
Phase coherence and disorder-induced wave propagation in micromotor arrays

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I. DESCRIPTION OF THE SUPPLEMENTARY VIDEOS

Following printing, we perform the post-exposure bake of the SU-8 according to the protocol detailed in [13]. We then place the printed structures vertically into a beaker filled with SU-8 developer. The beaker is positioned on an orbital shaker (PSU-10i) rotating at 200 rpm for 20 minutes. Finally, we rinse the developed structures with fresh SU-8 developer and clean the device by agitating it in isopropyl alcohol for 2 minutes.

2. Microfluidic device

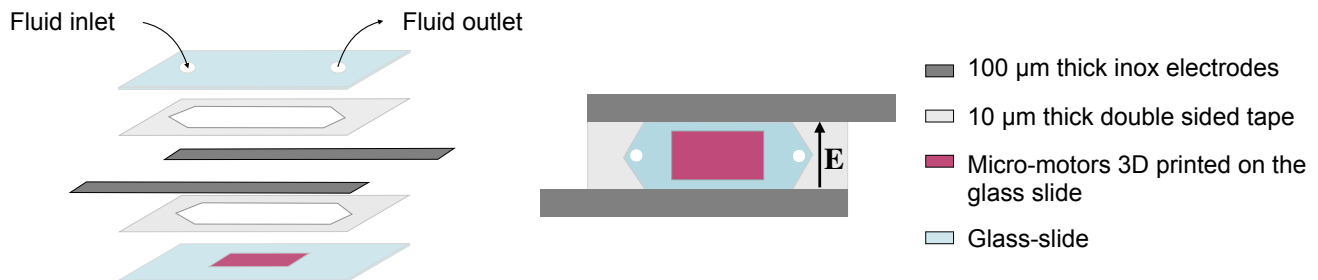


FIG. S2. **Microfluidic chip.** Our microfluidic channels are made of double sided tapes. The two electrodes are made of stainless steel.

We place the micromotors in 1 cm wide microfluidic channels (Fig. S2). We bond two electrodes made of 100 μm thick stainless steel plates to two glass slides using double-sided tapes with an approximate thickness of 10 μm . The total channel height is therefore approximately 120 μm . We seal the edges of the resulting microfluidic device with epoxy resin to ensure a leak-proof assembly. For all the experiments described in this article, the distance between the electrodes was between 4 mm and 6 mm.

Once the microfluidic device is assembled, we inject a 0.025 mol $\cdot$ L $^{-1}$ Dioctyl sulfosuccinate sodium salt (AOT)/hexadecane solution inside the microfluidic chamber. To release the mobile parts of the micromotors after SU-8 development, we rely on ultrasonic shaking. We place the sealed microfluidic device in a glass box (Wheaton staining dishes) filled with a water-glycerol mixture containing 40% glycerol by volume. We sonicate the microfluidic cell using an Electris UC317MD sonication bath in 2-second intervals. After each sonication cycle, we check whether the rotors are free to move. The process is terminated once all mobile parts are functional, or when a rotor is damaged.

3. Quincke motorization setup and principle

3.1 . Electric setup

To actuate our micromotors, we take advantage of the Quincke electroration instability, which we briefly recall in the next section [3, 14, 18]. Using a voltage amplifier (TREK 610E), we apply an electric field $\mathbf{E}$ transverse to the electrodes and parallel to one of the axes of the micromotor network, Fig. S2. Above a critical field amplitude, the rotors are mechanically unstable, leading to the steady rotation of the rotors [14]. At the end of each experiment, we turn off the electric field and renew the solvent in the microchannel to reset the experiment and avoid unwanted memory effects [25].

3.2 . Quincke electroration in a nutshell

We provide a short qualitative description of the Quincke rotation mechanism, which we use to motorize our micromachines. A comprehensive description of this electrohydrodynamic instability can be found e.g. in [3, 14].

Consider an axisymmetric rigid body immersed in a conducting fluid, and subjected to a uniform DC electric field $\mathbf{E}$. After a transient period, the ohmic transport of the ions under the action of the electric field results in the formation of a dipolar charge distribution at the surface of the solid particle. The resulting dipole moment $\mathbf{p}$ is oriented in the direction opposite to the $\mathbf{E}$ field (Fig. S3a).

