

Topological orbital angular momentum extraction and twofold protection of vortex transport

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Supplementary Information

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1. Experimental setup and method

In our experiment, we established three photonic disclination structures with C_3 -, C_4 - and C_5 -symmetries in topologically nontrivial configurations using the setup schematically shown in Fig. S1. The disclination lattices are written (optically induced) site-by-site via a continuous-wave laser (Coherent Verdi: $\lambda = 532$ nm) in a 20-mm-long electrically biased nonlinear crystal (SBN:61) through the photorefractive nonlinearity¹. The coupling coefficients for the lattices can be easily controlled by adjusting the waveguide spacing. Due to the photorefractive “memory effect”, all written waveguides remain intact during the time interval necessary for experimental measurements¹.

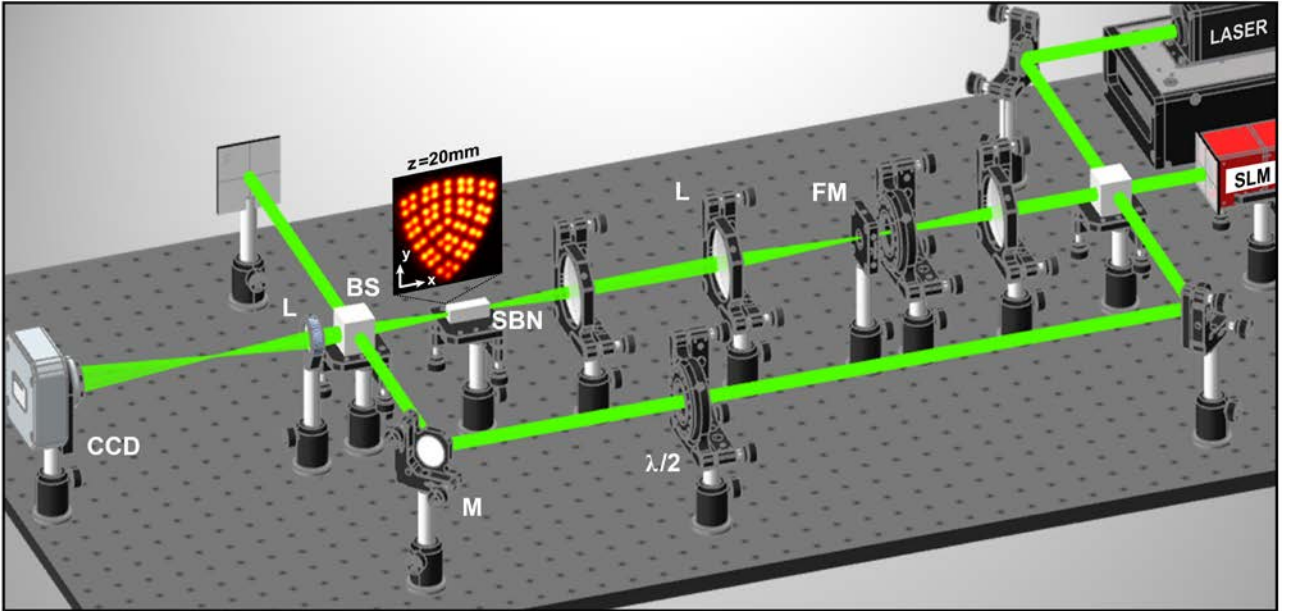


Fig. S1 Schematic of the experimental setup for writing and probing a photonic disclination structure. SLM: spatial light modulator; BS: beam splitter; FM: Fourier mask; L: spherical lens; SBN: strontium barium niobate crystal; M: mirror; $\lambda/2$: half-wave plate; CCD: charge-coupled device. The image in the back shows a typical laser-written C_3 -symmetric disclination structure.

A written photonic lattice can be easily erased using incoherent white-light illumination and re-written into another one, thus the lattice structures are reconfigurable, differently from the femtosecond laser-written waveguides in glass media². In addition, when needed for nonlinear control of topological states, both self-focusing and -defocusing nonlinearities can be readily obtained at low laser power³. Once the disclination structures are written, they are “visualized” by probing the lattice site-by-site with a set of Gaussian beams (see Fig. 2a for C_3 -, Fig. 4e1 for C_4 -, and

Fig. 4f1 for C_5 -symmetric lattices in the main text). To examine the vortex transport through the written disclination structure, we launch an extraordinarily-polarized optical vortex beam carrying a topological charge l into the single-site core (central defect waveguide) of the disclination lattice.

To prevent nonlinear self-action of the vortex probe beam during propagation, its power is set to a low value of ~ 20 nW. The vortex beam is generated by a spatial light modulator device (Holoeye Pluto). The phase information for the vortex beam is retrieved by interferogram measurements using a quasi-plane-wave reference beam. An imaging system formed by a spherical lens ($f = 100$ mm) and a CCD camera is used to record the input and output patterns of the vortex beam before and after propagating through the disclination core.

2. Numerical simulations of vortex beam propagation via the continuous model

Optical vortex beam propagation in photonic lattices, as established in our experiment, can be described by the continuous model of the nonlinear Schrödinger-like equation (NLSE) - see Eq. (M2) in the main text. The optical nonlinearity is used mainly for inducing the photonic lattices, although the vortex beam undergoes linear propagation through the lattices in this study. Illustrated in Extended Data Fig. 2 are the NLSE simulation results reproducing measurements in Fig. 2 and Fig. 4 by using parameters from the experiment. The initial input probe beam is an optical vortex beam with a topological charge $l = 1$ or $l = 2$ launched into the core of the C_3 -, C_4 - and C_5 -symmetric disclination lattices under a topologically nontrivial geometry (featuring a dimerization ratio $d_2/d_1 = 0.74$). The coefficients d_1 and d_2 are the intracell and intercell waveguide distances, respectively, as shown in Fig. 3b1. These numerical results are in good agreement with experimental observations. Both the low and high-order vortex beams are stably guided in the center waveguide channel of C_3 - and C_5 -disclination structures, preserving their typical doughnut shape and phase singularity even for a propagation distance much longer than the crystal length [Extended Data Fig. 2a1-a5 and 2c1-c5]. A *topological vortex guide* (TVG) is also achieved for the $l = 1$ vortex in the C_4 -lattice, as shown in Extended Data Fig. 2b1-b3. More importantly, the NLSE simulations confirm the vortex guidance for these examples because all of them experience a “double protection” - the nontrivial momentum- (k -space) and real-space topological protection. Conversely, in the C_4 -disclination, the $l = 2$ vortex beam breaks up into a quadrupole-like pattern during propagation (see

Extended Data Fig. 2b4-b5) due to the lack of real-space topological protection. Here, the real-space topology is characterized by the *vorticity-coordinated rotational symmetry* (VRS), which requires $q(l, n) = 2l/n \notin \mathbb{Z}$ for a given topological charge l and a rotational symmetry order n . For the $l = 2$ vortex in the C_4 -disclination ($n = 4$), clearly this condition is not satisfied. The interference fringes for the numerical interferograms in the insets display a fork structure stemming from the phase singularity when a TVG is realized. However, the fork-shaped fringe pattern is lost when the quadrupole forms without vorticity, consistent with the corresponding measurements. The VRS condition is necessary (but not sufficient) for stable TVG as further discussed below.

