

Synthetic reflection self-injection-locked microcombs

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1 Coupled mode equations and pump mode hybridization

The normalized coupled mode equations (CMEs) for the pump in forward and backward directions read

$$\partial_t a_0 = -(1 + i\zeta_0)a_0 + i|a_0|^2 a_0 + 2i|b_0|^2 a_0 + i\beta b_0 + f \quad (1)$$

$$\partial_t b_0 = -(1 + i\zeta_0)b_0 + i|b_0|^2 b_0 + 2i|a_0|^2 b_0 + i\beta a_0 \quad (2)$$

where for convenience the normalized coupling rate $\beta = 2\gamma/\kappa \geq 0$ has been introduced. Without loss of generality, $f \geq 0$.

The coefficient matrix of the system of equations (without the pump)

$$M = \begin{pmatrix} -(1 + i\zeta_0) + i|a_0|^2 + 2i|b_0|^2 & i\beta \\ i\beta & -(1 + i\zeta_0) + i|b_0|^2 + 2i|a_0|^2 \end{pmatrix} \quad (3)$$

has the following Eigenvalues

$$\lambda_{\pm} = - \left(1 + i\zeta_0 - \frac{3}{2}i(|a_0|^2 + |b_0|^2) \pm i\sqrt{\beta^2 + \left(\frac{1}{2}(|a_0|^2 - |b_0|^2)\right)^2} \right) \quad (4)$$

$$= - \left(1 + i(\zeta_0 - \delta\zeta_{\text{NL}} \pm \sqrt{\beta^2 + \delta\beta_{\text{NL}}^2}) \right) \quad (5)$$

and is diagonalized in the following Eigenbasis of hybridized forward-backward modes

$$\left\{ \frac{1}{2}(|a_0|^2 - |b_0|^2) \pm \sqrt{\beta^2 + \left(\frac{1}{2}(|a_0|^2 - |b_0|^2)\right)^2}; -\beta \right\} \quad (6)$$

$$= \left\{ \delta\beta_{\text{NL}} \pm \sqrt{\beta^2 + \delta\beta_{\text{NL}}^2}; -\beta \right\} \quad (7)$$

where $\delta\beta_{\text{NL}} = \frac{1}{2}(|a_0|^2 - |b_0|^2)$ and $\delta\zeta_{\text{NL}} = \frac{3}{2}(|a_0|^2 + |b_0|^2)$. The transformation matrices are:

$$T = \begin{pmatrix} \delta\beta_{\text{NL}} + \sqrt{\beta^2 + \delta\beta_{\text{NL}}^2} & \delta\beta_{\text{NL}} - \sqrt{\beta^2 + \delta\beta_{\text{NL}}^2} \\ -\beta & -\beta \end{pmatrix} \quad (8)$$

and

$$T^{-1} = \begin{pmatrix} \frac{1}{2\sqrt{\beta^2 + \delta\beta_{\text{NL}}^2}} & \frac{\delta\beta_{\text{NL}} - \sqrt{\beta^2 + \delta\beta_{\text{NL}}^2}}{2\beta\sqrt{\beta^2 + \delta\beta_{\text{NL}}^2}} \\ -\frac{1}{2\sqrt{\beta^2 + \delta\beta_{\text{NL}}^2}} & \frac{-\delta\beta_{\text{NL}} - \sqrt{\beta^2 + \delta\beta_{\text{NL}}^2}}{2\beta\sqrt{\beta^2 + \delta\beta_{\text{NL}}^2}} \end{pmatrix} \quad (9)$$

so that

$$\begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = T \begin{pmatrix} a_+ \\ a_- \end{pmatrix} \quad (10)$$

and

$$\begin{pmatrix} a_+ \\ a_- \end{pmatrix} = T^{-1} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} \quad (11)$$

where $a_{\pm}$ denote the (not specifically normalized) field amplitudes of the hybrid modes. The steady state equations for the hybrid modes are

$$0 = - \left(1 + i(\zeta_0 - \delta\zeta_{\text{NL}} + \sqrt{\beta^2 + \delta\beta_{\text{NL}}^2}) \right) a_+ + \frac{1}{2\sqrt{\beta^2 + \delta\beta_{\text{NL}}^2}} f \quad (12)$$

$$0 = - \left(1 + i(\zeta_0 - \delta\zeta_{\text{NL}} - \sqrt{\beta^2 + \delta\beta_{\text{NL}}^2}) \right) a_- - \frac{1}{2\sqrt{\beta^2 + \delta\beta_{\text{NL}}^2}} f \quad (13)$$