The stability of this dynamical steady state is determined by the competition between two opposing effects. On one hand, $\mathbf{p}$ tends to align with $\mathbf{E}$, exerting an electric torque that amplifies any angular fluctuation of the particle

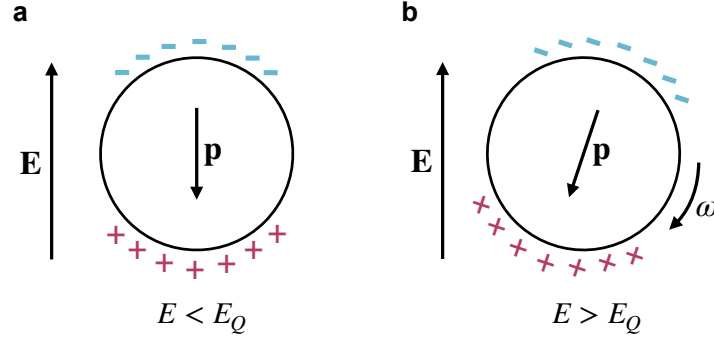


FIG. S3. **Quincke electroration: principle.** **a.** Charge distribution around a dielectric sphere immersed in a conducting fluid. Below E_Q , when the difference in electric permittivity between the fluid and the solid media is not too large, the Ohmic transport of the free ions in the solution induces the formation of a net electric dipole pointing in the direction opposite to $\mathbf{E}$. **b.** Above E_Q , the system reaches a different dynamic steady state where the particle spins at a constant rate ω . This steady rotation is sustained by the competition between the electric and the viscous torques acting on the particle.

orientation, and initiates rotation. On the other hand, the Ohmic current in the electrolyte continuously redistributes the surface charges, stabilizing the dipole orientation in the direction opposite to $\mathbf{E}$. This competition leads to a supercritical instability: below the Quincke threshold $\mathbf{E}_Q$, the dipole orientation is stable and remains anti-aligned with $\mathbf{E}$. Above $\mathbf{E}_Q$, the configuration where $\mathbf{E}$ and $\mathbf{p}$ are collinear is unstable. The amplification of any minute orientational fluctuation leads the system towards another dynamical steady state where $\mathbf{p}$ makes a finite angle with $\mathbf{E}$. This charge distribution results in a constant electric torque balanced by the viscous torque caused by the continuous rotation of the rigid body (Fig. S3b). The rotation speed ω is determined by the balance between the two torques:

$$\omega = \tau_M^{-1} (1 - E/E_Q)^{1/2}, \quad (\text{S1})$$

where τ_M is the so-called Maxwell-Wagner charge relaxation time. It is a material property and depends solely on the conductivity and electric permittivity of the materials. It is of the order of 10 ms in our experiments.

A key aspect of the Quincke instability is that the charge distribution spontaneously breaks the rotation symmetry. While the magnitude of the angular speed can be controlled by adjusting the magnitude of E , the direction of rotation remains only determined by the initial fluctuations: the axis of rotation of the particle can be in any direction in the plane perpendicular to $\mathbf{E}$.

4. Measurements of the instantaneous phase and angular velocity

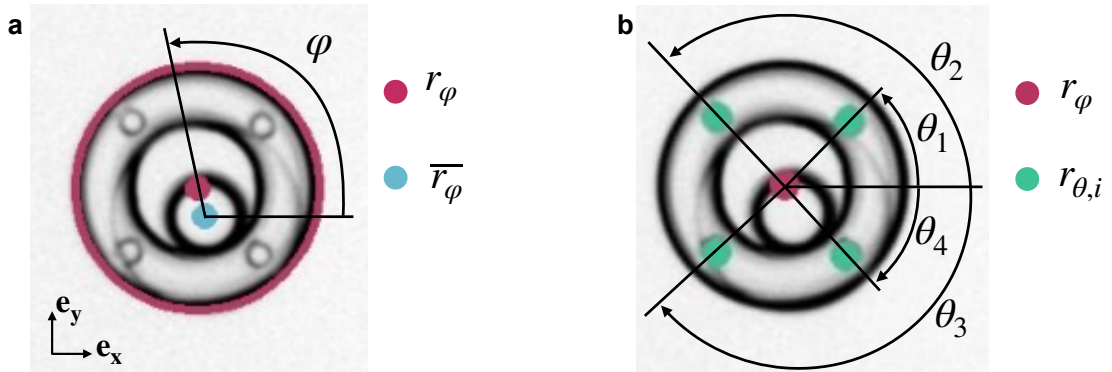


FIG. S4. **Tracking of the rotor phase and speed.** **a.** To measure the instantaneous phase $\varphi(t)$ we need to measure the instantaneous center of mass (pink dot) of the rotor and its time average (blue dot). **b.** We measure the instantaneous rotation angle θ by detecting the positions of the four marker holes on the rotor (green dots).