3. Numerical simulations of topological OAM extraction under mixed-mode excitation

We show here that the TVG can bring about a solution for vortex extraction since the mode mixing effect can be readily suppressed using the proposed disclination structure. Corresponding experimental results are presented in Fig. 5 of the main text. In Fig. S2, we show corresponding numerical simulations for a mixed-mode input beam consisting of the $l = 1$ mode and a portion of the undesired $l = 2$ mode (e.g., $\Psi(0) = \psi_{l=1} + 0.7\psi_{l=2}$). In the single waveguide, both $l = 1$ and $l = 2$ modes co-exist, leading to mode beating and unstable output (see Fig. S2 a1-a4), as observed in our experiments. To examine the mode quality, the mode occupation expressed as $|\langle \Psi(z) | \psi_i \rangle|^2$ is directly evaluated (Fig. S2a4), where $\Psi(z)$ is the probe-beam amplitude at the propagation distance z , and ψ_i is the eigenmode supported by a single waveguide. In contrast to a single waveguide, the $l = 2$ mode can be mostly “filtered” out through an appropriate design of our C_3 disclination lattice, leaving the $l = 1$ mode intact after a short propagation distance (see Fig. S2b1-b4). This specific test shows how the TVG can be employed for OAM-mode “filtering” through the disclination lattice, while a single waveguide cannot. From a physical viewpoint, the strategy for achieving vortex selection is to design a C_n -disclination structure where only a chosen OAM mode is topologically nontrivial, whereas the other OAM modes are instead topologically trivial. In our photonic lattices, this can be done by tuning the lattice parameters so that the disclination structure exhibits chiral symmetry or sub-symmetry⁹ only for the chosen mode, as different orders of vortex modes have different coupling strengths. (In Fig. S2, for instance, the structure is nontrivial for the $l = 1$ mode but trivial for the $l = 2$ mode). These simulations together with the experimental results presented in

Fig. 5 of the main text show clearly that the topological disclination can provide a solution for OAM-mode extraction, thus eliminating the effects of unwanted mode coupling during transport.

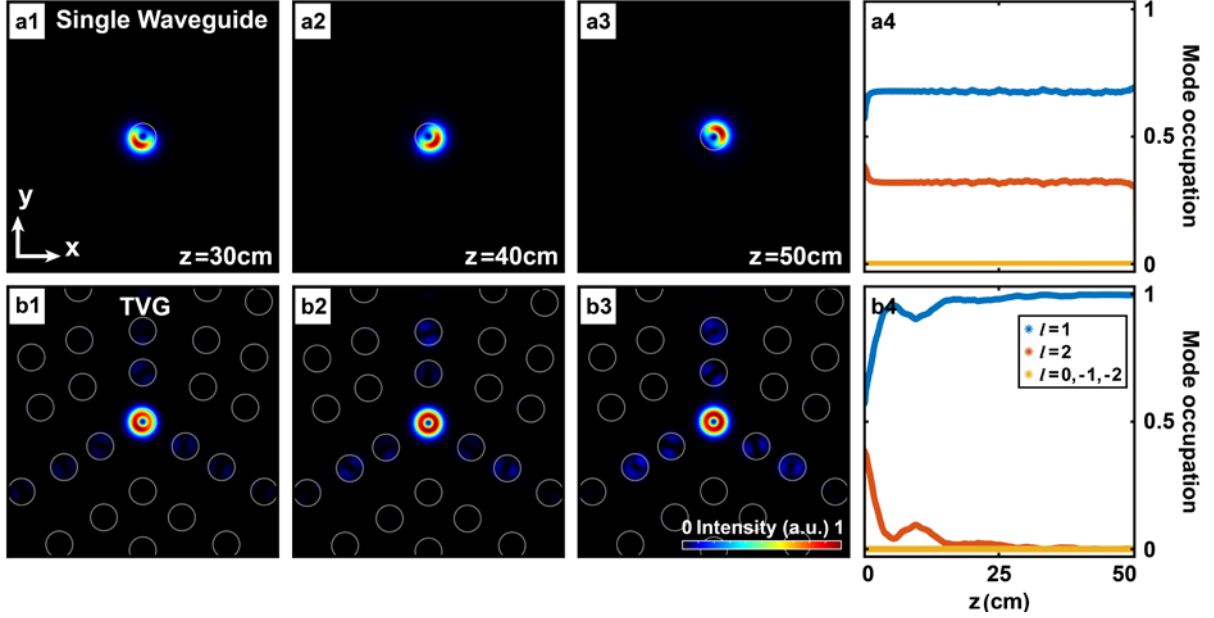


Fig. S2 Numerical simulations of topological OAM-mode extraction under a mixed-mode excitation. **a1-a3** Output intensity patterns from a single waveguide. **a4** Calculated mode-occupation as a function of propagation distance z . **b1- b4** Output from a topological C_3 disclination structure. As seen in (**a4**), the $l = 2$ vortex mode in the single waveguide cannot be eliminated during propagation, thus leading to a mixed-mode output characterized by a distorted “doughnut” pattern changing with propagation distance (**a1-a3**). However, the unwanted $l = 2$ vortex mode is “filtered” out by the TVG as shown in (**b4**), leading to only the $l = 1$ vortex mode that remains intact during propagation (**b1-b3**). (see Supplementary Media file). These results correspond to the experimental results presented in Fig. 5 of the main text.

4. The disclination Hamiltonian and vortex modes via the tight-binding model

Topological properties of a C_n -symmetric disclination lattice involving the OAM modes are better revealed through coupled mode theory (CMT) by applying the tight-binding approximation, wherein only the nearest-neighbor hopping is considered in the analysis. The discrete equation governing the light propagation reads as

$$i\partial_z \psi_{m_1, l_1} = \sum_{m_2, l_2} [H]_{m_1, l_1; m_2, l_2} \psi_{m_2, l_2} \quad (S1)$$

where $[H]_{m_1, l_1; m_2, l_2}$ are the elements of the real-space vortex Hamiltonian H accounting for the coupling between topological charge l_1 at site m_1 with topological charge l_2 at site m_2 . ψ_{m_1, l_1} (ψ_{m_2, l_2}) is the complex amplitude at the m_1 -th (m_2 -th) waveguide with a topological charge l_1 (l_2),