In non-normalized units, the effective resonance frequencies of the hybridized modes are

$$\omega_{\pm, \text{eff}} = \omega_0 - \frac{\kappa}{2} \left(\delta\zeta_{\text{NL}} \pm \sqrt{\beta^2 + \delta\beta_{\text{NL}}^2} \right) \quad (14)$$

2 Approximations for the forward pump mode under strong coupling

In what follows, it is assumed that

- the coupling is strong $\beta > 1$
- due to the strong coupling, the power levels in forward and backward directions are approximately equal $|a_0|^2 = |b_0|^2$. Note that due to symmetry breaking [1] this is only valid up to a certain power level. We validated, by numeric integration of the CMEs 1 and 2, that this approximation is valid.
- the detuning ζ_0 is such that approximately only the lower frequency hybrid mode a_- is driven, i.e. $|a_-| \gg |a_+|$.

Under these assumptions,

$$a_0 = \left(\delta\beta_{\text{NL}} - \sqrt{\beta^2 + \delta\beta_{\text{NL}}^2} \right) a_- \approx -\beta a_- \quad (15)$$

$$b_0 = -\beta a_- \quad (16)$$

and in consequence

$$(1 + i(\zeta_0 - \delta\zeta_{\text{NL}} - \beta))a_0 = \frac{f}{2} \quad (17)$$

Multiplying each side of the equation with its complex conjugate results in

$$(1 + (\zeta_0 - \delta\zeta_{\text{NL}} - \beta)^2)|a_0|^2 = \frac{f^2}{4} \quad (18)$$

An immediate insight is that the strong coupling between forward and backward waves limits the power in the forward (or backward) wave to values of

$$|a_0|^2 \leq f^2/4 \quad (19)$$

Expressing $\delta\zeta_{\text{NL}}$ via the field amplitudes gives

$$(1 + (\zeta_0 - 3|a_0|^2 - \beta)^2)|a_0|^2 = \frac{f^2}{4} \quad (20)$$

and for the detuning

$$\zeta_{0,\pm} = \beta + 3|a_0|^2 \pm \sqrt{\frac{f^2}{4|a_0|^2} - 1} \quad (21)$$

where $\zeta_{0,+}$ corresponds to an effective red-detuning and $\zeta_{0,-}$ to an effective blue-detuning with regard to the lower-frequency hybrid mode a_- .

3 Threshold condition and first oscillating sideband

We consider two initially zero-power (except for vacuum fluctuations) sidebands with mode number $\pm\mu$ relative to the pumped mode. Their CMEs are

$$\partial_t a_{+\mu} = -(1 + i(\zeta_\mu - 4|a_0|^2))a_{+\mu} + ia_0^2 a_{-\mu}^* \quad (22)$$

$$\partial_t a_{-\mu}^* = -(1 - i(\zeta_\mu - 4|a_0|^2))a_{-\mu}^* - ia_0^{*2} a_{+\mu} \quad (23)$$

where again $|a_0|^2 \approx |b_0|^2$ was assumed and $\zeta_\mu = \frac{2}{\kappa}(\omega_0 - \omega_p + \frac{1}{2}D_2\mu^2) = \zeta_0 + \frac{D_2}{\kappa}\mu^2$. The Eigenvalues of this set of equations are

$$\lambda_{\pm} = -1 \pm \sqrt{|a_0|^4 - (\zeta_\mu - 4|a_0|^2)^2} \quad (24)$$

The parametric gain experienced by the two sidebands therefore is

$$G = \kappa \sqrt{|a_0|^4 - \left(\zeta_0 + \frac{D_2}{\kappa}\mu^2 - 4|a_0|^2 \right)^2} \quad (25)$$

At least a intracavity power of $|a_0|^2 = 1$ is required to reach threshold. With Eq. 19 it follows that for strong coupling the threshold pump power $f^2 \geq 4|a_0|^2$ is at least four times the threshold power of a resonator without forward-backward coupling.