We observe the micro-motors using a Nikon AZ100 macroscope with a 2x objective (Nikon AZ Plan Fluor). A Ximea xiB-64 camera mounted on the macroscope via a DSC port records the images, introducing an additional 0.6x magnification factor. We adjust the magnification of the Nikon AZ100 macroscope according to experimental requirements, achieving a total magnification ranging from 3.6x to 8.4x.

Figure S4a and b illustrate how we measure the instantaneous phase and rotation speed of the motors. We use the HoughCircles function from the OpenCV cv2 module to detect the rotors, and their four circular holes. We denote the rotor's position as $\mathbf{r}_\varphi$, and the positions of the four holes as $\mathbf{r}_{\theta,i}$, where i ranges from 1 to 4.

To measure the phase φ , we first compute the center of the rotor's trajectory: $\overline{\mathbf{r}_\varphi} = \langle \mathbf{r}_\varphi \rangle_t$. We then define the phase as: $\varphi = \arctan((\mathbf{r}_\varphi - \overline{\mathbf{r}_\varphi}) \cdot \mathbf{e}_y, (\mathbf{r}_\varphi - \overline{\mathbf{r}_\varphi}) \cdot \mathbf{e}_x)$, where $\mathbf{e}_x$ and $\mathbf{e}_y$ are the unit vectors along and perpendicular to $\mathbf{E}$, respectively. To handle discontinuities in the trajectory caused by the periodic nature of the arctan2 function (confined to the interval $[-\pi, \pi]$), we add $\pm 2\pi$ to the phase when we detect phase jumps with a magnitude larger than π . We then compute the instantaneous rotation rate $\dot{\varphi}$ using a finite difference method.

To measure the angle θ , we calculate $\theta_i = \arctan2((\mathbf{r}_{\theta,i} - \mathbf{r}_\varphi) \cdot \mathbf{e}_y, (\mathbf{r}_{\theta,i} - \mathbf{r}_\varphi) \cdot \mathbf{e}_x)$, the angle of the i -th detected marker on the wheel. At each time, the angle θ is then defined as the mean of all θ_i modulo $\pi/2$, calculated for one rotor. As with φ , we correct the discontinuities in θ caused by its confinement to the interval $[0, \pi/2]$. If there is a jump in the trajectory greater than $0.42\frac{\pi}{2}$, we add $\pi/2$, and if the jump is smaller than $-0.42\frac{\pi}{2}$, we subtract $\pi/2$.

5. Experimental Protocols

Antiferromagnetic order To characterize the emergence of antiferromagnetic order as the lattice spacing ℓ decreases, we conducted experiments both on lattices made of 20×11 small motors, and on 13×12 large motors. For each geometry, and each value of ℓ , we repeated ten independent experiments. The order parameter Ω plotted in Fig. 2a in the main text is averaged over these ten measurements.

Dynamics of motor pairs To investigate the dynamics of the motor pairs presented in Fig. 2d of the main article, we printed series of four large motors as illustrated in Fig. S5. The leftmost and rightmost structures cannot rotate, as we added rigid links to immobilize the rotors. The idea is to measure the dynamics of isolated pairs of motors feeling an electric field close to that of the motors in the bulk of the square lattices. All measurements were performed on large motors.

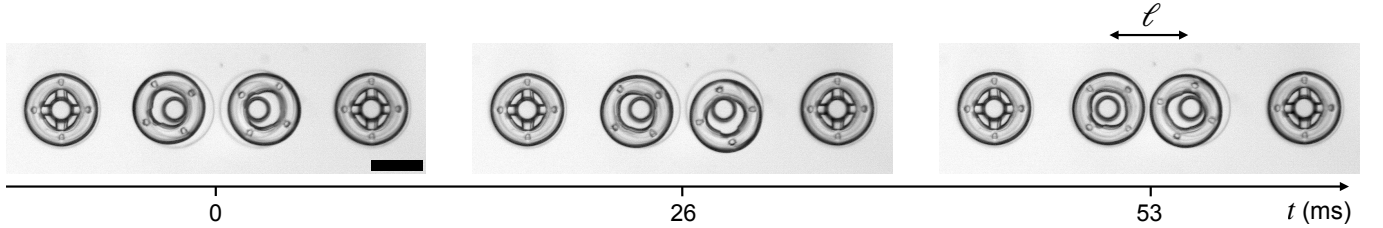


FIG. S5. **Dynamics of a single pair of motors.** Three subsequent snapshots showing two functional motors surrounded by two immobile rotors. Scale bar: $100 \mu\text{m}$.