and Z is the dimensionless propagation distance. The OAM modes with different orders of topological charges are assumed to be completely independent and thus uncoupled, which implies $l_{1,2} = \pm l$. Under these conditions, the Hamiltonian H , associated with a C_n -disclination lattice and a topological charge l , is explicitly reported in Eq. (M1) of the main text. We introduce the coupling coefficient $t_{OV}(\theta) = \xi(\mathbf{R} - \mathbf{R}')\kappa_{OV}e^{2il\theta(\mathbf{R}-\mathbf{R}')}$ to describe the *opposite* vorticity mode coupling (OVMC). The OVMC is in general complex and orientation dependent, accounting for hopping between two vortex modes with opposite phase circulations (l and $-l$) at the $\mathbf{R}$ and $\mathbf{R}'$ locations, with $\theta(\mathbf{R} - \mathbf{R}')$ denoting the azimuthal angle between the spacing vector $\mathbf{R} - \mathbf{R}'$ and a reference axis. The real and azimuthally isotropic coefficient $t_{SV}(\theta) = \xi(\mathbf{R} - \mathbf{R}')\kappa_{SV}$ describes the *same* vorticity mode coupling (SVMC), which refers to hopping between two vortex modes with the same circulations (l and l , or $-l$ and $-l$). In general, κ_{SV} and κ_{OV} are not equal, and κ_{OV} is typically larger than κ_{SV} ¹⁰. We plot in Fig. 3 of the main text the vortex band structures of the low- and high-order topological charges with anti-clockwise ($l > 0$) and clockwise ($l < 0$) phase circulations (i.e., $|l| = 1$ and 2) for a nontrivial C_3 -disclination lattice (with 61 lattice sites, $\kappa_{SV}/\kappa_{OV} = 0.5$, and $d_2/d_1 = 0.85$). Here, we supplement these results with the same CMT calculations but for a C_4 -disclination lattice with 81 sites. As illustrated in Fig. S3a1, a2, the nontrivial vortex band structures for this case can also support a pair of zero-energy defect modes with opposite phase circulations. Nevertheless, for $|l| = 1$, the eigenvalues associated with defect vortex modes are isolated at the mid-gap, while for some high-order vortices such as $|l| = 2$, they are bound states in the continuum (BICs)¹¹. Intensity and phase distributions associated with zero-energy defect modes exhibit again the characteristic features of topological states: they only reside in one sublattice, with exponential decay of the amplitudes towards the bulk. The k -space topology of zero-energy states is further confirmed by the calculation of the multipole chiral number (MCN) (see details in Section 5).

We note that, although the topological vortex modes for $l = 2$ and $l = -2$ are supported by the C_4 -disclination structure, they are not stable during long-distance transport (see Fig. 1c3). As in the continuous-model simulations, the discrete CMT evolution demonstrates that a high-order vortex launched into the central defect site undergoes an unstable propagation. This is because, under the C_4 -symmetry and for the topological charge $|l| = 2$, we do not have the VRS condition because

$q(2,4) = 1 \in \mathbb{Z}$. In other words, the influence of all other waveguides with OVMC on the central defect waveguide does not cancel out, resulting in a breakup of the vortex pattern into a quadrupole-like shape. A theoretical explanation of this phenomenon as well as a numerical analysis via the discrete CMT calculations are provided in the sections below. Results show that the TVG necessarily requires “double” or two-fold protection, coming from both the VRS-mediated real-space topology and the k -space band topology.

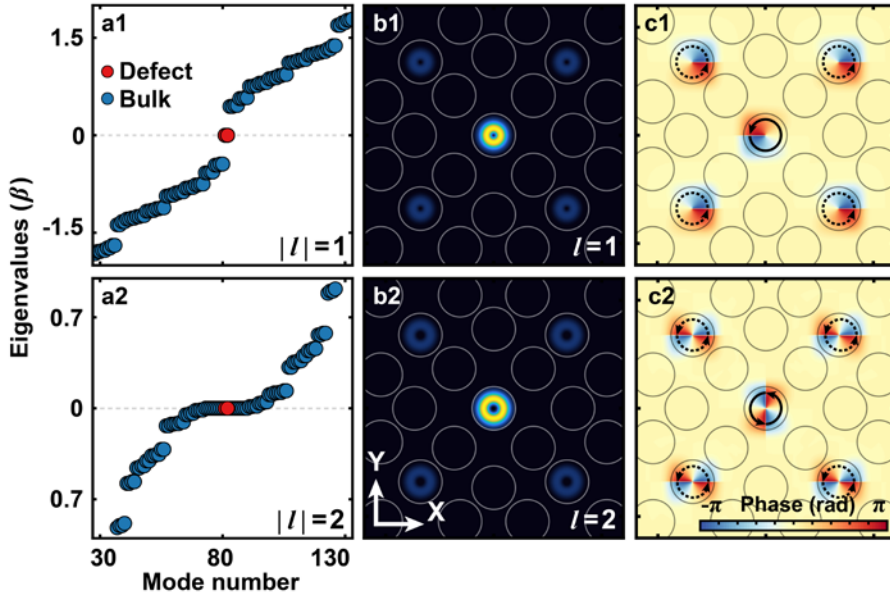


Fig. S3 Zero-energy vortex modes in a C_4 -disclination lattice. **a1** Calculated eigenvalues for the $|l| = 1$ vortex, where two degenerate in-gap defect modes (red dots, nearly overlapping) appear exactly at zero energy but with opposite vorticity. The intensity and phase distributions of the anti-clockwise vortex mode ($l = 1$) are plotted in **b1** and **c1**, respectively. The vortex mode is mostly confined at the central defect waveguide, featuring four weak-amplitude tails distributed over the next-nearest-neighbor sites in the same sublattice and a π -phase difference between the central vortex and the four satellite vortices - a characteristic of the topological vortex states. **a2-c2** Corresponding results obtained for the high-order vortex modes with $l = 2$.

5. Calculation of the generalized multipole chiral numbers

The main obstacle in determining the topological properties of a C_n -symmetric disclination structure is that it does not possess translational symmetry, i.e., it does not have a well-defined k -space. The higher-order topological insulator (HOTI) modes in disclination lattices cannot be simply predicted by estimating the fractional charge density¹²⁻¹⁵ as commonly done for periodic systems, because all unit cells do not have the same number of sites¹⁶. A rigorous study of the topological features manifested by the C_n lattice in the OAM domain would require finding an appropriate

quantized invariant, capable of correctly describing the topological phase of the zero-energy defect vortex modes under any lattice dimerization conditions. Recently, the concept of MCNs has been introduced¹⁷, where the HOTI phase of any structures respecting chiral symmetry can be predicted by a bulk integer topological invariant, exclusively defined in real space without the need for periodic boundary conditions. Here, we employ this concept to analyse the disclination structures featuring a C_n -symmetry. The generalized MCN is expressed as

$$N = \frac{1}{2\pi ni} \text{Tr} \log (\bar{Q}_A \bar{Q}_B^\dagger), \quad (\text{S2})$$

which is a quantized integer if the chiral symmetry is preserved. In Eq. (S2), $\bar{Q}_S^\dagger = U_S^\dagger Q_S U_S$, where S indicates A and B sublattices, Q_S is the sublattice multipole moment operator, and U_S is derived by the singular value decomposition from $h = U_A \Sigma U_B^\dagger$ (h is the off-diagonal element of the chiral Hamiltonian $H = \begin{pmatrix} 0 & h \\ h^\dagger & 0 \end{pmatrix}$). The sites in the disclination lattice are divided into n groups according to the C_n -symmetry and distinct coordinates are used for each group (see illustration in Extended Data Fig. 4a). The operator Q_S is defined as

$$Q_S = \sum_{j=1}^n \sum_{\mathbf{R}_c, \alpha \in S} |\mathbf{R}_c, \alpha\rangle \text{Exp} \left(-i \frac{2\pi x_j y_j}{x_j y_j} \right) \langle \mathbf{R}_c, \alpha|, \quad (\text{S3})$$