3.1 First oscillating sideband (Condition for single-DKS)

We are interested in finding the condition under which the first oscillating modulation instability (MI) sidebands will appear in the resonances directly adjacent to the pump mode; as described in the main text, this would seed a single DKS pulse.

The phase mismatch between the pump wave and the resonator modes can be quantified via their effective (including nonlinear frequency shifts) detuning $\zeta_{\mu,\text{eff}}$ from an equidistant D_1 -space frequency grid. A smaller $\zeta_{\mu,\text{eff}}$ implies better phase matching.

$$\zeta_{\mu,\text{eff}} = \zeta_0 + \frac{D_2}{\kappa} \mu^2 - 4|a_0|^2 \quad (26)$$

$$= \beta - |a_0|^2 \pm \sqrt{\frac{f^2}{4|a_0|^2} - 1} + \frac{D_2}{\kappa} \mu^2 \quad (27)$$

For DKS the resonator is characterized by anomalous dispersion $D_2 > 0$ ($\beta \gg D_2/\kappa$). It can therefore be guaranteed, that the first generated sideband pair (best phase matching) will be $\mu \pm 1$, if

$$\beta - |a_0|^2 - \sqrt{\frac{f^2}{4|a_0|^2} - 1} > 0 \quad (28)$$

Assuming $|a_0|^2 \leq f^2/4$ we find

$$\boxed{\beta > f^2/4 \quad \Leftrightarrow \quad \gamma/\kappa > f^2/8} \quad (29)$$

as a condition that guarantees that the first sideband pair will be generated at $\mu = \pm 1$.

3.2 Threshold power

The threshold power f_{th} is the power level where the parametric threshold $G > \kappa$ can be reached. Inserting Eq. 21 for the detuning into Eq. 25, we obtain for the threshold condition

$$|a_0|^4 - \left(\beta + 3|a_0|^2 \pm \sqrt{\frac{f_{\text{th}}^2}{4|a_0|^2} - 1} + \frac{D_2}{\kappa} \mu^2 - 4|a_0|^2 \right)^2 = 1 \quad (30)$$

$$\Leftrightarrow |a_0|^4 - 1 = \left(\beta \pm \sqrt{\frac{f_{\text{th}}^2}{4|a_0|^2} - 1} + \frac{D_2}{\kappa} \mu^2 - |a_0|^2 \right)^2 \quad (31)$$

Under the assumption that $\beta - |a_0|^2 - \sqrt{\frac{f_{\text{th}}^2}{4|a_0|^2} - 1} > 0$ (condition for first oscillating sidebands $\mu = \pm 1$), this results in

$$\boxed{f_{\text{th}}^2 = 4|a_0|^2 + 4|a_0|^2 \left(\beta - |a_0|^2 + \frac{D_2}{\kappa} \mu^2 - \sqrt{|a_0|^4 - 1} \right)^2, \quad (|a_0|^4 > 1)} \quad (32)$$

This equation can be solved numerically. For example, $\beta = 4$ results in $f_{\text{th}}^2 \approx 8.4$

3.3 Threshold power assuming zero-effective detuning

A simplified threshold condition may be derived assuming that the threshold will be reached at zero effective detuning so that

$$\zeta_0 = \beta + 3|a_0|^2 \quad \text{and} \quad (33)$$

$$f_{\text{th}}^2 = 4|a_0|^2 \quad (34)$$

In this case the threshold condition is

$$|a_0|^4 - \left(\beta + 3|a_0|^2 + \frac{D_2}{\kappa} \mu^2 - 4|a_0|^2 \right)^2 = 1 \quad (35)$$

$$\Leftrightarrow \frac{f_{\text{th}}^4}{16} - \left(\beta - \frac{1}{4} f_{\text{th}}^2 + \frac{D_2}{\kappa} \mu^2 \right)^2 = 1 \quad (36)$$

$$\Leftrightarrow f_{\text{th}}^2 = \frac{2 + 2 \left(\beta + \frac{D_2}{\kappa} \mu^2 \right)^2}{\beta + \frac{D_2}{\kappa} \mu^2} \quad (37)$$

Assuming that $\beta \gg \frac{D_2}{\kappa} \mu^2$ this simplifies to

$$\boxed{f_{\text{th}}^2 \approx \frac{2 + 2\beta^2}{\beta} = 2\beta + \frac{2}{\beta} \quad \Leftrightarrow \quad f_{\text{th}}^2 \approx 4\frac{\gamma}{\kappa} + \frac{\kappa}{\gamma}} \quad (38)$$

For $\beta = 4$ we find $f_{\text{th}}^2 = 8.5$, almost equal to what is obtained through Eq. 32.