Emergence of phase-coherence We investigate the phase coherence of the metachines using lattices of 48×44 small micromotors with $\ell = 99.6 \mu\text{m}$. Our observation window includes only the 36×33 rotors at the center of the lattice. We focus on this large bulk region to avoid boundary effects. Indeed, on the left and right edges of the lattice, the local electric field differs from the bulk, leading to lower Quincke thresholds and faster rotations speed as seen in Figure S6. We can provide a qualitative explanation of this phenomenon. Whatever the E field magnitude, the motors locally hinder the Ohmic transport of the ionic current between the electrodes (they reduce the available space in the z direction). At large scales, this effect can be modeled as a localized reduction of the Ohmic conductivity σ . As captured by the leaky dielectric model of Taylor and Melchner [14], a local reduction of the conductivity translates into a dipolar perturbation to the $\mathbf{E}$ field centered on each motor. The total $\mathbf{E}$ field that powers the Quincke rotation is the sum of the bare $\mathbf{E}$ field and of the dipolar perturbations caused by all the other rotors. In a square geometry the sum of all these perturbations cancel out everywhere but at the left and right hand side of the lattice, where we indeed observed localized speed heterogeneities over short distances Figure S6. All the measurements reported in Figure 4 therefore correspond to bulk data only.

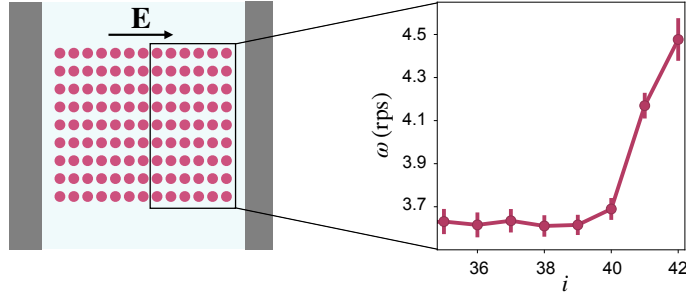


FIG. S6. **Speed heterogeneities at the edges of the lattice.** Left: sketch of the experimental geometry. The two vertical electrodes induce a $\mathbf{E}$ field in the x direction to power the Quincke motors. Right: Map of the rotor speed measured close to the lattice edge. The speed of the motors is increased in a localized vertical strip. Experiments on small rotors, lattice spacing $\ell = 101 \mu\text{m}$, $E = 0.66 \text{ V}/\mu\text{m}$. Error bars denote standard deviations calculated across rows.

6. Geometric control of the motor speed

We here describe a practical method to control the spatial variations of the motor speed.

The motors being powered by the Quincke instability, controlling their rotation speed amounts to controlling the magnitude of the electric field. To achieve a local control of E we take advantage of the Ohmic transport in the microchannel. The electric charge conservation imply that in steady state $\nabla \cdot \mathbf{j} = 0$, where $\mathbf{j}$ is the electric current. Combined with Ohm's law and integrating over the z direction, we find that the magnitude of the E field and the local height of the channel are related by

$$\nabla \cdot (\mathbf{E}h) = 0, \quad (\text{S2})$$

where $h(x, y)$ is the local channel thickness. This simple relation readily tells us how to pattern the electric field in space. Consider the simple case of a linear decrease of h along the field direction x . In the limit of small gradients, Eq. S2 implies that E increases linearly in the x direction, thereby patterning the rotation speed of the motors.

To demonstrate this effective design principle, we place a series of distant motors in a microfluidic cell, where the left and right electrodes have different thicknesses. Figure S7a confirms that the motors spin at a higher rate as they are closer to the thinnest electrode. For a more quantitative confirmation, we increase the voltage applied to the electrodes and measure its value $V_Q(x)$ when the motor located at x reaches the Quincke threshold and start spinning. We find a linear relationship between $V_Q(x)$ and $h(x)$ in excellent agreement with our model that predicts $V_Q(x) = Ah(x)$, where $A = -E_Q \int_0^L \frac{dx}{h(x)}$, see Figure S7b.