where $x_j = 1, 2, \dots, X_j$ and $y_j = 1, 2, \dots, Y_j$ are the coordinates with the origin located at the disclination core defined for each j^{th} section, while X_j, Y_j are the number of unit cells along the corresponding x_j - and y_j -directions. $|\mathbf{R}_c, \alpha\rangle = c_{\mathbf{R}_c, \alpha}^\dagger |0\rangle$, where $c_{\mathbf{R}_c, \alpha}^\dagger$ is the creation operator at the α orbital of the unit cell located at $\mathbf{R}_c$ in the S sublattice, and $|0\rangle$ is the vacuum state¹⁷. Numerical calculations are carried out for a large disclination structure ($X_j = Y_j = 40$) to reduce the influence of finite-size effects. Note that in Ref.¹⁷, the MCNs are only applied to systems where the number of sites belonging to A and B sublattices are equal. Nevertheless, Eq. (S3) can be appropriately used and quantized in our system, where the number of A and B sublattices sites is different. Geometrically, MCNs connect the phase winding of the wave function on the B sublattice with that on the A sublattice. From a physical viewpoint, MCNs describe the difference of multipole momenta originating from B and A sublattices. In Extended Data Figs. 4b1-c3, we compare numerically

calculated values of N as a function of the dimerization ratio d_2/d_1 for three different disclination lattices and for topological charges $l = 1$ and $l = 2$. The slight deviation of the N value from a quantized integer around $d_2/d_1 = 1$ is attributed to the finite-size effects.

6. Single vortex-mode selection: theoretical analysis

In this section, we theoretically study the requirement of single vortex-mode selection, which enables robust propagation of a vortex through the central defect waveguide in the presence of the VRS. We write an eigenvector $|\psi\rangle$ of the vortex Hamiltonian H as

$$|\psi\rangle = \begin{pmatrix} \varphi_{+d} \\ \varphi_{+b} \\ \varphi_{-d} \\ \varphi_{-b} \end{pmatrix}, \quad (\text{S4})$$

where the labels d and b indicate defect and bulk waveguides, respectively. Specifically, φ_{+d} and φ_{-d} are numbers corresponding to the complex amplitude of the mode at the central defect waveguide, for the $+l$ and the $-l$ component, respectively. φ_{+b} and φ_{-b} are vector columns formed by $N_{ls} - 1$ entries (N_{ls} is the total number of lattice sites), corresponding to the complex amplitude at bulk sites. Overall, the vector in Eq. (S4) has $2N_{ls}$ entries. Thus, we arrange the eigenvector in such a way as to distinguish the complex amplitudes corresponding to the central defect waveguide from those corresponding to the bulk. In particular, the initial condition is a single vortex that excites the central defect waveguide with the $+l$ component, reading as

$$|\psi_{+d}\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \sum_i |\psi_i\rangle \langle \psi_i | \psi_{+d} \rangle. \quad (\text{S5})$$

To realize a TVG for the $+l$ component, it is necessary to verify that the $-l$ component is not present at the central defect waveguide throughout propagation. This is valid only if all eigenvectors $|\psi_i\rangle$ for which $\langle \psi_i | \psi_{+d} \rangle \neq 0$ have the mathematical form

$$|\psi_+\rangle = \begin{pmatrix} 1 \\ \varphi_{+b} \\ 0 \\ \varphi_{-b} \end{pmatrix}. \quad (\text{S6})$$

In what follows, we prove that this requirement is fully equivalent to the VRS. The analysis starts by introducing the C_n -symmetry operator of H , expressed as

$$R_{nv} = \begin{pmatrix} \exp\left(-i\frac{2\pi l}{n}\right) & 0 \\ 0 & \exp\left(i\frac{2\pi l}{n}\right) \end{pmatrix} \otimes R_{ns}, \quad (\text{S7})$$

where R_{ns} is the C_n symmetry operator of the associated s -state Hamiltonian given by

$$R_{ns} = 1 \oplus \left[\begin{pmatrix} 0 & I_{n-1} \\ 1 & 0 \end{pmatrix} \otimes I_{(N_{ls}-1)/n} \right]. \quad (\text{S8})$$

Here, R_{ns} is an $N_{ls} \times N_{ls}$ matrix, the symbol $\otimes$ denotes the Kronecker product, $\oplus$ is the direct sum, and I_α represents the α -dimensional identity matrix; number 1 in Eq. (S8) corresponds to the central defect waveguide. The C_n -symmetry operator R_{nv} is a $2N_{ls} \times 2N_{ls}$ matrix, where the index v denotes that it is the operator corresponding to the vortex Hamiltonian H . H obeys the C_n symmetry, i.e., $[H, R_{nv}] = 0$, which implies that H and R_{nv} share the same eigenstates. The eigenvalues (denoted by λ) and eigenvectors (denoted by u) of R_{nv} are found to be

$$\begin{aligned} \lambda_{1p,\xi} &= \exp\left(\frac{2i\pi(p-l)}{n}\right), \quad u_{1p,\xi} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes s_{p,\xi} \\ \lambda_{2p,\xi} &= \exp\left(\frac{2i\pi(p+l)}{n}\right), \quad u_{2p,\xi} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes s_{p,\xi} \\ \lambda_{1d} &= \exp\left(-\frac{2i\pi l}{n}\right), \quad u_{1d} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \mathbb{I}_d \\ \lambda_{2d} &= \exp\left(\frac{2i\pi l}{n}\right), \quad u_{2d} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \mathbb{I}_d \end{aligned}, \quad (\text{S9})$$

where $s_{p,\xi}$ and $\mathbb{I}_d$ are eigenvectors of R_{ns} ($p = 1, 2, \dots, n$), expressed as

$$s_{p,\xi} = \begin{pmatrix} 0 \\ P_1 e_\xi \\ P_2 e_\xi \\ \vdots \\ P_n e_\xi \end{pmatrix}, \quad \mathbb{I}_d = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (\text{S10})$$

Here, $P_j = \exp\left(\frac{2i\pi p(j-1)}{n}\right)$, $j = 1, 2, \dots, n$, e_ξ is the ξ -th eigenvector of $I_{(N_{ls}-1)/n}$. If an eigenstate of R_{nv} is non-degenerate, then it is also an eigenstate of H . When the eigenstates of R_{nv} are degenerate,

the H eigenstates can be expressed as a linear combination of degenerate eigenstates of R_{nv} . From the third and the fourth line of Eq. (S9), we find that under the preserving VRS condition ($q(l, n) = 2l/n \notin \mathbb{Z}$), we have $\lambda_{1d} \neq \lambda_{2d}$, meaning that any linear combination of u_{1d} and u_{2d} is not an eigenstate of H . In the presence of VRS, all eigenstates of H only have either the $+l$, or $-l$ component at the central defect waveguide, or alternatively both of these components are zero at the central defect waveguide. Instead, in the absence of the VRS ($q(l, n) = 2l/n \in \mathbb{Z}$), we have $\lambda_{1d} = \lambda_{2d}$, and both u_{1d} and u_{2d} , thus the $+l$ and $-l$ components can readily combine at the defect site, leading to unstable transport of a single vortex disclination mode.