4 Pump mode hybridization above threshold

The derivations of Section 1 are only valid below threshold, where only the forward and backward pump mode are excited. Above threshold, and in particular in presence of DKS, the effective frequencies of the hybrid modes resulting from the avoided mode crossing of the coupled forward and backward modes are

$$\omega_{\pm, \text{eff}}(\mu) = \frac{1}{2}(\omega_{\text{a,eff}}(\mu) + \omega_{\text{b,eff}}(\mu)) \pm \frac{1}{2}\sqrt{(\omega_{\text{a,eff}}(\mu) - \omega_{\text{b,eff}}(\mu))^2 + 4\gamma(\mu)^2} \quad (39)$$

where $\omega_{\text{a,eff}}$ and $\omega_{\text{b,eff}}$ are the effective (i.e. taking nonlinear frequency shifts into account) resonance frequencies of the forward and backward modes, respectively; $\gamma(\mu)$ denotes the mode dependent coupling.

$$\omega_{\text{a,eff}}(\mu) = \omega_0 - \frac{\kappa}{2} \left(\text{Re}(\hat{\mathcal{F}}[|\psi_a(\theta)|^2 \psi_a(\theta)]_{\mu}/a_{\mu}) + 2 \sum_{\eta} |b_{\eta}|^2 \right) \quad (40)$$

$$\omega_{\text{b,eff}}(\mu) = \omega_0 - \frac{\kappa}{2} \left(\text{Re}(\hat{\mathcal{F}}[|\psi_b(\theta)|^2 \psi_b(\theta)]_{\mu}/b_{\mu}) + 2 \sum_{\eta} |a_{\eta}|^2 \right) \quad (41)$$

$$(42)$$

where $\psi_a(\theta) = \hat{\mathcal{F}}^{-1}[a_{\mu}]$ and $\psi_b(\theta) = \hat{\mathcal{F}}^{-1}[b_{\mu}]$ are the spatio-temporal field profiles and $\hat{\mathcal{F}}[\cdot]_{\mu}$ stands for the component corresponding to the μ -th mode ($\hat{\mathcal{F}}$ denotes the Fourier transform). Thus the nonlinear integrated dispersion for the hybrid modes can be defined as follows

$$D_{\pm}^{\text{int}}(\mu) = 2(\omega_{\pm, \text{eff}}(\mu) - \omega_0 - \mu D_1) / \kappa \quad (43)$$

An example of the nonlinear integrated dispersion is shown in Fig. S1 for a PhCR in a single DKS state with $f = 3$, and $2\gamma/\kappa \approx 3.7$ for the pumped mode $\mu = 0$ (zero-forward backward coupling is assumed for all other modes). In addition, the DKS existence range (DKS range) and the range of detunings accessible via SIL (SIL range) are indicated. Both, DKS and SIL ranges, are obtained numerically (cf. main text). For comparison, also the linear integrated dispersion (i.e. in the absence of nonlinear mode shifts) is shown.