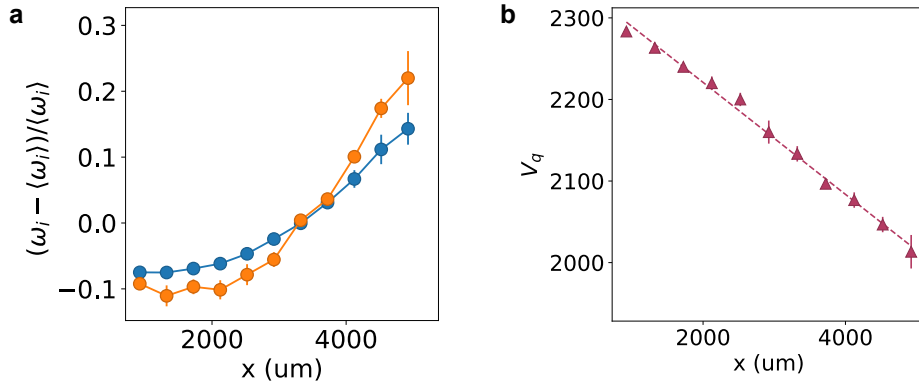


FIG. S7. **Local control of the motor speed.** **a.** Plot of the motor speed normalized by their average value as a function of their position in the channel for two values of the imposed voltage V . Orange dots: $V = 3000 \text{ V}$, Blue dots: 5000 V . Distance between the electrodes: $6100 \mu\text{m}$, height of the left electrode: $140 \mu\text{m}$, height of the right electrode: $240 \mu\text{m}$. **b.** Voltage V_q needed to trigger the rotation of the motors at position x . The experiments correspond to small motors separated by a distance of $400 \mu\text{m}$. Error bars denote standard deviations calculated across rows.

