarrow}^\dagger f_{i,-\mathbf{p},\downarrow} f_{i,\mathbf{p},\uparrow} + V_s^{h,e} h_{\mathbf{k},\uparrow}^\dagger h_{-\mathbf{k},\downarrow}^\dagger (f_{1,-\mathbf{p},\downarrow} f_{1,\mathbf{p},\uparrow} + f_{2,-\mathbf{p},\downarrow} f_{2,\mathbf{p},\uparrow}) \quad (40)$$

$$+ V_d^{h,e} h_{\mathbf{k},\uparrow}^\dagger h_{-\mathbf{k},\downarrow}^\dagger (f_{2,-\mathbf{p},\downarrow} f_{2,\mathbf{p},\uparrow} - f_{1,-\mathbf{p},\downarrow} f_{1,\mathbf{p},\uparrow}) \cos 2\theta_{\mathbf{k}}. \quad (41)$$

Here, V_0^h and V_0^e are the intra-pocket pairing interaction for the hole and electron pockets (same for $X-$ and $Y-$) respectively with the expression $V_0^h = \frac{U_4 + \bar{U}_4}{2} + \frac{U_4 - \bar{U}_4}{2} \cos 2\theta_{\mathbf{k}} \cos 2\theta_{\mathbf{p}} + \frac{\tilde{U}_4 + \tilde{\bar{U}}_4}{2} \sin 2\theta_{\mathbf{k}} \sin 2\theta_{\mathbf{p}}$ and $V_0^e = U_5$. On the other hand, $V_s^{h,e}$ and $V_d^{h,e}$ are the $s-$ and $d-$ wave components of the inter-pocket pairing interactions respectively with the expression $V_s^{h,e} = \frac{U_3 + \bar{U}_3}{2}$ and $V_d^{h,e} = \frac{U_3 - \bar{U}_3}{2}$. We neglect the pairing interaction between the electron pockets. Using Eqs. (9-12), one finds the pairing interactions are repulsive and can only lead to a superconducting instability if the inter-pocket repulsion overcomes intra-pocket repulsion. We define the superconducting gap on the hole pocket as Δ_h , on $X-$ pocket as Δ_x and on $Y-$ pocket as Δ_y . In the tetragonal phase, $s-$ and $d-$ wave components of the superconducting gap equations decouple. For the $s-$ wave component (labeled by a suffix s) the set of linearized equations for the gap is obtained straightforwardly and reads as

$$\begin{pmatrix} \Delta_h^s \\ \Delta_e^s \end{pmatrix} = L \begin{pmatrix} -u_h^s & -u_{he}^s \\ -2u_{he}^s & -u_e \end{pmatrix} \begin{pmatrix} \Delta_h^s \\ \Delta_e^s \end{pmatrix}. \quad (42)$$

Here, $\Delta_e^s = \Delta_x + \Delta_y$, $u_h^s = N_0 \frac{U_4 + \bar{U}_4}{2}$, $u_{he}^s = N_0 \frac{U_3 + \bar{U}_3}{2}$, $u_e = N_0 U_5$, N_0 is the density of states, $L = \log \frac{1.13\Lambda}{T_c}$ and Λ is the upper cut-off for the pairing. Eq. (42) has a solution only if the largest eigen-value $\lambda_s = \frac{-(u_e + u_h^s) + \sqrt{(u_e - u_h^s)^2 + 8(u_{he}^s)^2}}{2}$ is positive. For simplicity, if we assume all intra-pocket pairing interactions are same $u_h^s = u_e^s = u$, then this criteria translates to $u_{he}^s > \frac{u}{\sqrt{2}}$. For $u_{he}^s < \frac{u}{\sqrt{2}}$, there is no $s-$ wave superconducting instability. For the d -wave, the set of linearized equation become

$$\begin{pmatrix} \Delta_h^d \\ \Delta_e^d \end{pmatrix} = L \begin{pmatrix} -u_h^d/2 & -u_{he}^d \\ -u_{he}^d & -u_e \end{pmatrix} \begin{pmatrix} \Delta_h^d \\ \Delta_e^d \end{pmatrix}. \quad (43)$$

Here, $\Delta_e^d = \Delta_x - \Delta_y$, $u_h^d = N_0 \frac{U_4 - \bar{U}_4}{2}$, $u_{he}^d = N_0 \frac{U_3 - \bar{U}_3}{2}$. Eq. (43) has a solution only if the largest eigen-value $\lambda_d = \frac{-(u_e + u_h^d/2) + \sqrt{(u_e - u_h^d/2)^2 + 4(u_{he}^d)^2}}{2}$ is positive. For simplicity, if we assume $u_h^d = u_h^s = u_e = u$ and $u_{he}^s = u_{he}^d = v$, then $\lambda_s = -u + \sqrt{2}v^2 > \lambda_d = -3u + \sqrt{u^2 + 16v^2}/4 > 0$ when $v > u/\sqrt{2}$: $s-$ wave will be the leading instability. In the next section, we show that near a nematic instability order parameter fluctuations renormalize the intra-pocket pairing interactions u_s^h, u_s^e attractive and scale with the nematic susceptibility χ_{nem} which makes λ always positive.

SUPPLEMENTARY DISCUSSION IV: RENORMALIZATION OF THE PAIRING INTERACTION NEAR THE NEMATIC TRANSITION POINT

We find that the bare intra-pocket pairing interactions for both the hole and electron pockets, V_0^h and V_0^e respectively are repulsive, and in order to have superconductivity, inter-pocket repulsion, $V_{s,d}^{h,e}$ has to overcome the intra-pocket repulsion. In this section, we derive the corrections to the intra-pocket and inter-pocket pairing interaction due to the proximity of a nematic order. We show that the dressed intra-pocket pairing interactions labeled as V_{eff}^h (for hole) and V_{eff}^e (for electron pockets) get an attractive singular component proportional to the nematic susceptibility χ_{nem} besides its regular repulsive part near the nematic order. On the other hand, the large momentum transfer ($\mathbf{q} = (0, \pi)$ or $(\pi, 0)$) type interaction pair hopping/inter-pocket pairing interaction is mostly unaffected by the low momentum nematic fluctuations.

A. Intra-pocket pairing interaction

We first focus on the pairing interaction on the hole pocket describing scattering between fermion pair states $(\mathbf{k}, \alpha; -\mathbf{k}, \gamma) \rightarrow (\mathbf{p}, \beta; -\mathbf{p}, \delta)$ where $\mathbf{k}, \mathbf{p}$ are the momentum and $\alpha, \beta, \gamma, \delta$ are the spin components of the incoming

and outgoing fermions. Because of the orbital degrees of freedom for the hole pocket fermions, it is convenient to cast Eqs. (14-16) in a slightly different format

$$H_{hh} = \frac{1}{2} \sum_{\mathbf{k}, \mathbf{p}, \mathbf{q}} b^T(\mathbf{k}, \mathbf{k} + \mathbf{q}) \Gamma_{hh} b(\mathbf{p}, \mathbf{p} - \mathbf{q}) h_{\mathbf{k}, \sigma}^\dagger h_{\mathbf{k} + \mathbf{q}, \sigma} h_{\mathbf{p}, \sigma'}^\dagger h_{\mathbf{p} - \mathbf{q}, \sigma'} \quad (44)$$

$$H_{h,X} = \sum_{\mathbf{k}, \mathbf{p}, \mathbf{q}} d^T \Gamma_{hx} b(\mathbf{p}, \mathbf{p} - \mathbf{q}) f_{2, \mathbf{k}, \sigma}^\dagger f_{2, \mathbf{k} + \mathbf{q}, \sigma} h_{\mathbf{p}, \sigma'}^\dagger h_{\mathbf{p} - \mathbf{q}, \sigma'} \quad (45)$$

$$H_{h,Y} = \sum_{\mathbf{k}, \mathbf{p}, \mathbf{q}} d^T \Gamma_{hy} b(\mathbf{p}, \mathbf{p} - \mathbf{q}) f_{1, \mathbf{k}, \sigma}^\dagger f_{1, \mathbf{k} + \mathbf{q}, \sigma} h_{\mathbf{p}, \sigma'}^\dagger h_{\mathbf{p} - \mathbf{q}, \sigma'}. \quad (46)$$

Here, b and d are the bare vertex for the intra-hole(Γ_{hh}) and inter-pocket(Γ_{hx} and Γ_{hy}) density-density interactions defined as

$$b^T(\mathbf{k}, \mathbf{k} + \mathbf{q}) = (\cos \theta_{\mathbf{k}} \cos \theta_{\mathbf{k} + \mathbf{q}} \quad \sin \theta_{\mathbf{k}} \sin \theta_{\mathbf{k} + \mathbf{q}} \quad \cos \theta_{\mathbf{k}} \sin \theta_{\mathbf{k} + \mathbf{q}} \quad \sin \theta_{\mathbf{k}} \cos \theta_{\mathbf{k} + \mathbf{q}}), \quad (47)$$

$$d^T = (1 \quad 1 \quad 0 \quad 0), \quad (48)$$

and

$$\Gamma_{hh} = \begin{pmatrix} u_4 & \tilde{u}_4 & 0 & 0 \\ \tilde{u}_4 & u_4 & 0 & 0 \\ 0 & 0 & \bar{u}_4 & \tilde{\bar{u}}_4 \\ 0 & 0 & \tilde{\bar{u}}_4 & \bar{u}_4 \end{pmatrix} \quad \Gamma_{hx} = \begin{pmatrix} U_1 & 0 & 0 & 0 \\ 0 & \bar{U}_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \Gamma_{hy} = \begin{pmatrix} \bar{U}_1 & 0 & 0 & 0 \\ 0 & U_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (49)$$

The lower 2×2 diagonal block of Γ_{hx} and Γ_{hy} is zero as a consequence of the orbital preserving interaction between hole and electron pockets. To simplify the presentation, we first ignore the inter-electron pocket density interaction $\bar{U}_5$ in our microscopic derivation of the dressed interaction, and later give our full result when one includes it.

1. Singular part of the pairing interaction

We argue below that the singular part of the renormalized intra-pocket pairing interaction, V_{eff}^h near the onset of the nematic order is captured by the Dyson equation illustrated in Fig.2 and we call it V_{singular}^h . Each shaded boxes in Fig.2 labeled as B_h , B_x and B_y are an infinite series of bubble diagrams(defined below) made of hole, X - and Y - pocket fermions respectively as illustrated in Fig.3. Each bubble is dressed by an insertion of a ladder series of interactions at one of the vertices labeled as γ_h, γ_x and γ_y for the hole, X - and Y - pocket bubbles respectively shown in Fig.4. We also dress the bare external vertex b by including a ladder series of wine-glass (second diagram of the right hand side of the equation of Fig.5) and bubble diagram (third diagram of the right hand side of the equation of Fig.5) shown in Fig.5 and label it as $\tilde{\gamma}_h$. With these preliminary definitions, the computation of the singular part of the effective pairing interaction for the hole pocket fermions as depicted in Fig.2 goes as

$$-V_{\text{singular}}^h(\mathbf{k}, \mathbf{p}) = \tilde{\gamma}_h^T(\mathbf{k}, \mathbf{p}) \cdot [S + S \circ B_h \circ S + \dots] \cdot \tilde{\gamma}_h(-\mathbf{k}, -\mathbf{p}), \quad (50)$$

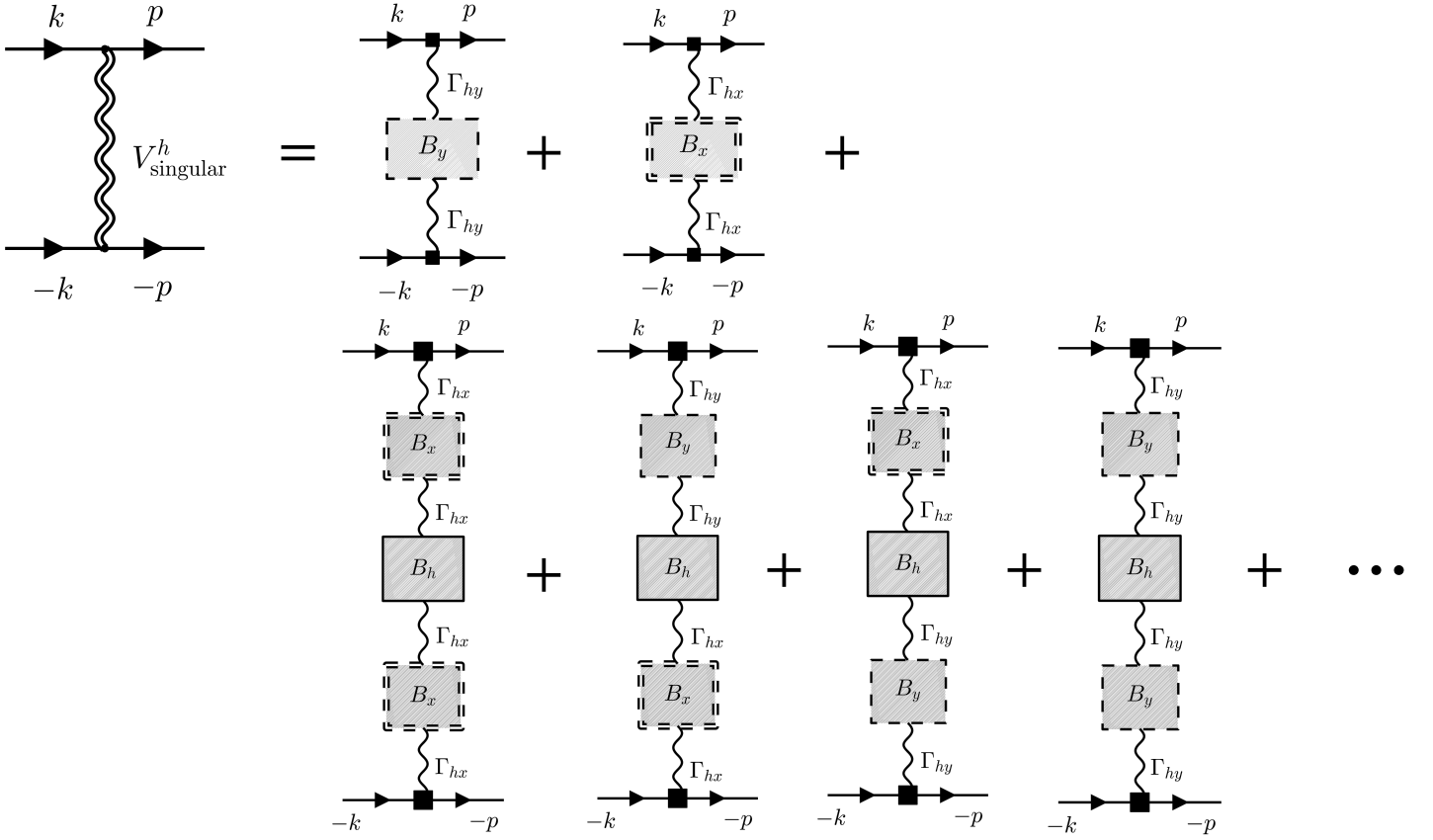
$$= \tilde{\gamma}_h^T(\mathbf{k}, \mathbf{p}) \cdot S \circ \frac{1}{1 - B_h \circ S} \cdot \tilde{\gamma}_h(-\mathbf{k}, -\mathbf{p}), \quad (51)$$

where "... " represents the higher order terms of the series and the spin component of each diagram is $\delta_{\alpha\beta}\delta_{\gamma\delta}$. S is the sum of the first two diagrams of the right hand side of Fig.2(without the external vertex corrections represented by the black square) which translates to

$$S = \Gamma_{hy} \cdot d \cdot d^T \cdot \Gamma_{hy} B_y + \Gamma_{hx} \cdot d \cdot d^T \cdot \Gamma_{hx} B_x. \quad (52)$$