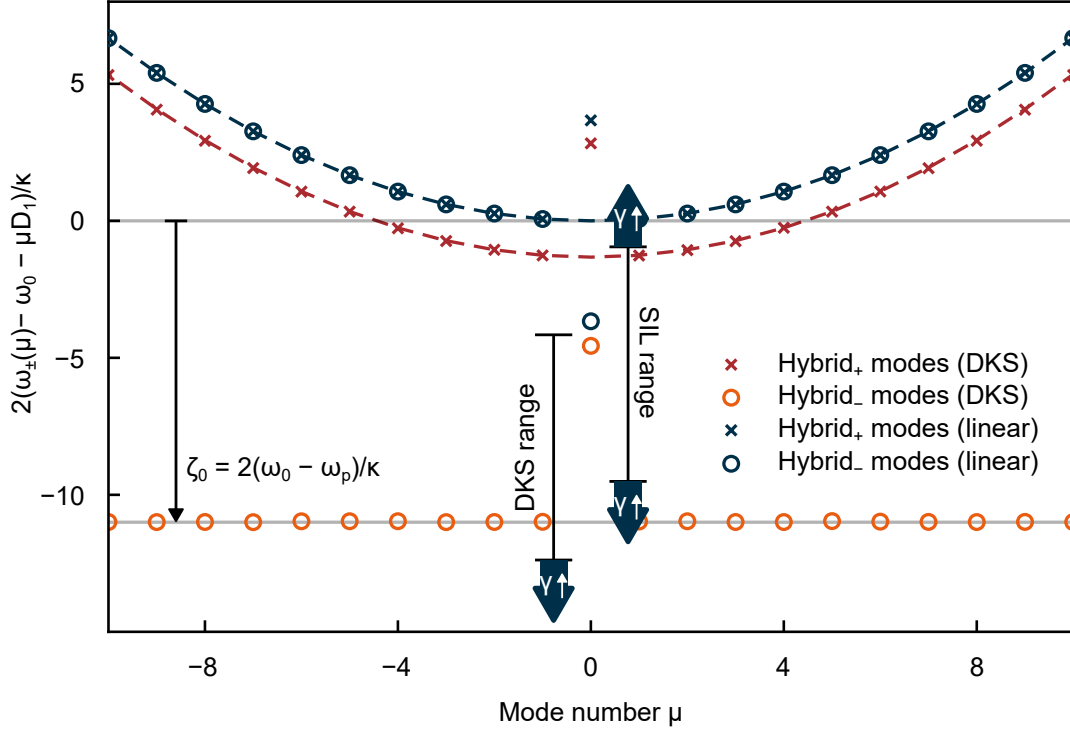


Figure S1 | Nonlinear integrated dispersion of the hybridized modes. Linear (blue circles and crosses) and nonlinear (orange circles and red crosses) integrated dispersions of a PhCR as detailed in the text.

5 Comb generation in the backward direction

Due to the initially similar power-levels in forward and backward pump modes, combs may in principle not only be generated in the forward, but also in the backward direction. Indeed, when the pump laser detuning is between the (effective) resonance frequencies of the hybridized modes, the backward-wave is usually stronger (despite the forward-pumping). For the parameters considered in this work, we found that this range of detuning does not overlap with the soliton existence range and backward combs were not observed. However, backward combs represent an interesting opportunity for additional research. Aside from backward comb generation we note, that backward modulation instability, can trigger forward modulation instability (and comb generation) and vice versa, through a non-zero forward-backward coupling of the modulation instability sidebands. This can readily be included in the numeric model by introducing non-zero γ also for modes with $\mu \neq 0$.

6 SIL DKS generation for different backscattering strengths

We investigate experimentally SIL-based DKS generation in samples with different values of backscattering. The first sample has no corrugation pattern, but exhibits a strong random backscattering of $2\gamma/\kappa \approx 0.44$. The two remaining samples have a synthetic reflection of $2\gamma/\kappa \approx 0.98$ and 2.7 . The characterization of the pumped resonances is shown in Fig. S2a. Total linewidths κ and coupling rate γ are extracted via lineshape fitting. Similar to Figure 3 in the main text, we explore self-injection locked DKS generation in these samples. The results are shown in Fig. S2b. Provided a certain minimal back-scattering is present, all samples support a single-DKS state. However, the current range over which the single-DKS is observed is negligible for the weakest back-reflection and remains marginal for the medium level of back-reflection (compared to the current range over which multi-DKS and noisy comb states are generated). Only for the strongest back-reflection, a significant, readily accessible current range is achieved for the, in this case, exclusive single-DKS state. Note that we validated the single and multi-DKS regimes via their characteristic optical spectra (not shown).