III. SINGLE-MOTOR DYNAMICS

1. Quincke electrotation of micromotors: spin and precession

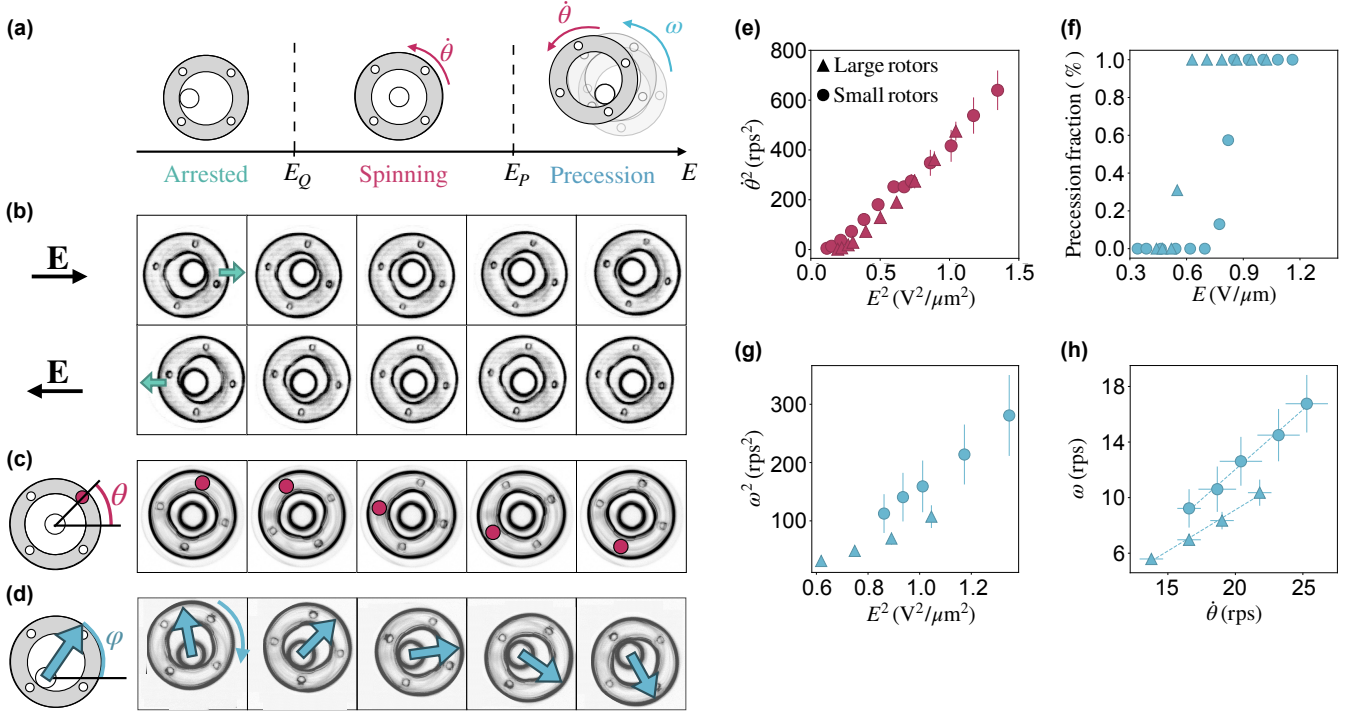


FIG. S8. **a.** Sketch of the three states of the motor dynamics: Arrested, Spinning, Precession. **b.** $E = 0.6E_Q$: The motors do not rotate. Electrophoresis drives the rotors towards the stators. **c.** $E = 1.3E_Q$: Above the onset of electro-rotation, the rotors are centered and spin at a constant rate. **d.** $E = 2.1E_Q$: At large field amplitude, the rotors are not centered anymore. In contact with the stators, the rotors roll on the stator and precess. It hula-hoops around the stator. **e.** Square of the Spinning rate, $\dot{\theta}^2$, plotted as a function of E^2 . **f.** Fraction of rotors in the precession state. **g.** Square of the precession rate, $\omega^2 = \dot{\varphi}^2$, plotted as a function of E^2 . **h.** Precession rate ω plotted as a function of the spinning rate $\dot{\theta}$. A linear fit indicates that the rotors both roll and slip on the stators. Circles: small rotors, triangles: large rotors. Error bars denote standard deviations over measurements from at least 15 motors in a single cell.

In the main text we focus on the collective dynamics of interacting motors powered by an electric field with a magnitude much higher than the Quincke threshold E_Q . In this section, we provide a quantitative characterization of the dynamics of isolated motors over a broader range of E fields (Fig. S8a).

At low field $E < E_Q$, the rotors do not spin. They are however driven towards the stator as a result of small electrophoretic forces (Fig. S8b). The direction of motion indicates that the rotors have a positive charge. As E exceeds E_Q , right above the instability threshold, the motors rotate at a constant rate $\dot{\theta}$, but do not precess. Instead the rotors are centered and spin around the stator (Fig. S8c). In this regime, $\dot{\theta}^2$ increases linearly with E^2 as predicted by Quincke-rotation theory [14] (Fig. S8e). Further increasing the value of E , the motors abruptly change their dynamics (Fig. S8f): the rotor contacts the stator thereby leading to a steady precession akin to hula-hoop motion (Fig. S8d). In this regime, as reported in the main text, the precession speed $\omega \equiv \dot{\varphi}$ increases with E in agreement with Quincke-electrorotation theory [14] (Fig. S8g). The value of the slope of the linear relation between ω and $\dot{\theta}$ in Fig. S8h implies that the rotor both slips and rolls on the stator. Three concluding comments are in order. Firstly, we stress that unlike in most active spinner experiments, see e.g. [5, 21, 22], here the parity symmetry of the dynamics is spontaneously broken: the motors spin, or precess, with equal probability in the clockwise and counterclockwise directions. Secondly, we note that the spinning and precession regimes do not coexist. Every single rotor is either in one state or the other unlike in the starfish embryos experiments of [6]. Thirdly, given the motor geometry we are using, the range of field amplitude over which all the motors rotate and remain in the spinning regime is too narrow to systematically investigate the collective organization of interacting pure spinners.