In Eqs. (50-52), for convenience we omit the momentum($q = k - p$) dependence of S and B_h , and use "." to denote the product between a matrix and vector and "o" to denote the product between two matrices. Because of the orbital structure of the hole pocket, $\tilde{\gamma}_h, \gamma_h, B_h$ and S are matrix in nature. Below we compute the analytical expression for $S, \gamma_h, \tilde{\gamma}_h, B_h, B_x$ and B_y to give the final form of Eq. (51). We first calculate γ_h , the vertex correction in the bubble with an insertion of the ladder series as shown in Fig.4. A straightforward calculation for γ_h gives

$$\gamma_h(\mathbf{k}, \mathbf{k} + \mathbf{q}) = b(\mathbf{k}, \mathbf{k} + \mathbf{q}) - \int_p b(\mathbf{p}, \mathbf{p} + \mathbf{q}) G_p^h G_{p+q}^h b^T(\mathbf{k}, \mathbf{p}) \cdot \Gamma_{hh} \cdot b(\mathbf{p} + \mathbf{q}, \mathbf{k} + \mathbf{q}) + \dots \quad (53)$$



Supplementary Figure 2. **Singular part of the dressed intraband density interaction on the hole pocket:** Dyson equation for the singular part of the dressed pairing interaction labeled as V_{sing}^h represented by the double curly line between the fermions with paired state $(\mathbf{k}, -\mathbf{k})$ scattering to $(\mathbf{p}, -\mathbf{p})$ within the hole pocket. The black straight arrow represents hole fermionic Green's function. The shaded boxes labeled as B_h , B_x and B_y are defined in Fig.3. The black rectangular vertex is defined in Fig.5. Single curly line represents the bare interaction between the hole and electron pocket. "... " represents higher order terms of series.

where ... represents higher order terms in the ladder series. We define

$$\int_p b(\mathbf{p}, \mathbf{p} + \mathbf{q}) G_p^h G_{p+q}^h b^T(\mathbf{k}, \mathbf{p}) \cdot \Gamma_{hh} \cdot b(\mathbf{p} + \mathbf{q}, \mathbf{k} + \mathbf{q}) = -\alpha(q) \cdot b(\mathbf{k}, \mathbf{k} + \mathbf{q}). \quad (54)$$

Here $\alpha(q)$ is a 4×4 matrix defined below

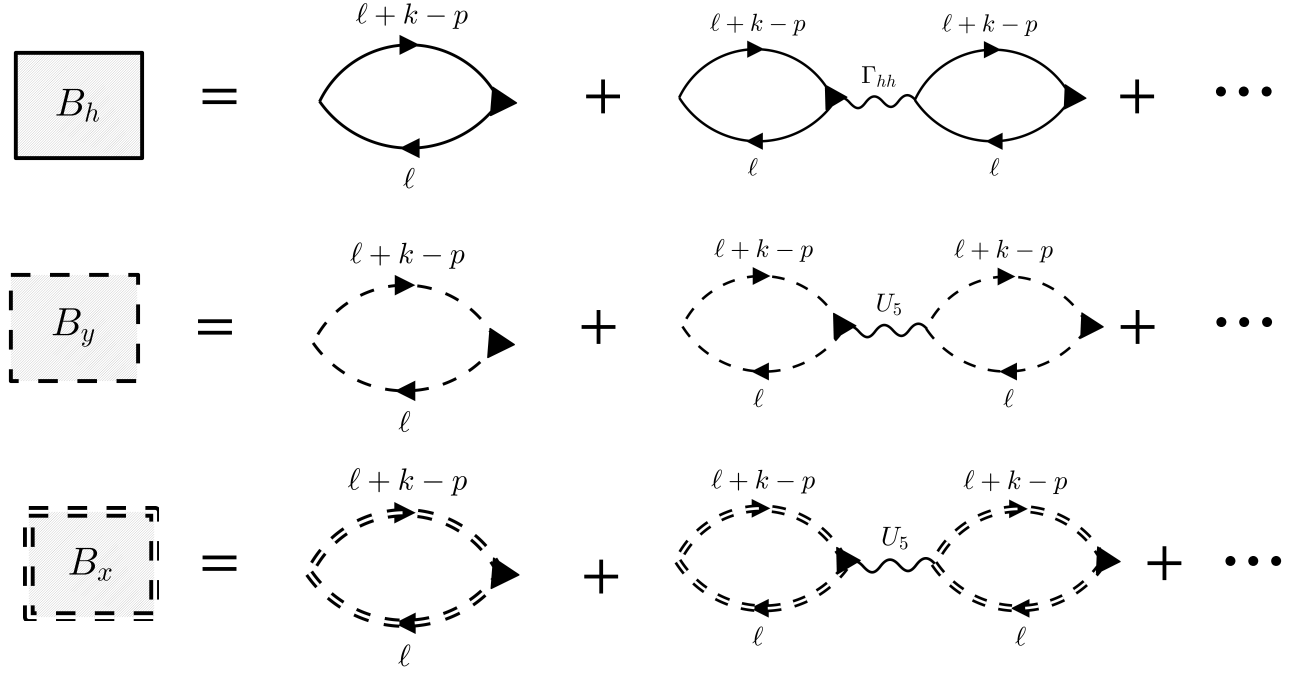
$$\alpha = \begin{pmatrix} U_4 \Pi_{cc} + \tilde{U}_4 \Pi_m & U_4 \Pi_m + \tilde{U}_4 \Pi_{ss} & 0 & 0 \\ U_4 \Pi_m + \tilde{U}_4 \Pi_{cc} & U_4 \Pi_{ss} + \tilde{U}_4 \Pi_m & 0 & 0 \\ 0 & 0 & \tilde{U}_4 \Pi_{cs} + \bar{U}_4 \Pi_m & \tilde{U}_4 \Pi_m + \bar{U}_4 \Pi_{cs} \\ 0 & 0 & \tilde{U}_4 \Pi_m + \bar{U}_4 \Pi_{cs} & \tilde{U}_4 \Pi_{cs} + \bar{U}_4 \Pi_m \end{pmatrix} \quad (55)$$

with

$$\Pi_{cc}(q) = - \int_l G_l^h G_{l+q}^h \cos^2 \theta_l \cos^2 \theta_{l+q}, \quad \Pi_{ss}(q) = - \int_l G_l^h G_{l+q}^h \sin^2 \theta_l \sin^2 \theta_{l+q} \quad (56)$$

$$\Pi_m(q) = - \int_l G_l^h G_{l+q}^h \cos \theta_l \sin \theta_l \cos \theta_{l+q} \sin \theta_{l+q} \quad \Pi_{cs}(q) = - \int_l G_l^h G_{l+q}^h \cos^2 \theta_l \sin^2 \theta_{l+q} \quad (57)$$

In Eq. (55) we omit the q dependence of α and Π 's for the simplicity of the representation. We put a negative sign in Eq. (54) to make α numerically positive. For $q = 0$, $\Pi_{cc} = \Pi_{ss}$ and $\Pi_m = \Pi_{cs}$. After summing up the ladder series in



Supplementary Figure 3. **Definition of Box:** Dyson equation for the shaded boxes B_h , B_y and B_x used in Fig.2 are represented here. The solid, dashed and double-dashed line represent the Green's functions for the hole, Y - and X - pocket fermions. The Black triangle represents the dressed vertex γ for different pockets depicted in Fig.4. The curly lines represent the interactions defined in the main text."..." captures the higher order terms of the corresponding series.

Eq. (53) we get

$$\gamma_h(\mathbf{k}, \mathbf{k} + \mathbf{q}) = \frac{1}{1 - \alpha} \cdot b(\mathbf{k}, \mathbf{k} + \mathbf{q}). \quad (58)$$

For the electron pockets, the vertex corrections for X - and Y - pockets are same $\gamma_x = \gamma_y = \gamma_e$ and equal to

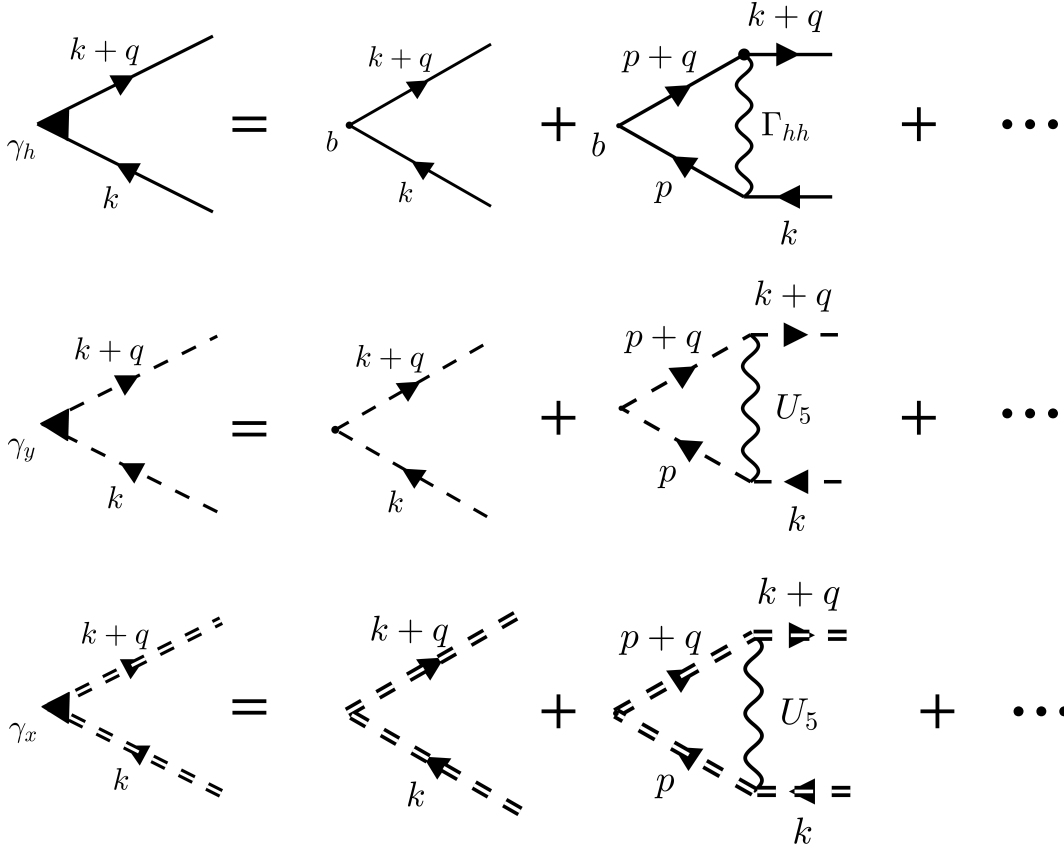
$$\gamma_e = \frac{1}{1 - \Pi_e U_5}, \quad (59)$$

where Π_e is defined in Eq. (38). Next we find the expression for the shaded boxes B_h , B_x and B_y which are the ladder series of dressed bubbles, shown in Fig.3. For the electron pockets, the insertion of one bubble without the vertex correction makes a contribution of $2\Pi_e$. A factor of 2 is coming from the spin. Because we define the polarization bubble (Π_e) with an overall minus sign, it makes the overall expression for B_i positive by canceling the effect of -1 coming from the fermionic loop. Summing up the bubbles with the vertex correction γ_e upto infinite order gives

$$B_e = 2\Pi_e\gamma_e - (2\Pi_e\gamma_e)^2 U_5 + \dots = \frac{2\Pi_e}{1 + U_5\Pi_e}, \quad (60)$$

where $B_x = B_y = B_e$. For the hole pocket, the inclusion of a bubble without the vertex correction (γ_h) makes a contribution of (see first diagram for B_h in Fig.3)

$$M(q) = -2 \int_l b(\mathbf{l}, \mathbf{l} + \mathbf{q}) G_l G_{l+q} b^T(\mathbf{l} + \mathbf{q}, \mathbf{l}), \quad (61)$$



Supplementary Figure 4. **Vertex correction diagram:** Ladder series for the polarization bubble vertex correction defined in Fig.3. We label the dressed vertex γ_h, γ_y and γ_x for the hole, Y- and X- pocket fermions.

where $M(q)$ is a 4×4 matrix defined below

$$M = -2 \int_l G_l G_{l+q} \times \begin{pmatrix} \cos^2 \theta_1 \cos^2 \theta_{1+q} & \cos \theta_1 \sin \theta_1 \cos \theta_{1+q} \sin \theta_{1+q} & \cos \theta_1 \sin \theta_1 \cos^2 \theta_{1+q} & \cos^2 \theta_1 \cos \theta_{1+q} \sin \theta_{1+q} \\ \cos \theta_1 \sin \theta_1 \cos \theta_{1+q} \sin \theta_{1+q} & \sin^2 \theta_1 \sin^2 \theta_{1+q} & \sin^2 \theta_1 \cos \theta_{1+q} \sin \theta_{1+q} & \cos \theta_1 \sin \theta_1 \sin^2 \theta_{1+q} \\ \cos^2 \theta_1 \cos \theta_{1+q} \sin \theta_{1+q} & \cos \theta_1 \sin \theta_1 \sin^2 \theta_{1+q} & \cos \theta_1 \sin \theta_1 \cos \theta_{1+q} \sin \theta_{1+q} & \cos^2 \theta_1 \sin^2 \theta_{1+q} \\ \cos \theta_1 \sin \theta_1 \cos^2 \theta_{1+q} & \sin^2 \theta_1 \cos \theta_{1+q} \sin \theta_{1+q} & \sin^2 \theta_1 \cos^2 \theta_{1+q} & \cos \theta_1 \sin \theta_1 \cos \theta_{1+q} \sin \theta_{1+q} \end{pmatrix}. \quad (62)$$