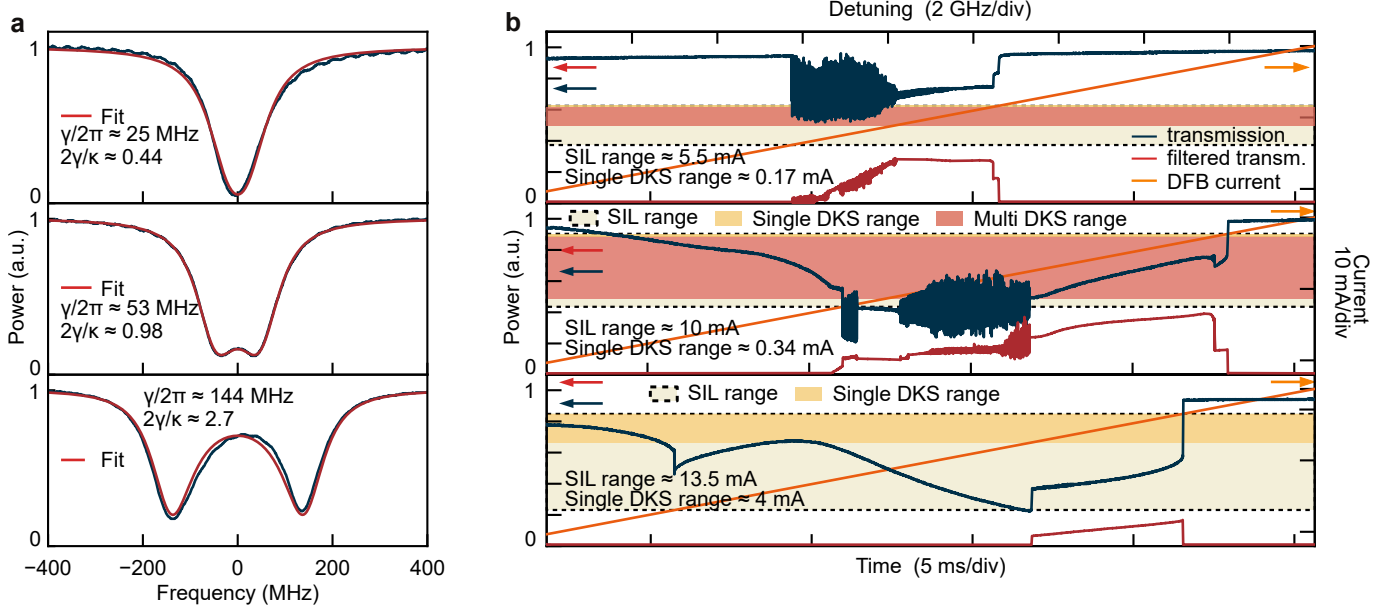


Figure S2 | SIL DKS generation for different backscattering strengths. **a**, Measured lineshapes of microresonators with different level of backscattering (cf. main text for details). The linewidths κ and coupling rates γ (mode splittings) are retrieved from the fit. **b**, Total transmission (blue) and bandpass-filtered microcomb power (cf. main text) (red; filter offset from the pump, indicates comb formation) measured during a laser scan towards longer wavelengths. The orange line corresponds to the driving current. SIL range, as well as single- and multi-DKS ranges on the current axis are highlighted. The horizontal axis indicates the free-running laser detuning and the experiment time.

7 Characterization of PhCRs

All samples are characterized via frequency comb calibrated laser scans [2], which allow us to determine the resonator's dispersion, the coupling rates γ , the resonance widths κ , over a wide spectral bandwidth. The integrated dispersion $D_{\text{int}} = (\omega_{\mu} - \omega_0 - \mu D_1)/2\pi$ for a representative sample (with relatively large backscattering (for visibility) of $2\gamma/\kappa \approx 10$) is shown in Fig. S3a along with a conventional sample without corrugation. The comparison between both shows that, while the corrugation leads to a resonance splitting for the mode $\mu = 0$ it has no effect on the overall dispersion profile. Fig. S3b shows the dependence of γ and the Q -factor ($Q = \omega_0/\kappa$) on the corrugation amplitude. No noticeable degradation of the Q -factor is observed up to $\gamma/2\pi \lesssim 5$ GHz; even for large coupling $\gamma \approx 45$ GHz, the Q -factor is only halved.