7. VRS: nontrivial winding and real-space topology

The OVMC terms t_{0V} introduced in Note 4 are in general complex numbers, being directly proportional to $e^{2il\theta}$. Let us consider the OVMC along some path from the disclination core to some lattice site in the j^{th} rotational sector of the C_n -disclination lattice, and along a fully equivalent path rotated by $2\pi/n$ from the disclination core to the equivalent lattice site in the $(j+1)^{\text{th}}$ rotational sector. The overall phase difference acquired along these two paths is $e^{2il2\pi/n}$. Real-space winding can be understood by examining the complex coupling coefficients $T_j \propto e^{2il\theta_j}$ (with $\theta_j = 2(j-1)\pi/n$, and $j \in N^+$), which account for these phase differences: that is, which account for vortex mode coupling from each j^{th} rotational sector of the C_n -disclination lattice to the disclination core. For the SVMCs, we have $t_{SV} = \kappa_{SV}$ which are isotropic along different orientations. They do not influence the vortex propagation in the core because the SVMC terms only introduce modes of the same vorticity, so they are not considered in the following analysis. Because of the C_n symmetry, T_j constructs discrete sets of vectors that have a periodic boundary condition. Real-space topology phases can be unveiled by defining the coupling winding number as

$$w = \frac{1}{2\pi i} \sum_{j=1}^n \ln \left(\frac{T_{j+1}}{T_j} \right). \quad (\text{S11})$$

One can straightforwardly verify that w is nonzero only when $q(l, n) = 2l/n \notin \mathbb{Z}$, that is, when our system possesses the VRS. A nonzero w indicates the existence of a topologically nontrivial phase. This in addition implies that the total complex coupling $\sum_j^n T_j = 0$ (note that for the trivial regime, q

is an integer, w remains zero, and $\sum_j^n T_j \neq 0$). We note that the total OVMCs on the complex plane between the central defect waveguide and all contributions originating from the n sectors cancel if and only if the topological invariant w is nonzero. Therefore, the winding of the complex coupling T_j is strictly related to single-mode selection and can be used to characterize real-space topology. Some examples of calculated w values are found in Fig. 4 of the main text, which confirms the theoretical prediction for single-mode selection in those disclination structures. We point out that the real-space winding introduced here can be beneficial for investigating topological properties in other systems. In particular, w is analogous to the topological index of magnetic skyrmion structures^{18,19}.

Any nontrivial C_n -symmetric lattice possessing real-space topology (i.e., $q(l, n) = 2l/n \notin \mathbb{Z}$) can support a single vortex mode at the core that maintains the same vorticity throughout propagation without being interfered by the opposite-vorticity components. Since this requirement is universal for any vortex charge l and for any order of rotational symmetry C_n , we name this symmetry as “vorticity-coordinated rotational symmetry” (VRS), for achieving protected vortex transport in C_n topological disclinations.

8. Stability of high-order vortex modes mediated by VRS in C_n -symmetric disclinations

In principle, any chiral-symmetric disclination lattice of C_n -symmetry with nontrivial band topology can support, for every l , a pair of degenerate zero-energy defect modes with opposite phase circulations. As illustrative examples, three typical nontrivial vortex band structures associated with three different C_n -symmetric disclinations ($n = 3, 4, 5$) for different high-order vortices are plotted in Fig. S4a and Extended Data Fig. 5b. Among them, only those lattices satisfying the VRS requirement are suitable to achieve a robust TVG where OAM is conserved. The plot in Fig. 4a3 summarizes the stability relation between values of n and l , highlighting the main result of our theory regarding real-space topology. It precisely illustrates the universal rule for single-mode selection originating from the VRS, i.e., $q = 2l/n \notin \mathbb{Z}$. Here, to corroborate our theoretical predictions, we perform discrete CMT calculations of Eq. (S1) for different C_n -symmetric lattices and different topological charges l . Initial conditions for all cases are the single vortex excitation at the defect site with high-order vortices ($l > 2$). Discrete evolutions to different dimensionless distances Z are found by means of a standard Runge-Kutta numerical method.

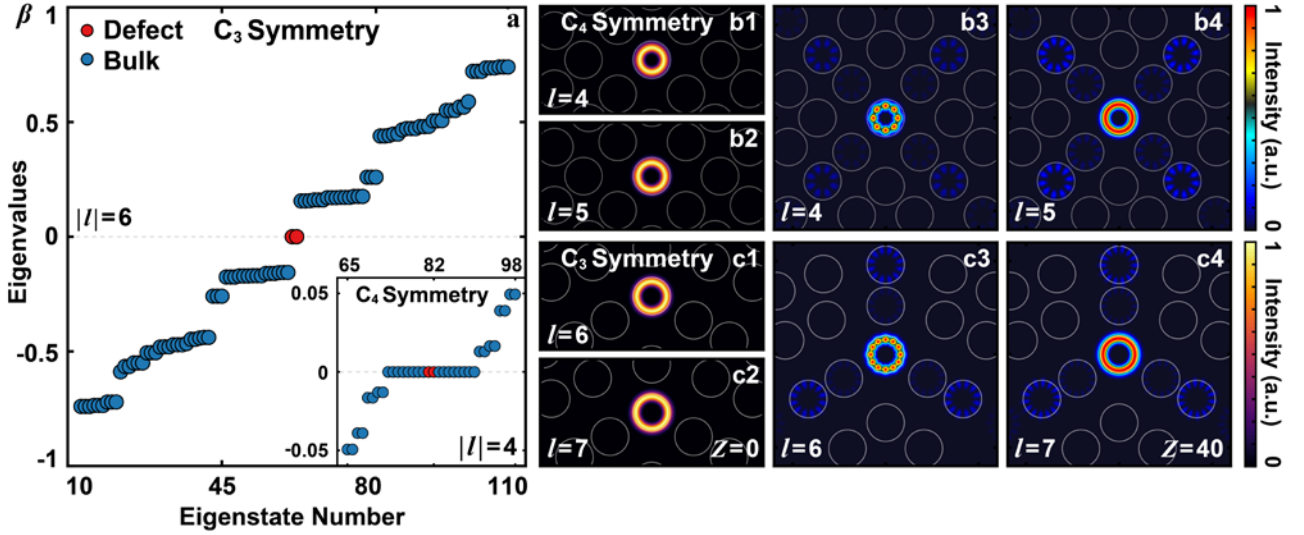


Fig. S4 Zero-energy high-order vortex modes protected by VRS in C_3 - and C_4 -symmetric disclinations.
a Calculated eigenvalues for the C_3 -lattice and $|l| = 6$ vortex, where two degenerate in-gap defect modes appear right at zero energy and carry opposite phase vorticity. Red and blue dots mark the disclination defect and bulk states, respectively. Inset in **(a)** plots an enlarged view of the calculated vortex band structure for the C_4 -lattice and $|l| = 4$ vortex. **b1-b4** Intensity distributions of defect vortex excitations with topological charge $l = 4$ (**b1, b3**) and $l = 5$ (**b2, b4**), obtained from CMT calculations and selected at the onset distance $Z = 0$ (**b1, b2**) and $Z = 40$ (**b3, b4**) for the C_4 -lattice. **c1-c4** Corresponding numerical results obtained for the C_3 -lattice and for the $l = 6$ (**c1, c3**) and $l = 7$ (**c2, c4**) vortices.