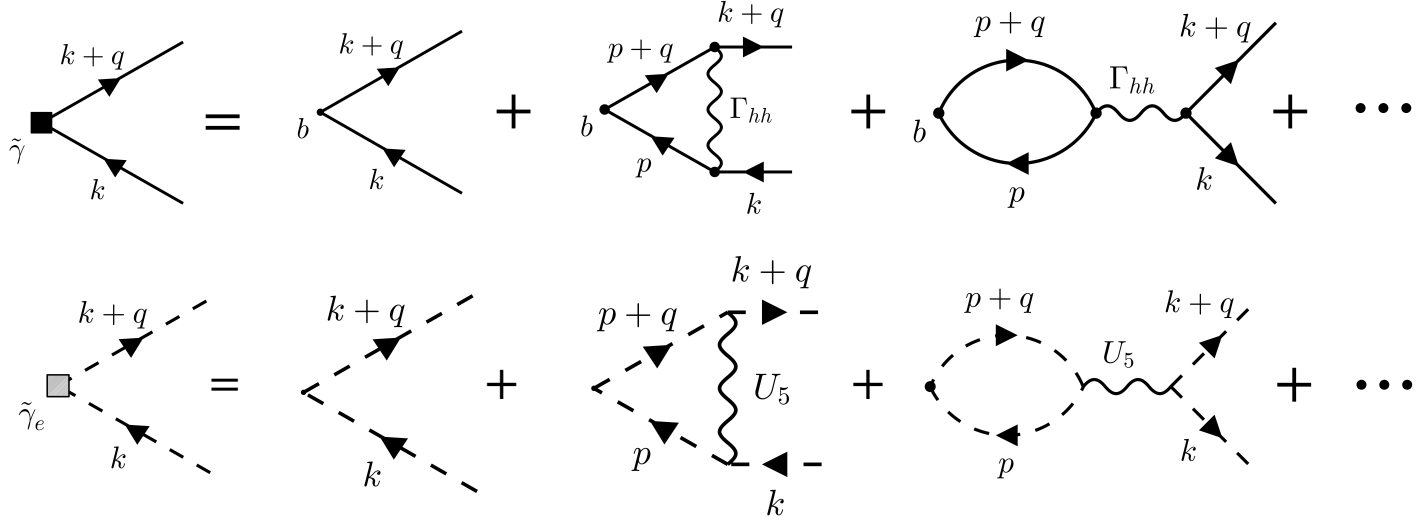
It is easy to check that in the presence of any odd number of "sine" term, the integration in Eq. (62) will give zero because of the parity symmetry of the underlying Hamiltonian. As a result, $M(q)$ becomes

$$M = 2 \begin{pmatrix} \Pi_{cc} & \Pi_m & 0 & 0 \\ \Pi_m & \Pi_{ss} & 0 & 0 \\ 0 & 0 & \Pi_m & \Pi_{cs} \\ 0 & 0 & \Pi_{cs} & \Pi_m \end{pmatrix}. \quad (63)$$

Using Eq. (58) for the vertex correction $\gamma_h^T = b^T \cdot \frac{1}{1 - \alpha^T}$, we sum up the bubbles upto infinite orders and get the expression for B_h as

$$B_h = M \circ \frac{1}{1 - \alpha^T} - M \circ \frac{1}{1 - \alpha^T} \circ \Gamma_{hh} \circ M \circ \frac{1}{1 - \alpha^T} + \dots \quad (64)$$

$$= M \circ \frac{1}{1 - \alpha^T} \circ \frac{1}{1 + \Gamma_{hh} \circ M \circ \frac{1}{1 - \alpha^T}}. \quad (65)$$



Supplementary Figure 5. **External vertex correction diagram:** Dyson equation for the dressed vertex between the external fermions on the hole pocket labeled as $\tilde{\gamma}_h$ (black square) and on the electron pockets labeled as $\tilde{\gamma}_e$ (gray square). b represents the bare vertex for the intra-hole interaction defined in Eq. (44) for hole pocket. The black and dashed straight arrow represent the Green's function for the hole and electron pocket (in this case same for both $X-$ and $Y-$) fermions. Wavy lines represent the interactions defined in the main text. "... " represent the higher order terms in the series.

Next we calculate the correction in the external vertex defined as $\tilde{\gamma}_h$ and depicted in Fig.5. Using Eqs. (54) and (61), we find the expression for $\tilde{\gamma}_h$

$$\tilde{\gamma}_h(\mathbf{k}, \mathbf{k} + \mathbf{q}) = b(\mathbf{k}, \mathbf{k} + \mathbf{q}) + \alpha \circ b(\mathbf{k}, \mathbf{k} + \mathbf{q}) - M \circ \Gamma_{hh} \cdot b(\mathbf{k}, \mathbf{k} + \mathbf{q}) + \dots \quad (66)$$

$$= \frac{1}{1 - \alpha + M \circ \Gamma_{hh}} \cdot b(\mathbf{k}, \mathbf{k} + \mathbf{q}) \quad (67)$$

Combining Eqs.(51,65,67), we find and further simplify the expression for V_{singular}^h

$$-V_{\text{singular}}^h = b^T \cdot \frac{1}{1 - \hat{\alpha}^T + \Gamma_{hh} \circ M} \circ S \circ \frac{1}{1 - B_h \circ S} \circ \frac{1}{1 - \alpha \circ + M \circ \Gamma_{hh}} \cdot b \quad (68)$$

$$= b^T \cdot \frac{1}{1 - \hat{\alpha}^T + \Gamma_{hh} \circ M} \circ [S^{-1} - B_h]^{-1} \circ \frac{1}{1 - \alpha \circ + M \circ \Gamma_{hh}} \cdot b \quad (69)$$

$$= b^T \cdot [1 - \hat{\alpha}^T + \Gamma_{hh} \circ M]^{-1} [S^{-1} - M \circ (1 - \alpha^T + \Gamma_{hh} \circ M)^{-1}]^{-1} \circ \frac{1}{1 - \alpha \circ + M \circ \Gamma_{hh}} \cdot b \quad (70)$$

$$= b^T \cdot [S^{-1} (1 - \alpha^T + \Gamma_{hh} \circ M) - M]^{-1} \circ \frac{1}{1 - \alpha \circ + M \circ \Gamma_{hh}} \cdot b \quad (71)$$

$$= b^T \cdot \frac{1}{1 - \alpha^T + \Gamma_{hh} \circ M - S \circ M} \circ S \circ \frac{1}{1 - \alpha \circ + M \circ \Gamma_{hh}} \cdot b \quad (72)$$

Here, for simplicity of the presentation we suppress the momentum dependence of V_{singular}^h , α , M and the vertices b and

b^T . Using Eqs. (60, 52) we find the expression for S as

$$S = \frac{2\Pi_e}{1 + U_5\Pi_e} \begin{pmatrix} U_1^2 + \bar{U}_1^2 & 2U_1\bar{U}_1 & 0 & 0 \\ 2U_1\bar{U}_1 & U_1^2 + \bar{U}_1^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (73)$$

To understand what qualitative results Eq. (72) yields, we can calculate order by order in inter-pocket interaction S while ignoring all intra-pocket interaction. To simplify the results we assume the external momenta are same $p = k$. To first order in S , direct computation of Eq. (72) produce $\Pi_e\{(U_1 + \bar{U}_1)^2 + (U_1 - \bar{U}_1)^2 \cos^2 2\theta_k\}$ while the second order produces $2\Pi_e^2\{(U_1 + \bar{U}_1)^4\Pi_h^s + (U_1 - \bar{U}_1)^4\Pi_h^d \cos^2 2\theta_k\}$. Π_h^s is the polarization bubble defined in the same way as Π_h^d , but with the vertex form factor 1.

$$\Pi_h^s = - \int_p G_p^h G_p^h. \quad (74)$$

Finding the pattern, we can add all the higher order terms and find

$$-V_{\text{singular}}^h = \frac{\Pi_e(U_1 - \bar{U}_1)^2}{1 - 2\Pi_h^d \Pi_e(U_1 - \bar{U}_1)^2} \cos^2 2\theta_k + \frac{\Pi_e(U_1 + \bar{U}_1)^2}{1 - 2\Pi_h^s \Pi_e(U_1 + \bar{U}_1)^2}. \quad (75)$$

Identifying with Eq. (39), we find the first term of Eq. (75) is nothing but the nematic susceptibility χ_{nem} with a prefactor proportional to $(U_1 - \bar{U}_1)^2 \cos^2 2\theta_k$ when one ignores the intra-pocket density interactions. The second term is non-singular as it is not related to any instability and can be ignored. Next we will include intra-pocket interactions and calculate Eq.(72) exactly. To make the flow of the calculation simple, we introduce a matrix function $Q(x)$ as

$$Q(x) = 1 - \alpha^T + \Gamma_{hh} \circ M - x S \circ M, \quad (76)$$

such that Eq. (72) can be written as (again suppressing the momentum dependence to simplify the presentation)

$$-V_{\text{singular}}^h = b^T \cdot [Q(1)]^{-1} \circ S \circ [Q^T(0)]^{-1} \cdot b. \quad (77)$$

Since, the lower 2×2 diagonal block of S is zero, to compute the effective interaction we need to compute only the upper block of $Q(x)$. Using Eqs. (76,73,63,49), we find the exact expression for $Q(x)$,

$$Q(x) = \begin{pmatrix} A_{11}(x) & A_{12}(x) \\ A_{21}(x) & A_{22}(x) \end{pmatrix}, \quad (78)$$

with

$$A_{11}(x) = 1 + U_4\Pi_{cc} + (2\tilde{U}_4 - \tilde{\tilde{U}}_4) \Pi_m - x \frac{2\Pi_e}{B} \left[(U_1 + \bar{U}_1)^2 (\Pi_{cc} + \Pi_m) + (U_1 - \bar{U}_1)^2 (\Pi_{cc} - \Pi_m) \right] \quad (79)$$

$$A_{22}(x) = 1 + U_4\Pi_{ss} + (2\tilde{U}_4 - \tilde{\tilde{U}}_4) \Pi_m - x \frac{2\Pi_e}{B} \left[U_1 + \bar{U}_1)^2 (\Pi_{ss} + \Pi_m) + (U_1 - \bar{U}_1)^2 (\Pi_{ss} - \Pi_m) \right] \quad (80)$$

$$A_{12}(x) = U_4\Pi_m + (2\tilde{U}_4 - \tilde{\tilde{U}}_4) \Pi_{ss} - x \frac{2\Pi_e}{B} \left[(U_1 + \bar{U}_1)^2 (\Pi_{ss} + \Pi_m) + (U_1 - \bar{U}_1)^2 (\Pi_{ss} - \Pi_m) \right] \quad (81)$$

$$A_{21}(x) = U_4\Pi_m + (2\tilde{U}_4 - \tilde{\tilde{U}}_4) \Pi_{cc} - x \frac{2\Pi_e}{B} \left[(U_1 + \bar{U}_1)^2 (\Pi_{cc} + \Pi_m) + (U_1 - \bar{U}_1)^2 (\Pi_{cc} - \Pi_m) \right] \quad (82)$$

where $B = 1 + U_5\Pi_e$. Using the expression of $Q(x)$, the effective pairing interaction in terms of $A_{i,j}$ becomes

$$-V_{\text{singular}}^h = \frac{1}{\text{Det}[Q(1)]} \frac{1}{\text{Det}[Q(0)]} b^T \cdot \begin{pmatrix} A_{22}(1) & -A_{12}(1) \\ -A_{21}(1) & A_{11}(1) \end{pmatrix} \circ S \circ \begin{pmatrix} A_{22}(0) & -A_{12}(0) \\ -A_{21}(0) & A_{11}(0) \end{pmatrix} \cdot b, \quad (83)$$

where "Det" represents the determinant. We express the form of determinant for $Q(x)$ using Eqs. (79-82) as a power series in B with coefficients Z_i ,

$$\text{Det}[Q(x)] = Z_1 + x \left(\frac{Z_2}{B} + \frac{Z_3}{B^2} \right) \quad (84)$$

with

$$Z_1 = (1 + U_h^d \Pi_h^d)(1 + U_h^s \Pi_h^s) - (\Pi_{cc} - \Pi_{ss})^2 U_h^d U_h^s, \quad (85)$$

$$Z_2 = -2\Pi_e(U_1 + \bar{U}_1)^2 \Pi_h^s (1 + U_h^d \Pi_h^d) - 2\Pi_e(U_1 - \bar{U}_1)^2 \Pi_h^d (1 + U_h^s \Pi_h^s) + 2\Pi_e (\Pi_{cc} - \Pi_{ss})^2 \times \\ ((U_1 + \bar{U}_1)^2 U_h^d + (U_1 - \bar{U}_1)^2 U_h^s) \quad (86)$$

$$Z_3 = 4\Pi_e^2(U_1 + \bar{U}_1)^2 (U_1 - \bar{U}_1)^2 [\Pi_h^d \Pi_h^s - (\Pi_{cc} - \Pi_{ss})^2]. \quad (87)$$