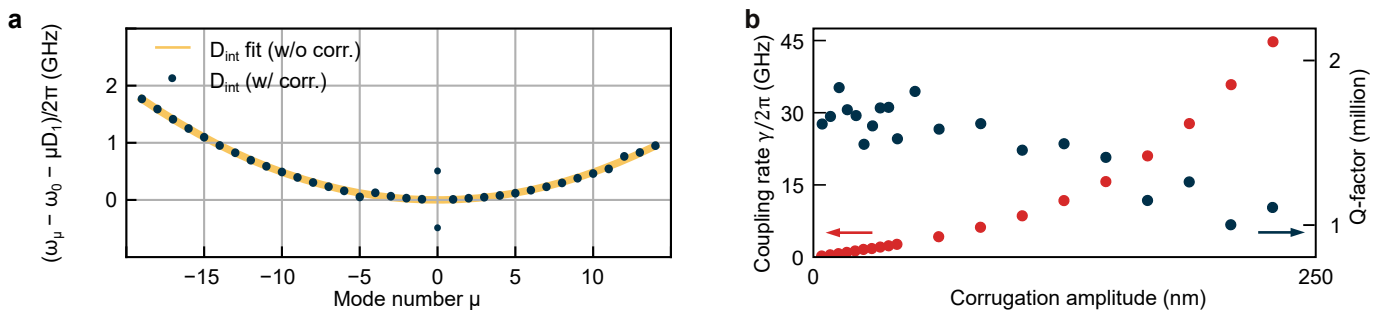


Figure S3 | PhCR characterization. **a**, Integrated dispersion D_{int} for two samples: without corrugation pattern (yellow line), and with single-period corrugation (blue dots). **b**, Measured backward coupling rates γ (red, left axis) and Q -factors (blue, right axis) of the PhCRs as a function of the estimated corrugation amplitude.