2. Modulation of the precession dynamics

Before explaining this variety of dynamics, it is worth taking a closer look at the precession regime. On average, the precession angle $\varphi(t)$ increases linearly with time and is controlled by E (Fig. S8g). However the instantaneous value of $\omega = \dot{\varphi}(t)$ is not constant. Even when a rotor is isolated, ω fluctuates periodically around a constant value. ω depends on φ (Fig. S9a). To study this phenomenon, we conducted experiments on more than 50 rotors. To minimize the interactions between the rotors, we set their distances to four diameters. We Fourier transform the $\omega(\varphi)$ signals, and plot their power spectra in Fig. S9b. We find a clear twofold modulation of the precession speed regardless of the rotor size: the motors periodically accelerate and decelerate over half a rotation. All our data are therefore well fitted by a function $\omega(\varphi) = \omega_0 - \omega_2 \cos(2\varphi + \chi)$, and Fig. S9c shows that the relative magnitude of the modulation ω_2/ω_0 increases with E . More importantly, we find that the sign of the phase of the twofold modulation depends on the direction of rotation of the rotors. Running experiments with the large motors, we find that the sign of χ mirrors that of the rotation direction (Fig. S9d). Rotors orbiting in the clockwise direction ($\omega < 0$) are associated with a positive phase ($\chi > 0$). Conversely, rotors orbiting in the counterclockwise direction ($\omega > 0$) are associated with a negative phase ($\chi < 0$). For the small rotors, it is the opposite: the sign of the phase is the same as the sign of the direction of rotation. The value of χ does not depend on the field strength (Fig. S9e). We will show that the phase χ of the speed modulation will prove crucial to identify the origin of the precession dynamics, and of the phase coherence of motor lattices.