Here, $U_h^s = \frac{U_4 + 2\tilde{U}_4 - \tilde{\tilde{U}}_4}{2}$ and we have used the following relations (suppressing the momentum dependence of the polarization bubble below)

$$\Pi_h^s = \Pi_{cc} + \Pi_{ss} + 2\Pi_m, \quad (88)$$

$$\Pi_h^d = \Pi_{cc} + \Pi_{ss} - 2\Pi_m. \quad (89)$$

In general, Π_{cc} and Π_{ss} differ from each other for $q \neq 0$. Since nematic order is $q = 0$ momentum order with strong fluctuations around it, the leading singularity in the effective pairing interaction comes from $q = 0$. As a result, we can ignore the term $\Pi_{cc} - \Pi_{ss}$ in the Eqs. (85-87) as the difference for $q \rightarrow 0$ is vanishingly small. With a little bit of algebraic manipulation Eq. (84) becomes $\text{Det}[Q(x)]^{-1} = (1 + U_5 \Pi_e)^2 K_d(x) \times K_s(x)$, where K_d and K_s are defined as

$$K_d(x) = \frac{1}{[(1 + U_5 \Pi_e)(1 + U_h^d \Pi_h^d) - 2x \Pi_h^d \Pi_e (U_1 - \bar{U}_1)^2]} \quad (90)$$

$$K_s(x) = \frac{1}{[(1 + U_5 \Pi_e)(1 + U_h^s \Pi_h^s) - 2x \Pi_h^s \Pi_e (U_1 + \bar{U}_1)^2]}. \quad (91)$$

Next, we compute the matrix multiplication in Eq. (83). we assert that the bare vertex $b(\mathbf{k}, \mathbf{p})$ has effectively 2 components relevant to our calculation with

$$b(\mathbf{k}, \mathbf{p}) = (\cos \theta_{\mathbf{k}} \cos \theta_{\mathbf{p}} \quad \sin \theta_{\mathbf{k}} \sin \theta_{\mathbf{p}}). \quad (92)$$

A straight forward calculation shows

$$\begin{aligned} & b^T(\mathbf{k}, \mathbf{p}) \cdot \begin{pmatrix} A_{22}(1) & -A_{12}(1) \\ -A_{21}(1) & A_{11}(1) \end{pmatrix} \circ S \circ \begin{pmatrix} A_{22}(0) & -A_{12}(0) \\ -A_{21}(0) & A_{11}(0) \end{pmatrix} \cdot b(\mathbf{k}, \mathbf{p}) \\ &= \frac{2\Pi_e}{1 + U_5 \Pi_e} \left[\frac{V_1(\mathbf{q})}{2} (1 + \cos 2\theta_{\mathbf{k}} \cos 2\theta_{\mathbf{p}}) + \frac{V_2(\mathbf{q})}{2} \sin 2\theta_{\mathbf{k}} \sin 2\theta_{\mathbf{p}} \right], \end{aligned} \quad (93)$$

with

$$V_1 = \frac{(U_1 + \bar{U}_1)^2}{2} (A_{11}(1) - A_{12}(1)) (A_{11}(0) - A_{12}(0)) + \frac{(U_1 - \bar{U}_1)^2}{2} (A_{11}(1) + A_{12}(1)) (A_{11}(0) + A_{12}(0)) \quad (94)$$

$$V_2 = \frac{(U_1 + \bar{U}_1)^2}{2} (A_{11}(1) - A_{12}(1)) (A_{11}(0) - A_{12}(0)) - \frac{(U_1 - \bar{U}_1)^2}{2} (A_{11}(1) + A_{12}(1)) (A_{11}(0) + A_{12}(0)). \quad (95)$$

All the momentum dependence of $V_{1,2}$ are hidden in the polarization bubbles used in $A_{i,j}$ functions. Using Eqs. (79-82), we find the following expression for

$$A_{11}(1) - A_{12}(1) = 1 + U_h^d \Pi_h^d - \frac{2\Pi_e \Pi_h^d}{1 + U_5 \Pi_e} (U_1 - \bar{U}_1)^2 = \frac{1}{K_d(1)(1 + U_5 \Pi_e)} \quad (96)$$

$$A_{11}(0) - A_{12}(0) = 1 + U_h^d \Pi_h^d = \frac{1}{K_d(0)(1 + U_5 \Pi_e)} \quad (97)$$

$$A_{11}(1) + A_{12}(1) = 1 + U_h^s \Pi_h^s - \frac{2\Pi_e \Pi_h^s}{1 + U_5 \Pi_e} (U_1 + \bar{U}_1)^2 = \frac{1}{K_s(1)(1 + U_5 \Pi_e)} \quad (98)$$

$$A_{11}(0) + A_{12}(0) = 1 + U_h^s \Pi_h^s = \frac{1}{K_s(0)(1 + U_5 \Pi_e)}. \quad (99)$$

Combining all the terms Eqs.(90-99), we find the expression for the singular pairing interaction

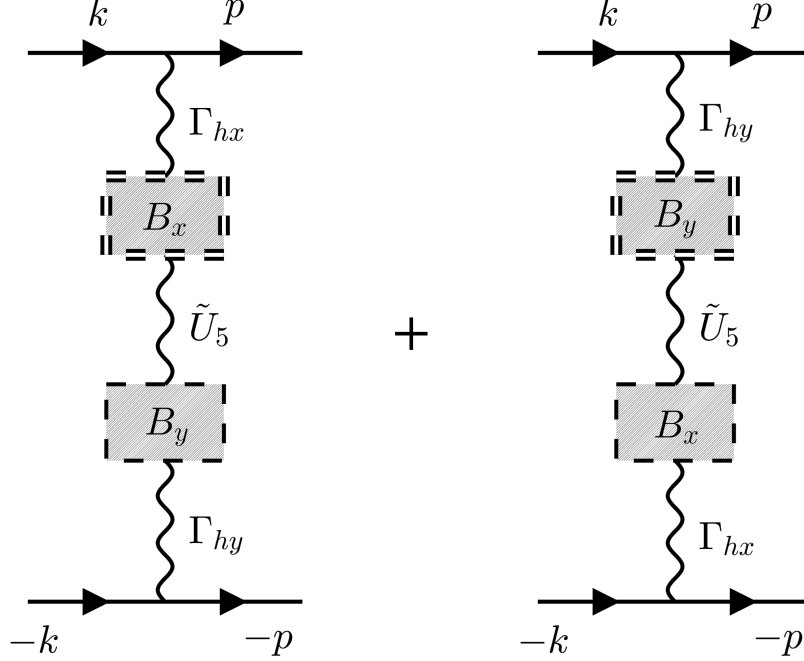
$$-V_{\text{singular}}^h(\mathbf{k}, \mathbf{p}) = \Pi_e (U_1 - \bar{U}_1)^2 \cos^2(\theta_{\mathbf{k}} + \theta_{\mathbf{p}}) (1 + U_5 \Pi_e) K_d(1) K_d(0) + \Pi_e (U_1 + \bar{U}_1)^2 \cos^2(\theta_{\mathbf{k}} - \theta_{\mathbf{p}}) (1 + U_5 \Pi_e) K_s(1) K_s(0). \quad (100)$$

In the presence of the inter-electron pocket density interaction $\tilde{U}_5$, one needs to add two more diagrams to S depicted in Fig.6. This changes the expression for S from Eq. (52) to

$$S = \Gamma_{hy} \cdot d \cdot d^T \cdot \Gamma_{hy} B_y + \Gamma_{hx} \cdot d \cdot d^T \cdot \Gamma_{hx} B_x + 2\Gamma_{hx} \cdot d \cdot d^T \cdot \Gamma_{hy} B_y B_x \tilde{U}_5. \quad (101)$$

Using Eq. (101) to perform the subsequent calculations outlined before, Eq. (72-100), one arrives at the final form for the V_{singular}^h as

$$-V_{\text{singular}}^h(\mathbf{k}, \mathbf{p}) = \frac{\Pi_e}{1 + U_h^d \Pi_h^d} (U_1 - \bar{U}_1)^2 \cos^2(\theta_{\mathbf{k}} + \theta_{\mathbf{p}}) \chi_{\text{nem}} + \frac{\Pi_e}{1 + U_h^s \Pi_h^s} (U_1 + \bar{U}_1)^2 \cos^2(\theta_{\mathbf{k}} - \theta_{\mathbf{p}}) \chi_{\text{nem}}^s, \quad (102)$$



Supplementary Figure 6. **Correction to the density interaction:** In the presence of the density interaction between the X - and Y - electron pockets, the zeroth order diagram for V_{singular}^h defined as S in the main text gets corrected with these two new sets of diagrams.

where $\chi_{\text{nem}}^s = 1 / [(1 + U_e^s \Pi_e)(1 + U_h^s \Pi_h^s) - 2 \Pi_h^s \Pi_e (U_1 + \bar{U}_1)^2]$ with $U_e^s = U_5 + 2\tilde{U}_5$. χ_{nem}^s represent the susceptibility of the total fermionic density in the A_{1g} channel and does not diverge at any temperature. On the other hand, near the onset of the d-wave nematic instability, the susceptibility χ_{nem} diverges. As a result, we can ignore the non-singular second term of the Eq. (102) and write the effective interaction near the nematic order as

$$V_{\text{eff}}(\mathbf{k}, \mathbf{p}) = -A_h (U_1 - \bar{U}_1)^2 \cos^2(\theta_{\mathbf{k}} + \theta_{\mathbf{p}}) \chi_{\text{nem}}(\mathbf{k} - \mathbf{p}), \quad (103)$$

where V_{eff} is the singular part of V_{singular}^h and $A_h = \frac{\Pi_e}{1 + U_h^d \Pi_h^d}$.

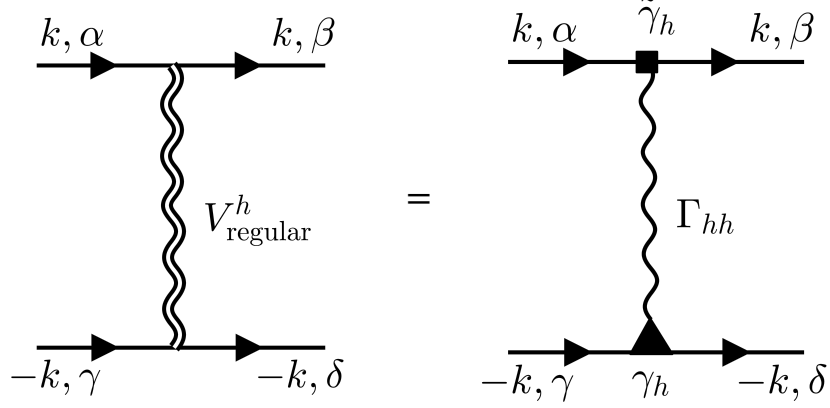
2. Non-singular part of the pairing interaction

We find the singular component of the dressed pairing interaction on the hole pocket. It scales with the nematic susceptibility χ_{nem} with an angular dependence of $\cos^2 2\theta_{\mathbf{k}}$ and exist as a consequence of the inter-pocket density interaction U_{he} . On the other hand, the regular part of the dressed pairing interaction comes purely from the intra-pocket (in this case hole pocket) density interaction as we will show in this section. For the simplicity of the calculation, we keep the momentum of the external fermions same as the internal ones and argue that the regular component termed as V_{regular}^h of the intra-pocket pairing interaction for the hole pocket is depicted in Fig.7

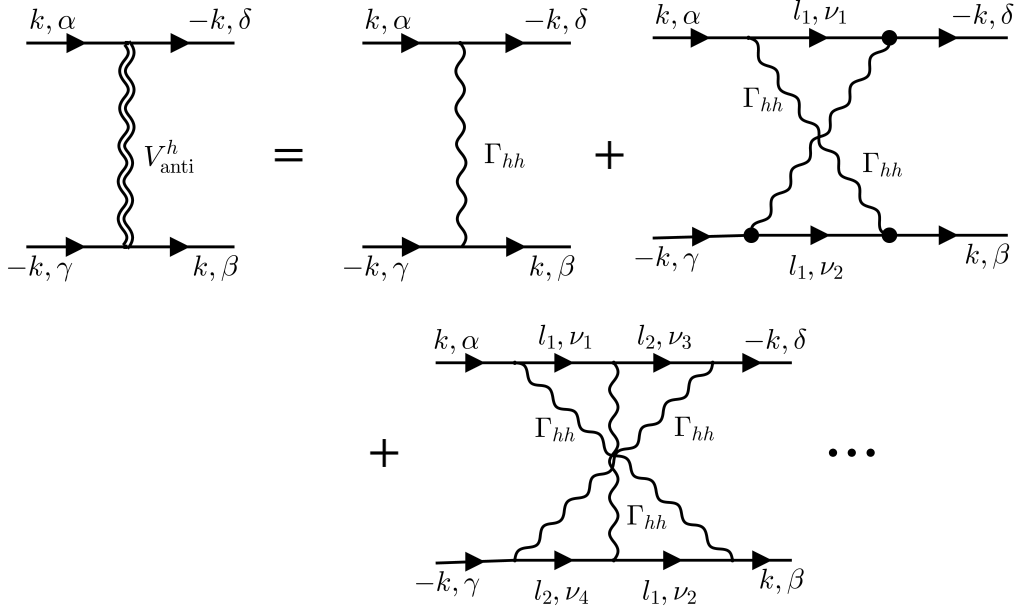
From Fig.7, $V_{\text{regular}}^h = \tilde{\gamma}_h \cdot \Gamma_{\text{hh}} \cdot \gamma_h$ where $\tilde{\gamma}_h$ and γ_h are defined in Fig.4 and 5. A straight forward calculation using Eqs.(58),(67) gives

$$V_{\text{regular}}^h = \frac{U_4 + \tilde{U}_4}{2} \chi_{\text{spin}}^s \chi_{\text{den}}^s + \frac{U_4 - \tilde{U}_4}{2} \chi_{\text{spin}}^d \chi_{\text{den}}^d \cos^2 2\theta_{\mathbf{k}} + \frac{\bar{U}_4 + \tilde{\bar{U}}_4}{2} \chi_{\text{spin}}^{\text{mix}} \chi_{\text{den}}^{\text{mix}} \sin^2 2\theta_{\mathbf{k}}, \quad (104)$$

with the spin component $\delta_{\alpha\beta}\delta_{\gamma\delta}$. Here, $\chi_{\text{spin}}^s, \chi_{\text{spin}}^d$ and $\chi_{\text{spin}}^{\text{mix}}$ are the spin susceptibilities with the form factor $1, \cos 2\theta$ and $\sin 2\theta$ respectively, while $\chi_{\text{den}}^s, \chi_{\text{den}}^d$ and $\chi_{\text{den}}^{\text{mix}}$ are the density susceptibilities with the form factor $1, \cos 2\theta$ and $\sin 2\theta$