In the first set of simulations, a single vortex is launched into the disclination core of the nontrivial C_3 -lattice (Fig. 3) and C_4 -lattice (Fig. S3). In the C_4 -lattice, a high-order vortex with $l = 4$ ($q(4,4) \in \mathbb{Z}$) displays a multipole intensity pattern with 8 lobes at $Z = 40$, indicative of unstable propagation, while another vortex with $l = 5$ ($q(5,4) \notin \mathbb{Z}$) preserves the vortex-ring pattern (see Fig. S4b1-b4). At the same distance in the C_3 -lattice, an $l = 6$ vortex ($q(6,3) \in \mathbb{Z}$) breaks into a 12-lobe multipole, but an $l = 7$ vortex ($q(7,3) \notin \mathbb{Z}$) sustains its shape [Fig. S4c1-c4]. For the C_5 -lattice (see Extended Data Fig. 5), the high-order vortices with $l = 5$ or 10 cannot maintain their shape during propagation (because of $q \notin \mathbb{Z}$), but other vortices with $l = 8$ or 12 (satisfying $q \notin \mathbb{Z}$) can be well guided in the disclination core with a preserved pattern.

These discrete simulations are in perfect agreement with the prediction from our theoretical analysis, following the VRS required for single-mode selection. All numerically investigated vortex excitations are protected by momentum-space topology in the chiral-symmetric disclination, but only

some of them are “doubly protected”. For example, the $l = 8$ vortex in the C_5 -lattice is doubly protected throughout propagation because it also satisfies real-space topology requirements. On the contrary, despite being localized to the core waveguide, the $l = 5$ vortex injected into the same disclination structure cannot maintain its intensity pattern throughout propagation due to a lack of real-space topological protection. The same arguments are valid for other choices of l and n .

9. Stability analysis under perturbations affecting chiral symmetry

A distinctive signature of symmetry-protected topological phases is their immunity to perturbations that respect certain (protecting) symmetries. Let us consider Fig. 3 of the main text, where the chiral symmetric lattice supports a pair of vortex modes with opposite vorticities pinned at zero energy. The chiral symmetry is expressed by the equation $\Gamma H \Gamma^{-1} = -H$, where the chiral operator Γ takes the following form: $\Gamma = P_A - P_B$, where P_A and P_B are the projection operators on the A and B sublattices⁹.

To unveil the higher-order nature of disclination vortex modes, we investigate their resilience against random perturbations that affect the original hopping values in the Hamiltonian H , which either maintain or break the chiral symmetry. Perturbations respecting the *sub-symmetry* (SubSy), a less restrictive condition than the chiral symmetry⁹, are also numerically tested. Without loss of generality, we only report the case of a C_3 -symmetric disclination structure for $l = 1$, although the analysis is also performed for different C_n lattices and different topological charges l . Typical numerical results are plotted in Fig. S5. Our calculations are performed (using the discrete model) by inducing random perturbations to the nearest-neighbor and next-nearest-neighbor hopping of H , as well as additional on-site potentials (refractive index changes) and long-range couplings among vortex modes. These perturbations can be selectively used to break a particular symmetry (chiral symmetry, A -SubSy, or B -SubSy)⁹. Vortex-mode robustness is explored by examining the evolution of the perturbed energy spectrum when increasing the magnitude of perturbation strengths δ up to 0.1- a threshold value after which a gap closing occurs. Figure S5a1 plots the overlap of twenty perturbed band structures with various random perturbations at different strengths δ (for a set of 100 perturbations), obtained for a chiral-symmetry-preserving lattice. In this stability test, random perturbations are only applied between A - B sublattice sites. One can clearly see that the two

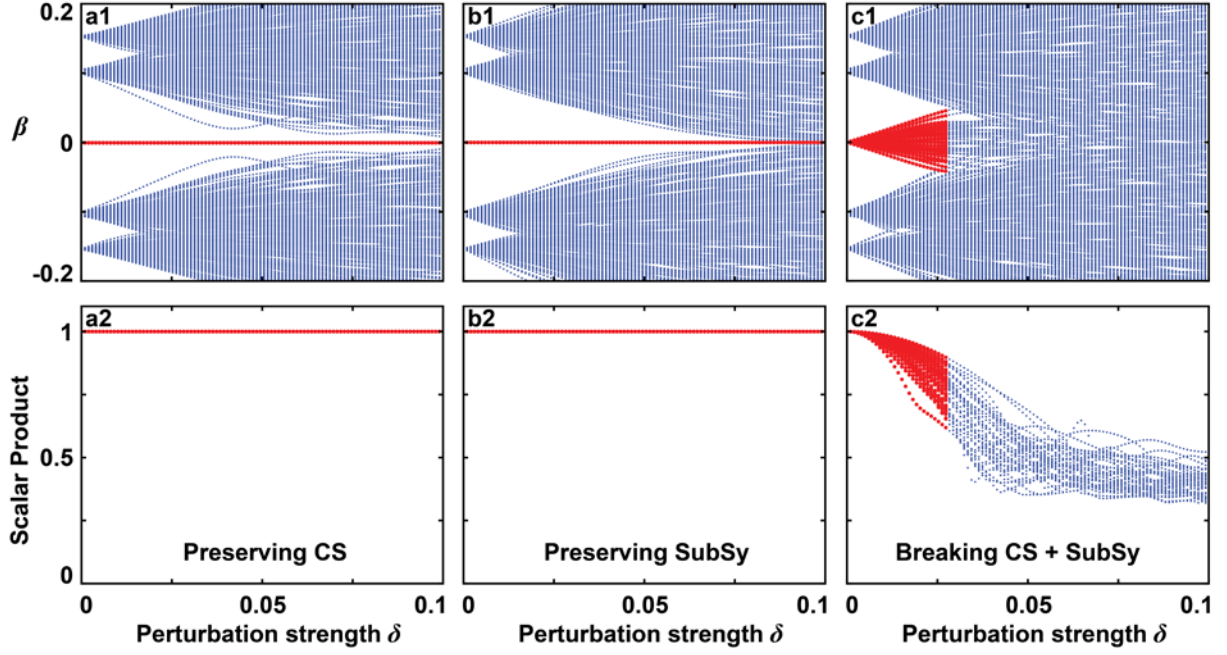


Fig. S5 Stability analysis of vortex modes under perturbations affecting chiral symmetry (CS) and sub-symmetry (SubSy). **a1** Vortex band-structure for perturbations preserving the CS in a C_3 -symmetric disclination lattice obtained by overlapping twenty spectra with different perturbation strengths δ . **a2** Calculated scalar product $|\langle\psi|P_A\psi\rangle|^2$ for the perturbed C_3 lattice, which evaluates the percentage of a vortex mode residing solely on the A -sublattice under perturbation. **b1-b2, c1-c2** Same as **a1-a2** but for perturbations preserving only A -SubSy (**b1-b2**) and for perturbations breaking the CS and both A - and B -SubSys (**c1, c2**). Red and blue dotted curves in each panel depict the disclination core and bulk vortex modes, respectively.

eigenvalues associated with the defect vortex modes remain stable at zero energy for increasing values of δ . In contrast, the bulk modes undergo significant spreading because they are affected by the applied perturbations. Since topological vortex modes are only distributed on the A -sublattice, any perturbation that respects the chiral symmetry does not destroy the vortex states regardless of its strength. By applying random perturbations between B - B sublattice sites, both the chiral symmetry and B -SubSy are broken. Numerical simulations show that the disclination vortex modes are still preserved under these perturbation conditions because A -SubSy is preserved, as the central defect site belongs to the A sublattice [Fig. S5b1]. On the other hand, if the same calculations are repeated for random perturbations that affect both the chiral symmetry and the A -SubSy, we see that for an increasing amplitude δ , the vortex-mode eigenvalues experience an evident distortion and mix with the bulk bands, thus losing topological protection (Fig. S5c1).