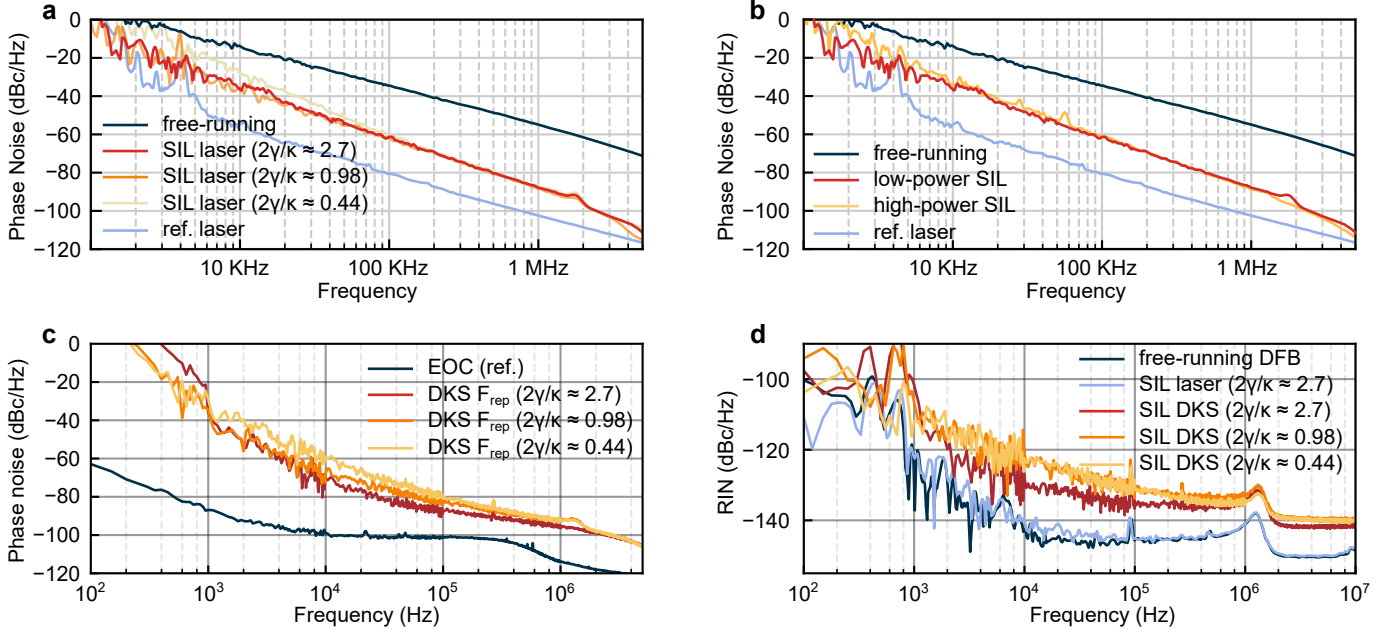


Figure S4 | Noise properties of SIL. **a**, Phase noise of the DFB laser in free-running and SIL regimes measured through heterodyne detection with a reference laser (cf. Figure 3, main text). The phase noise of the reference laser is provided as a baseline. **b**, Comparison of the phase noise of the DFB laser in SIL regimes at low and high output powers for the sample with $2\gamma/\kappa \approx 2.7$. **c**, Phase noise of the single DKS repetition rate. The phase noise of the electro-optic comb line used for heterodyning (cf. main text) is indicated for reference. **d**, Relative intensity noise (RIN) of SIL-based single DKS (red, orange, yellow), low-power continuous-wave SIL lasing (blue) and the free-running DFB laser (dark blue).

8 Noise properties of synthetic reflection SIL

Fig. S4 shows and compares noise properties continuous-wave lasing and DKS combs generated in samples with different levels of backscattering. Fig. S4a shows the phase noise of the self-injection locked continuous-wave laser at low power and without soliton formation for different values of backscattering (the phase noise of the free-running laser diode as well as the phase noise of the reference laser are also shown). Fig. S4b compares the phase noise of the self-injection locked continuous-wave laser at low power and high power (note that at high power a DKS is present). Fig. S4c shows the repetition rate phase noise, i.e. the phase noise of the 300 GHz tone. Reduced noise is observed for larger values of backscattering. As the 300 GHz signal cannot be detected, an electro-optic comb (EOC) is used for optical-down mixing, as described in the main text (the EOC's phase noise is given for reference). Fig. S4d compares the RINs of the free-running laser, the self-injection locked continuous-wave laser at low power, as well as the RIN of the DKS states.

9 Comparison of different diode current ramp speeds

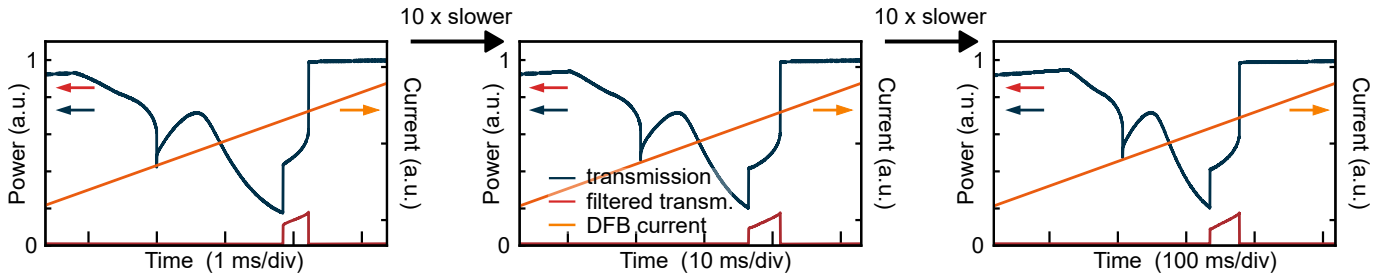


Figure S5 | SIL DKS formation at different diode current ramp speeds. Filtered and unfiltered transmission recorded while ramping the diode current with different speed over the same current interval. The observed transmission (filtered/unfiltered) is virtually independent of the diode current ramp speed.

References

- [1] Del Bino, L., Silver, J. M., Stebbings, S. L. & Del’Haye, P. Symmetry Breaking of Counter-Propagating Light in a Nonlinear Resonators. *Scientific Reports*, 7:43142 (2017).
- [2] Del’Haye, P., Arcizet, O., Gorodetsky, M. L., Holzwarth, R. & Kippenberg, T. J. frequency Comb Assisted Diode Laser Spectroscopy for Measurement of Microcavity Dispersion. *Nature Photonics*, 3:529–533 (2009).