Supplementary Figure 7. **Regular part of the dressed intraband pairing interaction:** Dyson equation for the the regular part of the intra-pocket pairing interaction, V_{regular}^h is shown. The solid black line, double curly line, single curly line represent the hole fermion Green's function, dressed interaction and bare interaction Γ_{hh} respectively. The black square and the triangle represent $\tilde{\gamma}_h$ and γ defined in the main text



Supplementary Figure 8. **Anti-symmetric part of the dressed pairing interaction:** Dyson equation for the anti-symmetric part of the pairing interaction, V_{anti}^h is shown

respectively for a one band model(in our case it is the hole pocket). They are defined as

$$\chi_{\text{spin}}^{\text{s}} = \frac{1}{1 - \frac{U_4 + \tilde{U}_4}{2} \Pi_{\text{h}}^{\text{s}}}, \quad \chi_{\text{den}}^{\text{s}} = \frac{1}{1 + \frac{U_4 + 2\tilde{U}_4 - \tilde{\tilde{U}}_4}{2} \Pi_{\text{h}}^{\text{s}}}, \quad (105)$$

$$\chi_{\text{spin}}^{\text{d}} = \frac{1}{1 - \frac{U_4 - \tilde{U}_4}{2} \Pi_{\text{h}}^{\text{s}}}, \quad \chi_{\text{den}}^{\text{d}} = \frac{1}{1 + \frac{U_4 - 2\tilde{U}_4 - \tilde{\tilde{U}}_4}{2} \Pi_{\text{h}}^{\text{s}}}, \quad (106)$$

$$\chi_{\text{spin}}^{\text{mix}} = \frac{1}{1 - 2\Pi_{\text{m}}(\bar{U}_4 + \tilde{U}_4)}, \quad \chi_{\text{den}}^{\text{mix}} = \frac{1}{1 - 2\Pi_{\text{m}}(\bar{U}_4 + 2\tilde{U}_4 - \tilde{\tilde{U}}_4)}. \quad (107)$$

To complete our calculation of the dressed pairing interaction, we also compute the fully anti-symmetrized pairing interaction. The anti-symmetric part of the fully anti-symmetrized pairing interaction describes scattering between fermion pair states $(\mathbf{k}, \alpha; -\mathbf{k}, \gamma) \rightarrow (-\mathbf{k}, \delta; \mathbf{k}, \beta)$. We depict the relevant diagrams contributing to $V_{\text{anti}}^{\text{h}}$ in Fig.8. Using Eq. (17) one finds

$$V_{\text{anti}}^{\text{h}} = \frac{U_4 + \bar{U}_4}{2} \frac{1}{1 - \frac{U_4 + \bar{U}_4}{2} \Pi_{\text{h}}^{\text{s}}} + \frac{U_4 - \bar{U}_4}{2} \frac{\cos^2 2\theta_{\mathbf{k}}}{1 - \frac{U_4 - \bar{U}_4}{2} \Pi_{\text{h}}^{\text{d}}} + \frac{\tilde{U}_4 + \tilde{\tilde{U}}_4}{2} \frac{\sin^2 2\theta_{\mathbf{k}}}{1 - \frac{\tilde{U}_4 + \tilde{\tilde{U}}_4}{2} \Pi_{\text{h}}^{\text{d}}} \quad (108)$$

with the spin component $\delta_{\alpha\delta}\delta_{\gamma\beta}$. To understand what Eqs. (104-108) produce, we combine both the terms and write the non-singular part of the fully anti-symmetrized pairing vertex $\Gamma_{\text{regular}}^{\text{h}}$ using the bare Hubbard-Hund interaction, Eq. (9-12) as

$$\Gamma_{\text{regular}}^{\text{h}} = \delta_{\alpha\beta}\delta_{\gamma\delta} \left[\frac{U - J}{\left(1 + \frac{3U - 5J}{2} \Pi_{\text{h}}^{\text{s}}\right) \left(1 - \frac{U + J}{2} \Pi_{\text{h}}^{\text{s}}\right)} + \frac{J}{\left(1 + \frac{5J - U}{2} \Pi_{\text{h}}^{\text{d}}\right) \left(1 - \frac{U - J}{2} \Pi_{\text{h}}^{\text{d}}\right)} \right] - \delta_{\alpha\delta}\delta_{\gamma\beta} \left[\frac{U + J}{2 \left(1 - \frac{U + J}{2} \Pi_{\text{h}}^{\text{s}}\right)} + \frac{U - J}{2 \left(1 - \frac{U - J}{2} \Pi_{\text{h}}^{\text{d}}\right)} \right] \quad (109)$$

$$= \Gamma_{\text{c}} \delta_{\alpha\beta} \delta_{\gamma\delta} + \Gamma_{\text{s}} \hat{\sigma}_{\alpha\beta} \cdot \hat{\sigma}_{\gamma\delta}. \quad (110)$$

Here, Γ_{c} and Γ_{s} are the spin and charge components of the antisymmetrized vertex with the form

$$\Gamma_{\text{c}} = \frac{3U - 5J}{4} \frac{1}{1 + \frac{3U - 5J}{2} \Pi_{\text{h}}^{\text{s}}} + \frac{5J - U}{4} \frac{1}{1 + \frac{5J - U}{2} \Pi_{\text{h}}^{\text{d}}} \quad (111)$$

$$\Gamma_{\text{s}} = -\frac{U + J}{4} \frac{1}{1 - \frac{U + J}{2} \Pi_{\text{h}}^{\text{s}}} - \frac{U - J}{4} \frac{1}{1 - \frac{U - J}{2} \Pi_{\text{h}}^{\text{d}}}. \quad (112)$$

From Eq. (109), one finds that the regular part does not scale with the nematic instability χ_{nem} and is angle independent unlike the singular part $V_{\text{singular}}^{\text{h}}$ which has an angular dependence as $\cos^2 2\theta_{\mathbf{k}}$. We calculate the effective intra-pocket interaction for the electron pockets depicted in Fig.9 in the similar way and find

$$V_{\text{eff}}^{\text{e}}(\mathbf{k}, \mathbf{p}) = -A_{\text{e}}(U_1 - \bar{U}_1)^2 \chi_{\text{nem}}(\mathbf{k} - \mathbf{p}), \quad (113)$$

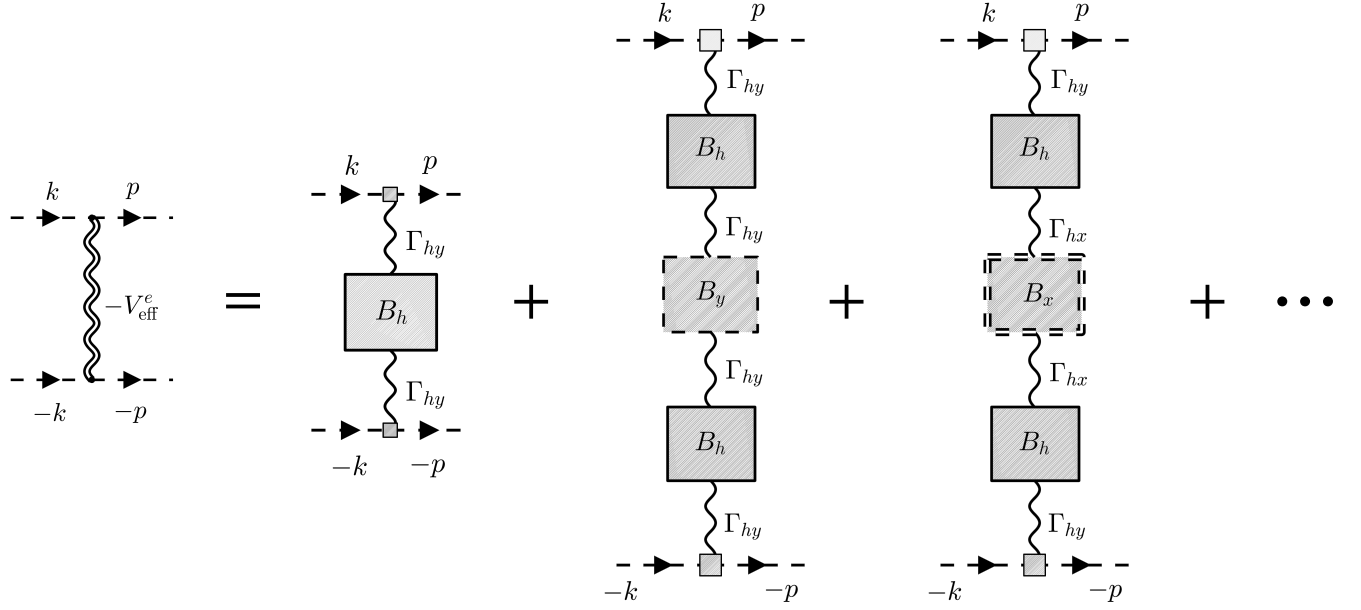
where $A_{\text{e}} = \frac{1}{2} \frac{\Pi_{\text{h}}^{\text{d}}}{1 + U_{\text{e}}^{\text{d}} \Pi_{\text{e}}}$. Because of the extra factor of 1/2 and $\Pi_{\text{h}}^{\text{d}} < \Pi_{\text{e}}$, $A_{\text{h}} > A_{\text{e}}$.

B. Inter-pocket pairing interaction

We argue that the inter-pocket pairing interaction a.k.a pair hopping interaction does not get affected by the strong nematic fluctuation. Pair hopping interaction is a large momentum($\mathbf{q}$) transfer interaction with $\mathbf{q} = (0, \pi)$ or $(\pi, 0)$. On the other hand, nematic fluctuations are peaked near zero momentum and can't influence pair hopping term much.

SUPPLEMENTARY DISCUSSION V: SOLVING THE SUPERCONDUCTING GAP EQUATION AT THE QCP ($\delta = 0$)

In the previous sections, we find that near the nematic instability the intra-pocket pairing interaction becomes attractive and divergent as it scales with the nematic susceptibility χ_{nem} compared to the finite repulsive inter-pocket interaction.



Supplementary Figure 9. **Singular part of the dressed intraband density interaction on the electron pockets:** Ladder series for the dressed pairing interaction represented by the double curly line between the fermions with paired state $(\mathbf{k}, -\mathbf{k})$ scattering to $(\mathbf{p}, -\mathbf{p})$ within the electron pocket. Here the shaded boxes labeled as B_h, B_x and B_y are defined in Fig.3. The black rectangular vertex is defined in Fig.5. Single curly line represents the bare interaction between the hole and electron pocket.

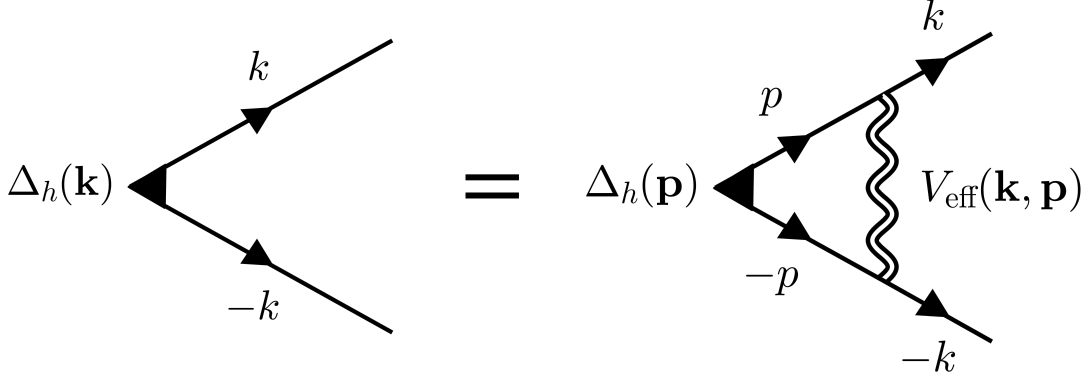
As a result, we can ignore the effect of the inter-pocket pairing interaction and study the superconducting gap equation for individual pockets led by the intra-pocket pairing interaction. The pairing interaction within the hole and the electron pockets presented in Eqs. (103,113) differ by the prefactor $A_{h,e}$. Since $A_h > A_e$, pairing interaction is larger on the hole pocket and superconductivity will first develop on it. In our further analysis, we keep the pairing interaction (Eq. (103)) static, predominantly between fermions near the Fermi surface and used the regular Ornstein-Zernike form for the $\chi_{\text{nem}}(\mathbf{q})$ ($= \chi_0/(\xi^{-2} + \mathbf{q}^2)$) and find

$$V_{\text{eff}}(\mathbf{k}, \mathbf{p}) = -A_h \chi_0 (U_1 - \bar{U}_1)^2 \frac{\cos^2(\theta_{\mathbf{k}} + \theta_{\mathbf{p}})}{\xi^{-2} + (\mathbf{k} - \mathbf{p})^2} \quad (114)$$

$$= -\bar{g} \frac{\cos^2(\theta_{\mathbf{k}} + \theta_{\mathbf{p}})}{\delta^2 + 4 \sin^2\left(\frac{\theta_{\mathbf{k}} - \theta_{\mathbf{p}}}{2}\right)}. \quad (115)$$

Here χ_0 is the bare susceptibility, $\delta = 1/k_f \xi$ is the inverse of the nematic correlation length ξ , $\bar{g} = A_h \frac{(U_1 - \bar{U}_1)^2}{k_F^2}$ is the effective coupling constant and k_F is the Fermi momentum. With the pairing interaction listed in Eq. (115) we write the non-linear gap equation $\Delta_h(\mathbf{k})$ on hole pocket depicted in Fig.10 as

$$\Delta_h(\mathbf{k}) = \bar{g} \int \frac{d^2 \mathbf{p}}{(2\pi)^2} \frac{\Delta_h(\mathbf{p}) \cos^2(\theta_{\mathbf{k}} + \theta_{\mathbf{p}})}{\delta^2 + 4 \sin^2(\theta_{\mathbf{k}} - \theta_{\mathbf{p}}/2)} \frac{\tanh(\beta E_h(\mathbf{p})/2)}{E_h(\mathbf{p})}, \quad (116)$$



Supplementary Figure 10. **SC vertex correction on the hole pocket:** Feynman diagram for the non-linear gap equation for the hole pocket. Δ_h represent the gap function. The double wavy line is the effective pairing interaction defined in Eq. (115) for the hole pocket fermions between momentum k and p .