To better describe the robustness of vortex transport in a perturbed disclination structure, we

calculate the scalar product parameter $|\langle\psi|P_A\psi\rangle|^2$, which basically projects an excited vortex mode onto the A -sublattice. The P_A operator returns a vortex mode amplitude $|P_A\psi\rangle$ that is identical to $|\psi\rangle$ in the A sublattice, and is null in the B sublattice. As shown in Fig. S5a2, b2, the values of the scalar product for the chiral symmetry and A -SubSy preserving lattices are always 1, indicating that the whole energy content of the vortex modes resides only on the A -sublattice sites. In contrast, for the chiral-symmetry-breaking case plotted in Fig. S5c2, the calculated scalar product deviates significantly from 1 when the perturbation strength is increased, which suggests that the vortex modes in this case are destroyed and therefore also spread to the B sublattice.

10. Stability analysis under perturbations affecting real-space topology

The perturbation analysis above shows that protection of zero-energy disclination vortex modes demands chiral symmetry or SubSy. Despite being necessary, this condition is not sufficient for stable TVG at a central defect waveguide of our C_n -disclination lattice structure since VRS is also required to be satisfied, which is a manifestation of real-space topology. The VRS implies C_n -symmetry is present in the system. To better understand what would happen to the vortex propagation if the perturbations broke the C_n -symmetry, let us numerically study here the vortex propagation robustness using *a discrete model* under random perturbations that affect the rotational symmetry of the C_3 -disclination lattice, with the chiral symmetry preserved.

To do such an analysis, we add random perturbations with magnitude $\delta = 0.05$ to the A - B hopping of the vortex Hamiltonian H , corresponding to the three sectors of the C_3 -lattice. The C_3 -rotational symmetry is preserved when the perturbations are distributed equally in every sector; otherwise, we have broken the C_3 -rotational symmetry. The results are presented in Fig. S6, where the overlap of twenty random perturbations for the same δ is displayed. The Hamiltonian H corresponds to the nontrivial C_3 -disclination lattice and for the topological charge $l = 1$. Simulations of the vortex-beam evolution are carried out by CMT calculations with propagation length $Z = 80$. Here, we introduce a normalized parameter γ as a figure of merit to distinguish stable and unstable vortex dynamics:

$$\gamma = \frac{|\psi_{d,+l}| - |\psi_{d,-l}|}{|\psi_{d,+l}| + |\psi_{d,-l}|}. \quad (\text{S12})$$

This parameter quantifies the difference between the amplitudes $|\psi_{d,+l}|$ and $|\psi_{d,-l}|$ with opposite vorticities ($+l$ and $-l$) at the central defect waveguide. When $|\gamma| = 1$, only one vorticity component is present in the vortex beam at the central defect waveguide (in this specific example $l = 1$). On the other hand, when both clockwise and anti-clockwise vortex components are present, we have $|\gamma| < 1$. As shown in Fig. S6a, in the C_3 -symmetry preserving lattice, $|\gamma|$ acquires unitary values throughout propagation, implying the presence of a single vorticity at the central defect waveguide, therefore supporting stable vortex transport. If the C_3 symmetry is broken, $|\gamma|$ decays significantly from 1 (especially after a long propagation distance), indicating that both (opposite) vorticities are present in the disclination core, leading to unstable vortex transport (see Fig. S6b). A similar analysis for other values of n and l satisfying VRS condition shows that, under any C_n -symmetry-preserving perturbations, the disclination core can sustain a single vortex mode during propagation. Such robust vortex transport originates from real-space topology. In brief, these results, presented in Fig. S6, further corroborate that real-space topology represents another condition for robust transport of a single vortex.

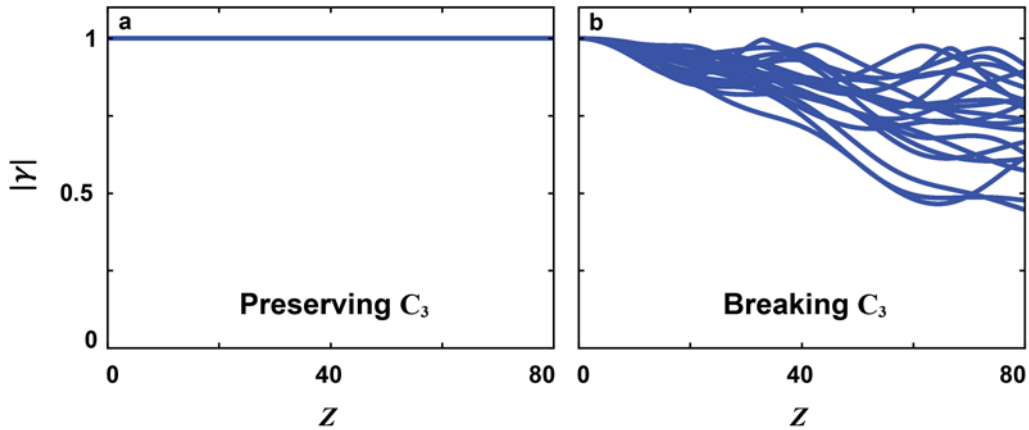


Fig. S6 Stability analysis of vortex transport under perturbations affecting C_3 -symmetry. a, b Coefficient $|\gamma|$ as a function of the dimensionless distance Z , obtained by overlapping twenty different random perturbations of strength $\delta = 0.05$. Calculations are performed for a C_3 -symmetric disclination, which preserves (a) or breaks (b) the lattice rotational symmetry.

Finally, for the sake of completeness, the TVG in the presence of randomly applied perturbations that preserve and break VRS in the C_3 disclination is also investigated using the *continuous model* of Eq. (M1). Numerical calculations are presented in Fig. S7. Specifically, the robustness analysis is carried out by introducing a random background noise on the C_3 -disclination structure that makes it

inhomogeneous over $z = 50$ cm. In one case, the background noise is fully random and breaks the VRS (Fig. S7b1-b4). In another case, the background noise is random but it preserves the VRS (Fig. S7a1-a4). Results for a single $l = 1$ vortex-mode excitation show that the probe beam remains stable when the random background noise maintains the VRS in a disclination lattice, while it loses stability as the VRS is broken. In this regard, one may raise some concerns because realistic perturbations should not obey the VRS condition. For a completely perturbed structure without satisfying any symmetry, it of course has no topological protection for stable vortex transport. Nevertheless, in practical applications, the VRS is easier to achieve than a global circular symmetry. Therefore, the VRS represents a unique and effective way to realize stable vortex transport due to the underlying topological protection.