with $\beta = 1/T$ and the excitation energy $E_h(\mathbf{p} = \sqrt{\xi_h(\mathbf{p})^2 + \Delta_h(\mathbf{p})^2})$. We shift the momentum $\mathbf{p} = \mathbf{k} + \tilde{\mathbf{p}}$ and rewrite the integration in Eq. (116) as

$$\Delta_h(\mathbf{k}) = \bar{g} \int \frac{d^2 \tilde{\mathbf{p}}}{(2\pi)^2} \frac{\Delta_h(\mathbf{k} + \tilde{\mathbf{p}}) \cos^2(2\theta_{\mathbf{k}} + \theta_{\tilde{\mathbf{p}}}) \tanh(\beta E_h(\mathbf{k} + \tilde{\mathbf{p}})/2)}{\delta^2 + 4 \sin^2(\theta_{\tilde{\mathbf{p}}}/2) E_h(\mathbf{k} + \tilde{\mathbf{p}})}. \quad (117)$$

Eq. (117) is analytically tractable in the limit $\delta \rightarrow 0$. In this limit, the integration is peaked near $\theta_{\tilde{\mathbf{p}}} = 0$. As a result, we can ignore the $\tilde{\mathbf{p}}$ dependence of the gap and the cosine factor in Eq. (117) and transform it from an integral equation to an algebraic equation

$$\Delta_h(\mathbf{k}) = \bar{g} N_0 \Delta_h(\mathbf{k}) \cos^2 2\theta_{\mathbf{k}} \int_0^{2\pi} \frac{d\theta_{\tilde{\mathbf{p}}}}{2\pi} \frac{1}{\delta^2 + 4 \sin^2(\theta_{\tilde{\mathbf{p}}}/2)} \int_0^\Lambda dx \frac{\tanh\left(\frac{\sqrt{x^2 + |\Delta_h(\mathbf{k})|^2}}{2T}\right)}{\sqrt{x^2 + \Delta_h(\mathbf{k})^2}}, \quad (118)$$

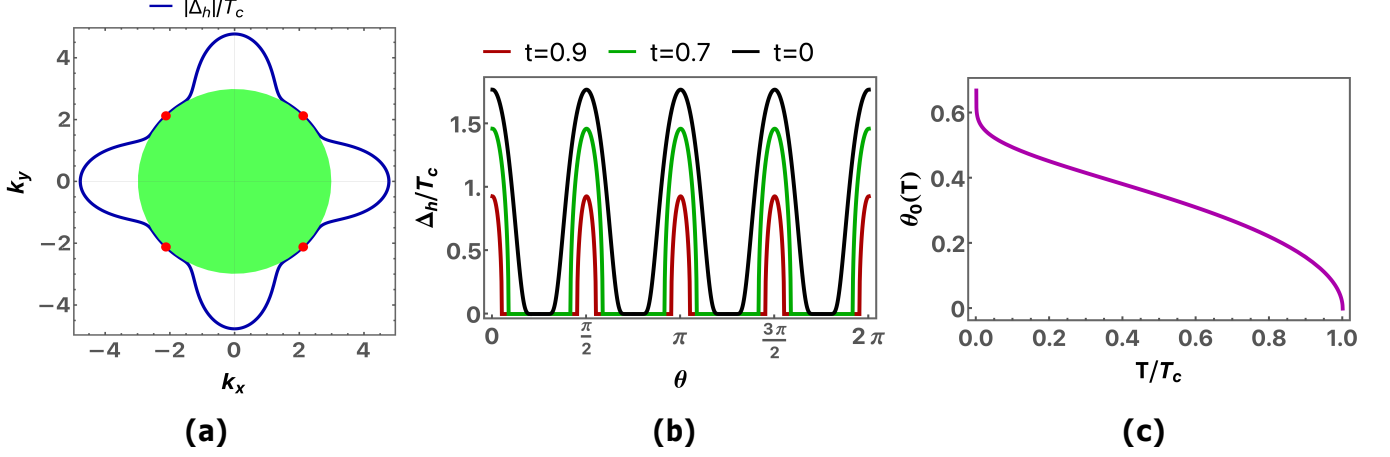
where N_0 is the density of states and Λ is the upper pairing cutoff. Performing the angular integration over $\theta_{\tilde{\mathbf{p}}}$ and defining the effective coupling $g = \frac{N_0 \bar{g}}{2\delta}$, Eq. (118) reduces to

$$1 = g \cos^2 2\theta_{\mathbf{k}} \int_0^\Lambda dx \frac{\tanh\left(\frac{\sqrt{x^2 + |\Delta_h(\mathbf{k})|^2}}{2T}\right)}{\sqrt{x^2 + \Delta_h(\mathbf{k})^2}}. \quad (119)$$

Since Eq. (119) depends only on the modulus of the gap Δ_h , its solution does not provide any information about the phase of that gap: $\Delta_h = |\Delta_h(\mathbf{k})|e^{i\phi(\mathbf{k})}$. Solving the linearized equation by putting $\Delta_h = 0$ in Eq. (119), we find an angle dependent critical temperature $\bar{T}_c(\theta_{\mathbf{k}}) = 1.13 \Lambda \exp(-1/g \cos^2 2\theta_{\mathbf{k}})$. The largest critical temperature happens when $\cos^2 2\theta_{\mathbf{k}} = 1$ and the superconducting gap will first develop at these discrete number of points on the Fermi surface and remain zero everywhere. The set of these points is $W = \{0, \pi/2, \pi, 3\pi/2\}$ and the true transition temperature of the system is $T_c = 1.13 \Lambda \exp(-1/g)$. At any lower temperature $T < T_c$, the gap develops around these 4 special points ($\in W$) and remains non-zero upto $\theta_{\mathbf{k}} < |\theta_0(T) - \theta_W|$ where $\theta_0(T)$ is the solution of Eq. (119) with $\Delta_h = 0$.

$$\frac{1}{g \cos^2 2\theta_0(T)} = \int_0^\Lambda dx \frac{\tanh(x/2T)}{x} = \log \frac{1.13 \Lambda}{T} = \frac{1}{g} + \log \frac{T_c}{T} \quad (120)$$

$$\rightarrow \theta_0(T) = \frac{1}{2} \arctan \sqrt{g \log \frac{T_c}{T}}. \quad (121)$$



Supplementary Figure 11. **Superconductivity at the nematic QCP ($\delta = 0$):** (a) We plot the absolute value of the SC gap scaled by the transition temperature T_c , Δ_h/T_c (blue) on the hole pocket at zero temperature as a function of the angle on the Fermi surface (green disk) with radius 3 unit. The gap is exponentially small near the cold spots (red dots) while maximum along the k_x and k_y axis. In (b) we plot the angular variation of the gap, $|\Delta_h|/T_c$ on the hole pocket for three different reduced temperatures, $t = 0.9, 0.7$ and 0 below the transition point T_c . At finite temperature, the gap vanishes on the four patches of the Fermi surface arcs (flat segments of the gap). In (c) we plot the width of the gap, $\theta_0(T)$ as a function of the reduced temperature $t = T/T_c$. We keep the coupling strength $g = 1$ here.

We plot $\theta_0(T)$ in Fig.11b as a function of the reduced temperature $t = T/T_c$. For $\theta_{\mathbf{k}} > |\theta_0(T) - \theta_W|$, the gap vanishes and the Fermi surface remains intact with the width $\Delta\theta(T) = \pi/2 - 2\theta_0(T)$ where $\theta_0(T)$ is defined in Eq. (121). At zero temperature, $\Delta\theta$ reduces to zero letting the gap open everywhere except on the cold spots (red dots in Fig.11a): $\theta_c = \pi/4 + n\pi/2$ where n is an integer. In Fig.11c, we plot the angular variation of the gap magnitude for a set of temperatures. At $T = 0$, the angular variation of the gap turns out to be

$$\left| \frac{\Delta_h(\mathbf{k})}{T_c} \right|_{T=0} \approx 1.76 \exp\{-\tan^2 2\theta_{\mathbf{k}}/g\}. \quad (122)$$

Unlike the conventional d - wave superconducting gap ($\propto \cos 2\theta_{\mathbf{k}}$), where near the cold spot the gap is linearly proportional to the angular deviation ($\propto (\theta - \theta_c)$), in this case the gap is exponentially suppressed near the cold spots and behaves as $\exp\left\{-\frac{1}{4g(\theta - \theta_c)^2}\right\}$.

SUPPLEMENTARY DISCUSSION VI: CALCULATION OF THE SPECIFIC HEAT C_v AT $\delta = 0$

The specific heat $C_v(T)$ for the hole band is given by the expression

$$C_v(T) = \frac{2}{T} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \left(-\frac{\partial n_{\mathbf{k}}}{\partial E_h(\mathbf{k})} \right) \left[E_h(\mathbf{k})^2 - \frac{T}{2} \frac{\partial |\Delta_h(\mathbf{k})|^2}{\partial T} \right] \quad (123)$$

where $n_{\mathbf{k}}$ is the Fermi function $1/[\exp(\beta E_h) + 1]$ and $E_h(\mathbf{k}) = \sqrt{\xi_h(\mathbf{k})^2 + |\Delta_h(\mathbf{k})|^2}$ is the excitation energy. Since $\frac{\partial n_{\mathbf{k}}}{\partial E_h(\mathbf{k})} = -\frac{1}{T} \text{sech}^2(\beta E_h(\mathbf{k})/2)$ is peaked near the Fermi surface at low temperature, we convert the momentum integration into the energy and angle variables and approximate the energy integral by keeping the density of state at the Fermi surface which gives

$$C_v(T) = \frac{N_0}{2T^2} \int_{-\infty}^{\infty} dx \int_0^{2\pi} \frac{d\theta}{2\pi} \text{sech}^2\left(\frac{\sqrt{x^2 + |\Delta_h(\theta)|^2}}{2T}\right) \left[x^2 + |\Delta_h(\theta)|^2 - \frac{T}{2} \frac{\partial |\Delta_h(\theta)|^2}{\partial T} \right]. \quad (124)$$

Here, N_0 is the density of states at the Fermi surface. In Sec.A0B, we solve the non-linear gap equation, Eq. (119) and use the solution to compute the specific heat in this section. Since the gap function $\Delta_h(\theta)$ vanishes for $\theta \geq \theta_0(T)$ at the temperature T for $\theta \in (0, \pi/4)$ (See Fig.11a), we can split the specific heat into two parts, one for the metallic

part called C_v^N for which the gap vanishes and another for the superconducting part called C_v^s for which the gap exist: $C_v(T) = C_v^N(T) + C_v^s(T)$ defined as

$$C_v^s(T) = 4 \frac{N_0}{T^2} \int_0^{\theta_0(T)} \frac{d\theta}{2\pi} \int_{-\infty}^{\infty} dx \operatorname{sech}^2\left(\frac{\sqrt{x^2 + |\Delta_h(\theta)|^2}}{2T}\right) \left[x^2 + |\Delta_h(\theta)|^2 - \frac{T}{2} \frac{\partial |\Delta_h(\theta)|^2}{\partial T} \right], \quad (125)$$

$$C_v^N(T) = 4 \frac{N_0}{T^2} \int_{\theta_0(T)}^{\pi/4} \frac{d\theta}{2\pi} \int_{-\infty}^{\infty} dx \operatorname{sech}^2\left(\frac{x}{2T}\right) x^2 \quad (126)$$

A straightforward calculation gives the normal part of the specific heat

$$C_v^N(T) = \frac{8 N_0 \pi T}{3} \left[\frac{\pi}{4} - \theta_0(T) \right]. \quad (127)$$

We plot the specific heat coefficient for the metallic part, C_v^N/T as a function of the reduced temperature $t = T/T_c$ in Fig.12a. At $t = 1$, C_v^N/T reduces to the normal metallic contribution $2\pi^2 N_0/3$. On the other hand, at low temperature, it behaves as

$$\left. \frac{C_v^N(T)}{T} \right|_{T \rightarrow 0} = \frac{4 N_0 \pi}{3 \sqrt{|g \log t|}} + 0((\log t)^{-3/2}). \quad (128)$$

To compute the superconducting part of the specific heat, we scale the temperature, gap and energy integration variable by the critical temperature $\bar{T}_c(\theta)$ inside the angular integration of Eq. (125) and write

$$C_v^s(T) = 4 \frac{N_0 T_c}{t^2} \int_0^{\theta_0(T)} \frac{d\theta}{2\pi} \left(\frac{\bar{T}_c(\theta)}{T_c} \right)^3 \int_{-\infty}^{\infty} d\tilde{x} \operatorname{sech}^2\left(\frac{\sqrt{\tilde{x}^2 + |\tilde{\Delta}_h(\theta)|^2}}{2t_\theta}\right) \left[\tilde{x}^2 + |\tilde{\Delta}_h(\theta)|^2 - \frac{t_\theta}{2} \frac{\partial |\tilde{\Delta}_h(\theta)|^2}{\partial t_\theta} \right], \quad (129)$$

where $t_\theta = T/\bar{T}_c(\theta)$, $\tilde{\Delta}_h(\theta) = \Delta_h(\theta)/\bar{T}_c(\theta)$ and $\tilde{x} = x/\bar{T}_c(\theta)$. We define the integration under $\tilde{x}$ variable in Eq. (129) as

$$f(t_\theta) = \int_{-\infty}^{\infty} d\tilde{x} \operatorname{sech}^2\left(\frac{\sqrt{\tilde{x}^2 + |\tilde{\Delta}_h(\theta)|^2}}{2t_\theta}\right) \left[\tilde{x}^2 + |\tilde{\Delta}_h(\theta)|^2 - \frac{t_\theta}{2} \frac{\partial |\tilde{\Delta}_h(\theta)|^2}{\partial t_\theta} \right], \quad (130)$$

such that

$$C_v^s(T) = 4 \frac{N_0 T_c}{t^2} \int_0^{\theta_0(T)} \frac{d\theta}{2\pi} \left(\frac{\bar{T}_c(\theta)}{T_c} \right)^3 f(t_\theta) \quad (131)$$

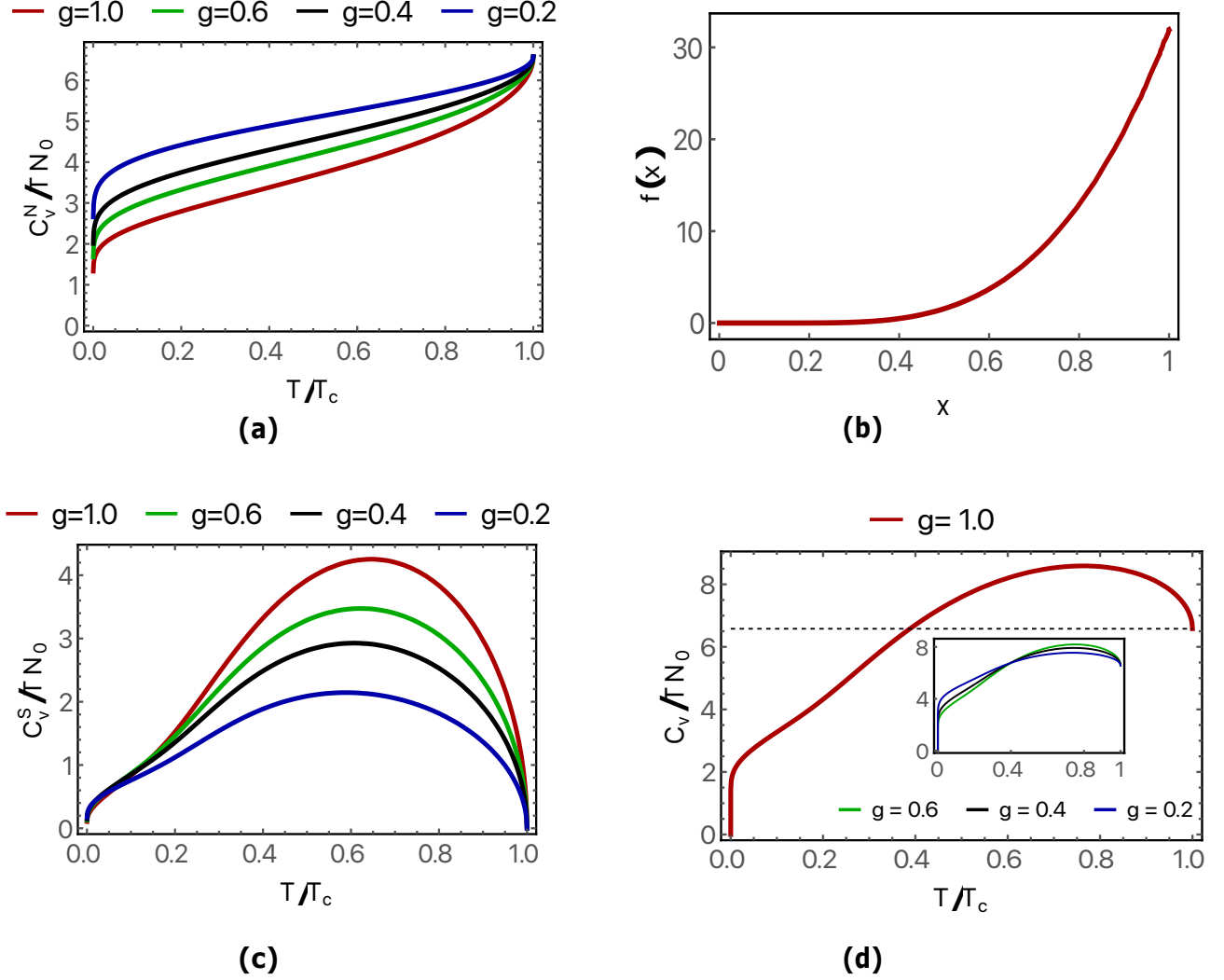
$$= 4 \frac{N_0 T_c}{t^2} \int_0^{\theta_0(T)} \frac{d\theta}{2\pi} e^{-3 \tan^2 2\theta/g} f(e^{\tan^2 2\theta/g} t) \quad (132)$$

$$= N_0 g T_c \int_t^1 dx \frac{f(x)}{x^4 (1 + g \log \frac{x}{t}) \sqrt{g \log \frac{x}{t}}}. \quad (133)$$

Here we have used the relations $\bar{T}_c(\theta)/T_c = \exp(-\tan^2 2\theta/g)$ and $f(t_\theta) \equiv f\left(t \times \frac{T_c}{\bar{T}_c(\theta)}\right)$. Eq. (133) has a log singularity near $x = t$ which approximates the integration to

$$\frac{C_v^s(T)}{T} = N_0 g \frac{f(t)}{t^3} \int_1^{1/t} \frac{dy}{\sqrt{g \log y}} = \sqrt{\pi} N_0 \sqrt{g} \frac{f(t)}{t^3} \operatorname{Erfi}[\sqrt{|\log t|}], \quad (134)$$

where $\operatorname{Erfi}(x)$ is the imaginary error function. We obtain the scaling function $f(x)$ numerically and plot in Fig.12b. We numerically compute Eq. (133) and present our result for the specific heat coefficient of the superconducting part, C_v^s/T in Fig.12c.



Supplementary Figure 12. **Specific heat at the nematic QCP ($\delta = 0$):** We plot the temperature variation of the metallic (C_v^N) and superconducting component (C_v^S) of the specific heat coefficient defined in Eq. (126) and (125) for a range of values of coupling constant g in (a) and (c). In (b), we plot the scaling function $f(x)$ defined in Eq. (130) as a function of its argument x . In (d), we add the contribution from part (a) and (c) and plot the total specific heat coefficient as a function of the reduced temperature T/T_c where T_c is the critical temperature for $g = 1$ in the main frame and $g = 0.6, 0.4$ and 0.2 in the inset.

In Fig.12d, we plot the total specific heat coefficient, C_v/T as a function of the temperature for a range of values for the coupling constant $\bar{g}$. The dashed black line represent the normal state result for the specific heat. We find there is no specific heat jump at the onset of the superconductivity because gap opens only at the discrete number of points on the Fermi surface and remains zero everywhere. This is captured by the fact that $C_v^S/T = 0$ as T goes to T_c . Near the critical temperature T_c , C_v/T first goes up, attains a maximum around $t = 0.8$ and then falls with lowering the temperature. At very low temperature, specific heat declines very rapidly as it goes as $\log(t)^{-1/2}$. This is a consequence of the existence of the exponentially suppressed gap near the cold spots, captured in Eq. (122).

SUPPLEMENTARY DISCUSSION VII: EFFECT OF THE PAIR-HOPPING INTERACTION AT $\delta = 0$

So far in our analysis we have neglected the effect of the inter-pocket pairing interaction a.k.a pair hopping defined in Eq. (16). In this section, we study if the inter-pocket pairing interaction can break the degeneracy between $s-$, $d-$ and $p-$ wave channel at $\delta = 0$ limit. For the simplicity of the calculation, we have neglected the pairing interaction within the electron pockets and keep the density of states N_0 same for all the pockets. In the presence of the pair-hopping interaction,

the non-linear gap equation for Δ_h, Δ_x and Δ_y become

$$\begin{aligned} \Delta_h(\theta_{\mathbf{k}}) = & g \Delta_h(\theta_{\mathbf{k}}) \cos^2 2\theta_{\mathbf{k}} \int_0^\Lambda d\xi_h \frac{\tanh\left(\frac{\sqrt{\xi_h^2 + |\Delta_h(\theta_{\mathbf{k}})|^2}}{2T}\right)}{\sqrt{\xi_h^2 + |\Delta_h(\theta_{\mathbf{k}})|^2}} - \Delta_x N_0 (U_s + U_d \cos 2\theta_{\mathbf{k}}) \int_0^\Lambda d\xi_x \frac{\tanh\left(\frac{\sqrt{\xi_x^2 + |\Delta_x|^2}}{2T}\right)}{\sqrt{\xi_x^2 + |\Delta_x|^2}} \\ & - \Delta_y N_0 (U_s - U_d \cos 2\theta_{\mathbf{k}}) \int_0^\Lambda d\xi_y \frac{\tanh\left(\frac{\sqrt{\xi_y^2 + |\Delta_y|^2}}{2T}\right)}{\sqrt{\xi_y^2 + |\Delta_y|^2}} \end{aligned} \quad (135)$$

$$\Delta_x = -N_0 \int_0^{2\pi} \frac{d\theta_{\mathbf{p}}}{2\pi} (U_s + U_d \cos 2\theta) \Delta_h(\theta_{\mathbf{p}}) \int_0^\Lambda d\xi_h \frac{\tanh\left(\frac{\sqrt{\xi_h^2 + |\Delta_h(\theta_{\mathbf{p}})|^2}}{2T}\right)}{\sqrt{\xi_h^2 + |\Delta_h(\theta_{\mathbf{p}})|^2}} \quad (136)$$

$$\Delta_y = -N_0 \int_0^{2\pi} \frac{d\theta_{\mathbf{p}}}{2\pi} (U_s - U_d \cos 2\theta) \Delta_h(\theta_{\mathbf{p}}) \int_0^\Lambda d\xi_h \frac{\tanh\left(\frac{\sqrt{\xi_h^2 + |\Delta_h(\theta_{\mathbf{p}})|^2}}{2T}\right)}{\sqrt{\xi_h^2 + |\Delta_h(\theta_{\mathbf{p}})|^2}}. \quad (137)$$

Because of the pair-hopping interaction, the gap on the hole pocket will induce uniform gap on the electron pockets which, as a feedback induces non-zero gap everywhere on the hole pocket. We break the inter-pocket pairing interaction into s - and d - wave components and call them U_s and U_d respectively. In terms of the bare interactions, $U_s = (U + J)/2$ and $U_d = (U - J)/2$. We first focus on the linearized gap equations

$$\Delta_h(\theta_{\mathbf{k}}) = g \Delta_h(\theta_{\mathbf{k}}) \cos^2 2\theta_{\mathbf{k}} L - (\Delta_x + \Delta_y) u_s L - (\Delta_x - \Delta_y) u_d \cos 2\theta_{\mathbf{k}} L, \quad (138)$$

$$\Delta_x + \Delta_y = -2L u_s \int_0^{2\pi} \frac{d\theta_{\mathbf{k}}}{2\pi} \Delta_h(\theta_{\mathbf{k}}), \quad (139)$$

$$\Delta_x - \Delta_y = -2L u_d \int_0^{2\pi} \frac{d\theta_{\mathbf{k}}}{2\pi} \Delta_h(\theta_{\mathbf{k}}) \cos 2\theta_{\mathbf{k}}, \quad (140)$$

where $L = \log \frac{\Lambda}{T}$, $u_s = N_0 U_s$ and $u_d = N_0 U_d$. We plug Eqs. (139-140) into Eq. (138) and find the effective linearized gap equation on the hole pocket as

$$\Delta_h(\theta_{\mathbf{k}}) = g \Delta_h(\theta_{\mathbf{k}}) \cos^2 2\theta_{\mathbf{k}} L + 2U_s^2 L^2 \int_0^{2\pi} \frac{d\theta_{\mathbf{k}}}{2\pi} \Delta_h(\theta_{\mathbf{k}}) + 2U_d^2 L^2 \cos 2\theta_{\mathbf{k}} \int_0^{2\pi} \frac{d\theta_{\mathbf{k}}}{2\pi} \Delta_h(\theta_{\mathbf{k}}) \cos 2\theta_{\mathbf{k}}. \quad (141)$$

In the absence of the pair hopping, Eq. (141) reduces to Eq. (119) and we get back the results of Section A0B. In its presence, we make the following observation from the Eq. (141). First, pair-hopping only affects the s - and d - wave components of the gap, not the p - wave. For the s - wave solution, the third term vanishes while for the d - wave the second term goes away. On the other hand, for the p wave solution, both the second and third term will vanish. Second, just like the isolated case, the first superconducting instability will happen at 4 points dictated by the relation $\cos^2 2\theta_{\mathbf{k}} = 1$ on the Fermi surface. These are the points along the k_x and k_y axis on the Fermi surface. Third, the critical temperature gets modified by the pair-hopping term. We argue that because the gap opens only at discrete number of points, to compute the critical temperature one needs to substitute the integration over angle in the second and third term of Eq. (141) by a summation over these 4 points. This gives

$$\Delta_h(\theta_i) = g \Delta_h(\theta_i) \cos^2 2\theta_i L + 2U_s^2 L^2 \sum_k \Delta_h(\theta_k) + 2U_d^2 L^2 \cos 2\theta_i \sum_k \Delta_h(\theta_k) \cos 2\theta_k, \quad (142)$$

where $\theta_k = \{0, \pi/2, \pi, 3\pi/2\}$. Writing Eq. (142) in s -, p - and d - wave channel gives the following conditions for the instability respectively

$$1 = gL + 8L^2 u_s^2, \quad (143)$$

$$1 = gL, \quad (144)$$

$$1 = gL + 8L^2 u_d^2. \quad (145)$$

Solving Eqs. (143-145) we find the renormalized critical temperatures $T_{0,s}^*$ (for s - wave), $T_{0,d}^*$ (for d - wave) and $T_{0,p}^*$ (for p - wave) such that

$$T_{0,s}^* \approx \Lambda e^{-1/g} e^{8u_s^2/g^3} \geq T_{0,d}^* \approx \Lambda e^{-1/g} e^{8u_d^2/g^3} > T_{0,p}^* = \Lambda e^{-1/g}. \quad (146)$$

Inter-pocket pairing interaction favors s - and d - wave channel compared to the p - wave. When $u_s = u_d$, the equality sign holds and s - and d - wave becomes degenerate. To realize what gap symmetry at low temperature prevails for $u_s = u_d$, we solve the non-linear gap equations listed in Eq. (135-137) numerically and present our result in Fig. Our results supports an s - symmetry at all low temperature down to zero within the numerical accuracy. To understand if there is a possibility of an $s + id$ state with a tiny d - wave component not captured by the numerical result, we perform a perturbative calculation around the numerical solution. We write the gap as

$$\Delta_h(\theta) = \Delta_h^s(\theta) + i \Delta_h^d(\delta), \quad \Delta_{x,y} = \Delta_{x,y}^s + i \Delta_{x,y}^d, \quad (147)$$

where Δ_h^s, Δ_x^s and Δ_y^s are the numerical solution of Eq. (135-137) for the hole, X - and Y - pocket respectively. Δ_h^d, Δ_x^d and Δ_y^d are the small perturbing fields. Expanding Eq. (135-137) in Δ^d 's, we find the linearized equation for $\Delta_{h,x,y}^d$ in the presence of the full grown $\Delta_{h,x,y}^s$ gap as

$$\Delta_h^d(\theta) = g \Delta_h^d(\theta) \cos^2 2\theta J(|\Delta_h^s(\theta)|, T) - \Delta_x^d \bar{u} \cos^2 \theta J(|\Delta_x^s|, T) - \Delta_y^d \bar{u} \sin^2 \theta J(|\Delta_y^s|, T) \quad (148)$$

$$\Delta_x^d = -\bar{u} \int_0^{2\pi} \frac{d\theta}{2\pi} \cos^2 \theta \Delta_h^d(\theta) J(|\Delta_h^s(\theta)|, T) \quad (149)$$

$$\Delta_y^d = -\bar{u} \int_0^{2\pi} \frac{d\theta}{2\pi} \sin^2 \theta \Delta_h^d(\theta) J(|\Delta_h^s(\theta)|, T), \quad (150)$$

where $\bar{u} = 2u_s = 2u_d$ and the function J defined as

$$J(|x|, T) = \int_0^\Lambda d\xi \frac{\tanh\left(\frac{\sqrt{\xi^2 + |x|^2}}{2T}\right)}{\sqrt{\xi^2 + |x|^2}}. \quad (151)$$