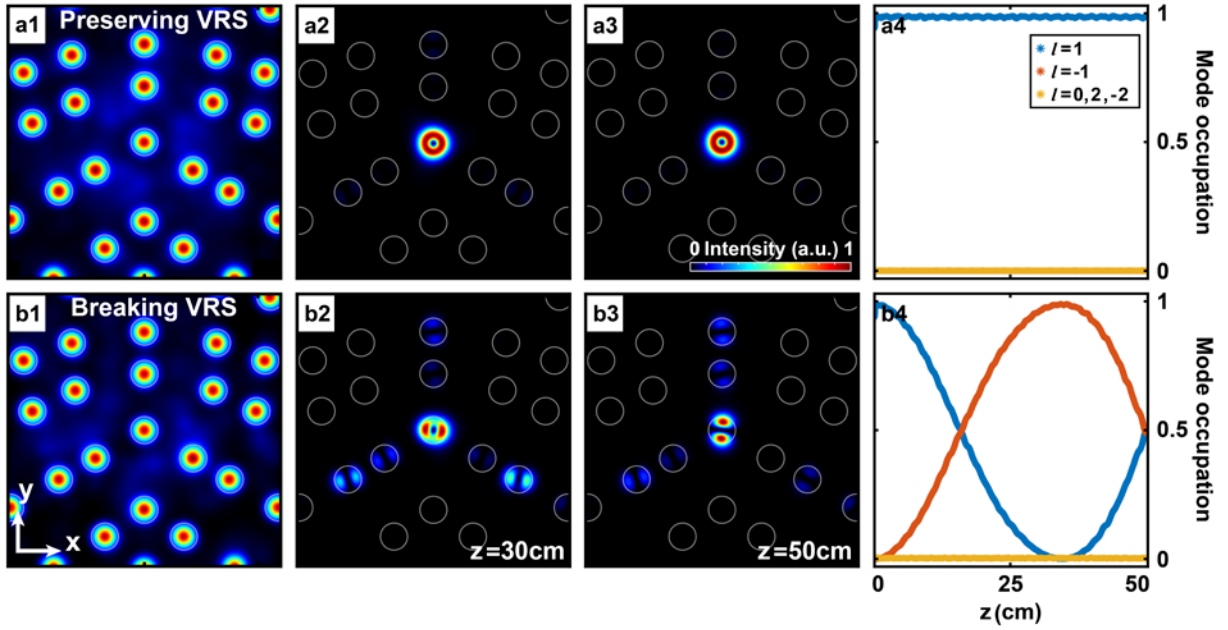


Fig. S7 Numerical simulations of vortex transport under perturbations preserving and breaking the VRS. **a1** Perturbed C_3 disclination lattice under a random background noise preserving the VRS. **a2, a3** Corresponding output profile at **(a2)** $z = 30\text{cm}$ and **(a3)** $z = 50\text{cm}$, respectively. **a4** Mode occupation calculations as a function of the propagation distance z . **b1-b4** The same as in **a1-a4** for a random background noise that breaks the VRS condition in the system. The $l = 1$ vortex remains a nearly perfect mode as the VRS is unaffected. However, if the background noise does not obey the VRS condition, the doughnut vortex amplitude is completely destroyed (leaking to the $l = -1$ mode), resulting in the dipole-shaped output pattern (**b2, b3**).

We also evaluate numerically the performance of our C_n -disclination lattice when some ellipticity is intentionally introduced into the core waveguide and compare to a single waveguide

with the same elliptical deformation. As expected, since the elliptical waveguide breaks the overall VRS of the disclination structure, robust vortex guidance cannot be ensured due to the loss of real-space topological protection. In both cases, when the eccentricity is increased to over 0.5%, we find that a vortex breaks up into a dipole mode during propagation, although this occurs a bit slower in the C_3 disclination as compared to the single waveguide. Our results suggest that the fundamental principle demonstrated in this work may find applications for robust OAM-based devices.

11. High-order vortex transport in free space vs. disclination TVG: a direct comparison

One may wonder how an unguided high-order vortex looks like upon propagation as compared to a well-guided vortex. In this last section, we show a direct comparison of a guided high-order vortex in the TVG as opposed to its respective dynamics in free space under random perturbation. A high-order vortex typically undergoes decomposition into multiple low-order vortices during propagation without waveguide, or when the lattice structure does not feature topological protection (see an example for $l = 3$ in Extended Data Fig. 3c1-c4). Experimentally, it is very difficult to induce a waveguide to support an even higher-order vortex in our platform, mainly because the index potential of each waveguide formed by laser writing is not deep and wide enough to support such a high-order vortex mode. As such, here, we numerically investigate the case of high-order vortex transport under perturbation (using the example of $l = 5$). Random noise is intentionally introduced as background perturbation, implemented in both free space and the C_3 -disclination lattice.

The magnitude of introduced background noise is quite small, only 1% when compared with that of the disclination lattice, but it is however sufficient to break up a high-order vortex beam. If an $l = 5$ vortex is excited at the core of the C_3 -disclination lattice, it can be integrally guided (Extended Data Fig. 6b1-b2) and the phase vorticity remains intact (Extended Data Fig. 6b3) under such perturbation. In contrast, when the same vortex beam is launched into other off-center lattice sites, or when it propagates in free space under the same perturbation, not only it expands dramatically and undergoes evident distortion (see Extended Data Fig. 6c1-c2), but its phase vorticity also splits into 5 low-order ($l = 1$) vortices as shown in Extended Data Fig. 6c3. For these simulation results, the lattice period is $105\mu\text{m}$, the dimerization parameter is $d_2/d_1 = 0.75$, the waveguide width is $15\mu\text{m}$, and the relative refractive index change is $\Delta n/n = 1.38 \times 10^{-4}$. These numerical results along with

the experimental results for $l = 3$ (Extended Data Fig. 3) clearly demonstrate the benefit of using C_n -topological disclination for guiding and protecting high-order vortices.

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Supplementary Media Files

(Suppl. Media File 1). Animation of vortex transport through a single waveguide and a TVG under mixed-mode excitation from numerical simulation.

(Suppl. Media File 2). Animation of vortex transport through a TVG under perturbations preserving and breaking the rotational symmetry from numerical simulation.

(Suppl. Media File 3). Animation of a high-order vortex ($l = 5$) transport showing guidance and preserved vorticity in a TVG, as opposed to diffraction and vortex splitting in free space, under same perturbation from numerical simulation.

Extended Data Figures

(Extended Data Fig. 1). Realization of C_3 -symmetric disclination lattices with a single-site core at the center.

(Extended Data Fig. 2). Numerical simulations of the topological vortex guide by chiral-symmetric disclinations.

(Extended Data Fig. 3). Experimental results comparing topologically protected and unprotected transport of a high-order ($l = 3$) vortex.

(Extended Data Fig. 4). Generalized multipole chiral numbers used for describing the topological phase of C_n -symmetric disclination lattices.

(Extended Data Fig. 5). Protected and unprotected high-order vortex modes in a C_5 -symmetric disclination lattice.

(Extended Data Fig. 6). Comparison of high-order vortex transport in free space vs. disclination TVG