Combining Eqs. (149-150) into Eq. (148) and doing a little bit of manipulation we write the effective linearized equation for Δ_h^d

$$\Delta_h^d(\theta) = g \Delta_h^d(\theta) \cos^2 2\theta J(|\Delta_h^s(\theta)|, T) + \frac{\bar{u}^2}{2} \cos 2\theta \int_0^{2\pi} \frac{d\theta'}{2\pi} \Delta_h^d(\theta') J(|\Delta_h^s(\theta')|, T) J(|\Delta_e^s|, T), \quad (152)$$

$$= g \Delta_h^d(\theta) \cos^2 2\theta J(|\Delta_h^s(\theta)|, T) + \frac{\bar{u}^2}{2} \cos 2\theta \Gamma(T) \quad (153)$$

where $\Gamma(T) = \int_0^{2\pi} \frac{d\theta'}{2\pi} \Delta_h^d(\theta') J(|\Delta_h^s(\theta')|, T) J(|\Delta_e^s|, T)$ and we use $|\Delta_x^s| = |\Delta_y^s| = \Delta_e^s$. We convert Eq. (153) into an algebraic equation using the definition of Γ and find the self-consistent equation for the existence of the d - wave component as

$$1 = J(|\Delta_e^s|) \frac{\bar{u}^2}{2} \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{\cos^2 2\theta J(|\Delta_h^s(\theta)|, T)}{1 - g \cos^2 2\theta J(|\Delta_h^s(\theta)|, T)}. \quad (154)$$

To complete our proof, we revisit Eq. (135-137) and write the non-linear integral equation for Δ_h^s

$$\Delta_h^s(\theta) = g \Delta_h^s(\theta) \cos^2 2\theta J(|\Delta_h^s(\theta)|, T) + \frac{\bar{u}}{2} J(|\Delta_e|, T) \int_0^{2\pi} \frac{d\theta'}{2\pi} \Delta_h^s(\theta') J(|\Delta_h^s(\theta')|, T). \quad (155)$$

With a little bit of manipulation, Eq. (155) can be rearranged as

$$1 - g \cos^2 2\theta J(|\Delta_h^s(\theta)|, T) = \frac{\bar{u}}{2} J(|\Delta_e|, T) \frac{\int_0^{2\pi} \frac{d\theta'}{2\pi} \Delta_h^s(\theta') J(|\Delta_h^s(\theta')|, T)}{\Delta_h^s(\theta)}. \quad (156)$$

Using Eq. (156), we further simplify the r.h.s of Eq.(154) and arrive

$$J(|\Delta_\epsilon^s|) \frac{\bar{u}^2}{2} \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{\cos^2 2\theta J(|\Delta_h^s(\theta)|, T)}{1 - g \cos^2 2\theta J(|\Delta_h^s(\theta)|, T)} = \frac{\int_0^{2\pi} \frac{d\theta}{2\pi} \cos^2 2\theta J(|\Delta_h^s(\theta)|, T) \Delta_h^s(\theta)}{\int_0^{2\pi} \frac{d\theta}{2\pi} J(|\Delta_h^s(\theta)|, T) \Delta_h^s(\theta)} < 1. \quad (157)$$

Because of the extra $\cos^2 2\theta$ factor in the numerator, the ratio will always be less than 1, and Eq. (154) can never be fulfilled below the critical temperature $T_{0,s}^*$. This proves that the gap symmetry remains s - wave at all temperature below the critical point.

SUPPLEMENTARY DISCUSSION VIII: NUMERICAL SOLUTION OF THE GAP EQUATION, THE POSSIBLE GAP SYMMETRY AND SPECIFIC HEAT AWAY FROM THE QCP ($\delta \neq 0$)

In the previous section, we solve the non-linear gap equation at the nematic quantum critical point ($\delta = 0$) by transforming it from an integral equation (Eq. (116)) to an algebraic equation (Eq. (119)). Even though this is right to do at $\delta = 0$, the price one has to pay is that the solution gives only the modulus of the gap function $\Delta_h(\mathbf{k})$ leading to a functional degeneracy to the obtained solution, as any function of the form $\Delta_h(k) e^{i\Phi(\mathbf{k})}$ is also another possible solution. Away from the critical point ($\delta \neq 0$), one has to solve the integral equation. In this section, we present our numerical result for the solution of the non-linear integral gap equation for non-zero values of δ . We first focus on the linearized gap equation

$$\frac{1}{\lambda} \Delta_h(\theta_{\mathbf{k}}) = \int \frac{d\theta_{\mathbf{p}}}{2\pi} \frac{\cos^2(\theta_{\mathbf{k}} + \theta_{\mathbf{p}})}{\delta^2 + 4 \sin^2(\theta_{\mathbf{k}} - \theta_{\mathbf{p}}/2)} \Delta_h(\theta_{\mathbf{p}}) = \int \frac{d\theta_{\mathbf{p}}}{2\pi} M(\theta_{\mathbf{k}}, \theta_{\mathbf{p}}) \Delta_h(\theta_{\mathbf{p}}), \quad (158)$$

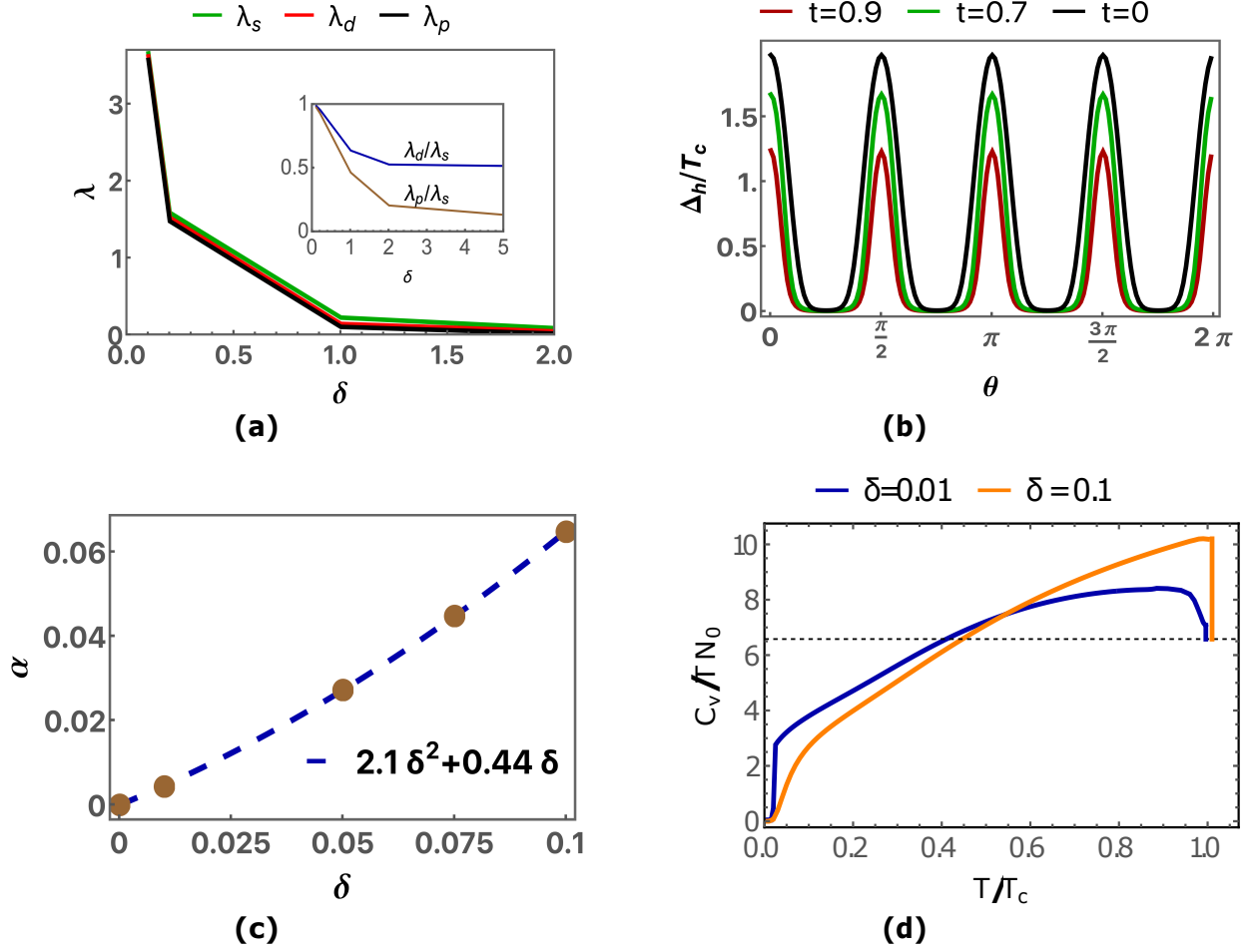
cast into a matrix equation in angular space with eigenvalue labeled as λ and kernel matrix

$$M(\theta_{\mathbf{k}}, \theta_{\mathbf{p}}) = \frac{\cos^2(\theta_{\mathbf{k}} + \theta_{\mathbf{p}})}{\delta^2 + 4 \sin^2(\theta_{\mathbf{k}} - \theta_{\mathbf{p}}/2)}. \quad (159)$$

Here, we suppress the δ dependence of the matrix M for the simplicity of the representation. We break Eq. (158) into different angular momentum channels and solve for the largest eigenvalue in each channel to find the leading instability. The relevant channels consistent with the pairing interaction, Eq. (115) are A_{1g} (eigen functions are like: $1, \cos 4\theta, \cos 8\theta$ etc), B_{1g} (eigen functions are like: $\cos 2\theta, \cos 6\theta, \cos 10\theta$ etc), and E_g (eigen functions are like: $\cos \theta, \cos 3\theta, \cos 5\theta$ etc). We will refer them as s, d and p - waves respectively. In Fig.13a, we plot how the largest eigenvalue λ_s, λ_d and λ_p in s, d , and p - wave channel respectively vary with the nematic mass parameter δ . We find the leading instability happens in the s -channel followed by d - and p - wave respectively from the linearized gap equation. In the inset, we plot the ratio of λ_d/λ_s and λ_p/λ_s with varying δ . As δ goes to zero, these eigen values get closer to each other and become degenerate at $\delta = 0$. Below the transition point, there could be 2 situations possible: (a) after the gap in s - wave channel develops, it acts against the subleading d and p waves and never let them win. As a result, s - wave remains down to zero temperature, or (b) at some low temperature there is a second phase transition from s - to either $s + id$ or p wave. We solve the full non-linear gap equation Eq. (116) numerically for $\delta = 0.01$ at different temperatures and show our result in Fig.13b. Our results support the first case of s - wave gap symmetry being intact at all the temperature below the critical temperature T_c .

Even though s - wave wins, we argue there is a possibility of having only p - wave gap symmetry if one includes the repulsive component of the pairing interaction presented in Eq. (110). The repulsive regular part of the intra-pocket pairing interaction does not scale with the nematic susceptibility and has only s - and d - wave components. As a result, the pairing strength for the p - wave is larger than for s - and d - wave when one includes both the attractive (V_{singular}^h) and repulsive (V_{regular}^h) components of the effective pairing interaction in the solution of the non-linear gap equation. The p - wave has to be of the form $p_x \pm i p_y$, as $|\Delta_h(\mathbf{k})|$ is same on the k_x and k_y axis found from the analysis of section.A0B.

We define the gap anisotropy $\alpha = \Delta_h(\theta = \pi/4)/\Delta_h(\theta = 0)$ as the ratio of the gap along the $k_x = \pm k_y$ axis ($\theta = \pi/4$) to k_x axis ($\theta = 0$) show its variation with the nematic mass parameter δ in Fig.13c. We fit our result upto second order in δ . We find a good fit with: $\alpha(\delta) = 2.12\delta^2 + 0.44\delta$. We numerically obtain the specific heat coefficient as a function of the reduced temperature T/T_c for two different values of $\delta = 0.01$ and 0.1 and show the result in Fig.13d. For small but non-zero δ , there will be jump at the transition point as mentioned before. On the other hand, at the low temperature the behavior of the specific heat coefficient will be qualitatively similar to the $\delta = 0$ case because of the presence of the exponentially small gap near the cold spot for small values of δ .



Supplementary Figure 13. **Superconductivity away from the nematic QCP ($\delta \neq 0$):** In (a) we plot the largest eigenvalue λ of the matrix M defined in Eq. (159) in different angular momentum channels labeled as λ_s, λ_d and λ_p for the s, d and p - wave respectively. We represent the ratio of these eigenvalues in the inset and find $\lambda_p/\lambda_s < \lambda_d/\lambda_s < 1$ which indicates that s - wave is the leading instability. With nematic mass parameter decreasing, the ratio of the eigenvalues becoming closer to each other and approaching unity which indicates degeneracy among different pairing channels. In (b) we plot the angular variation of the gap $\Delta_h(\theta)$ on the hole pocket for a set of temperatures. In (c) we plot (brown dots) the ratio of the gap along the $k_x = \pm k_y$ axis ($\theta = \pi/4$) to k_x axis ($\theta = 0$) defined as α in the main text as a function of δ and fit (blue line) our result upto second order in δ with the fitting parameters $\Delta(\pi/4)/\Delta(0) = 2.12\delta^2 + 0.44\delta$. In (d) we plot the variation of the specific heat coefficient divided by the density of state, N_0 with the reduced temperature T/T_c for a set of values of $\delta = 0.01$ (blue) and 0.1 (orange). The black dashed line represents the normal state result for the specific heat coefficient and is equal to $2\pi^2/3$.

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