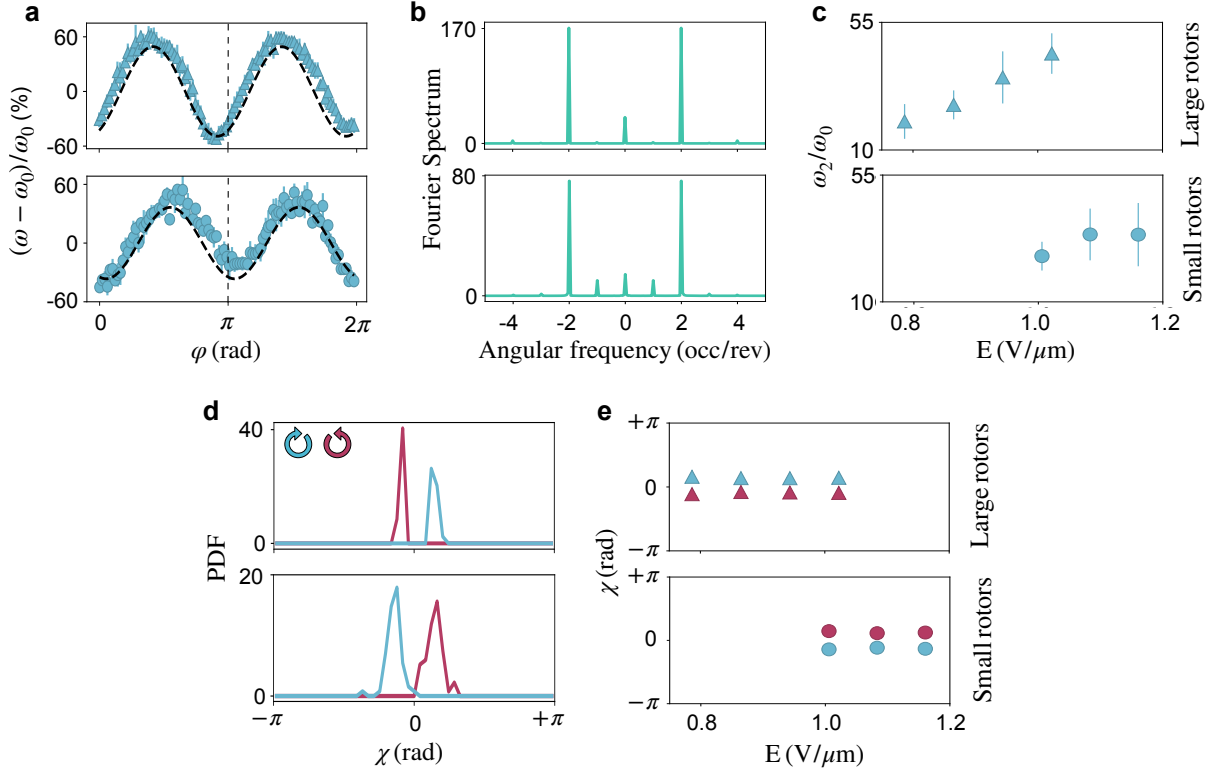


FIG. S9. **Twofold modulation of the hula-hoop dynamics** **a.** Relative precession speed $(\omega - \omega_0)/\omega_0$ for one rotor plotted versus the orientation angle φ . We wrapped the signal $\varphi(t)$ over the interval $[0, 2\pi]$ and average $\omega(\varphi)$ over bins of width $2\pi/100$. **b.** Power spectra associated with the Fourier transform of the signals shown in **a** (signal averaged over more than 50 different rotors). **c.** Evolution of the relative amplitude of the twofold modulation (ω_2/ω_0) with the field amplitude. **d.** Distribution of the twofold-modulation phase χ for motors spinning in the clockwise (blue) and counterclockwise (red) directions. **e.** Twofold-modulation phase χ as a function of the field amplitude. χ hardly depends on E . Top: Large rotors. Bottom: Small rotors. Error bars denote standard deviations over measurements from at least 15 motors in a single cell.

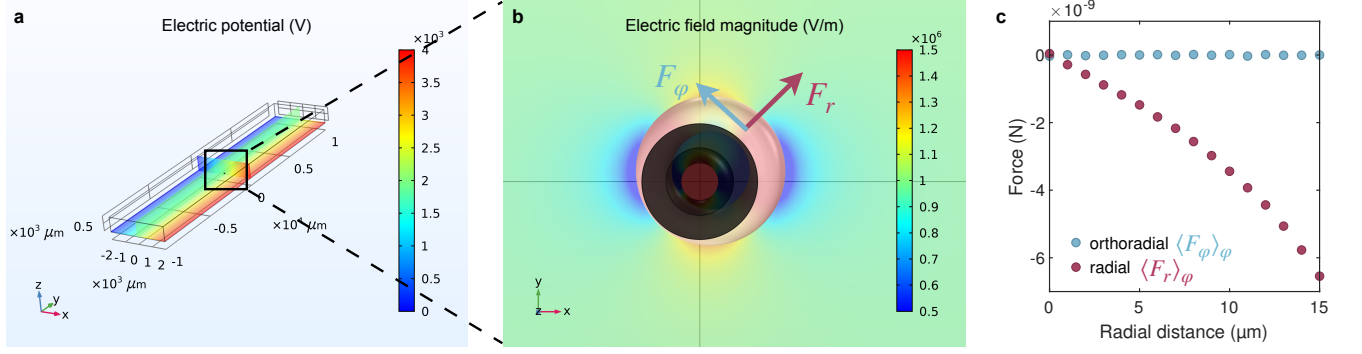


FIG. S10. **Dielectrophoresis centers the rotor's position** **a** A single rotor immersed in hexadecane is modeled with the finite element software COMSOL under an applied voltage of 4kV across 4mm along the x-direction according to the experimental conditions. The relative permittivities of the SU8 rotor and hexadecane solution are $\epsilon_{\text{SU8}} = 3.5$ and $\epsilon_{\text{hexa+AOT}} = 2$ respectively. Their conductivities are $\sigma_{\text{hexa+AOT}} = 10^{-9} \text{ S/m} \gg \sigma_{\text{SU8}} = 10^{-17} \text{ S/m}$. **b** Close-up on the Quincke rotor showing heterogeneities in the electric field magnitude. This heterogeneities cause a net dielectrophoresis force on the toroidal rotor (white) when it is off-centered from the stator (black). **c** The radial component of the electric force is negative and tends to center the rotor.

3. Origins of the spinning and hula-hoop dynamics

Providing a quantitative and predictive theory of single motor dynamics goes beyond the scope of our article. On the sole basis of our measurements, we can however pinpoint the physical phenomena that rule the transition from spinning to hula-hoop motion with twofold speed modulation. Before delving into our phenomenological reasoning, we summarize our main conclusions. In short, at low electric field electrophoresis drives the rotor in the direction of the $\mathbf{E}$ field. Increasing the value of E , the Quincke instability powers rotation, and dielectrophoresis stabilizes a motion centered around the stator. Further increasing the E field, the dipolar interactions between the rotor and stator overcome the dielectrophoretic repulsion, thereby resulting in a modulated precession dynamics.

We now explain in more detail how our experimental observations hint towards this three-steps scenario:

- At low E field, below E_Q , the translation of the rotors in the direction of the field indicates that they acquire a positive charge. Surface charge dissociation in apolar solvent containing AOT salt is a well established phenomenon [7, 20]. The drift of the rotors below E_Q is a mere consequence of the net electrophoretic mobility of insulating bodies immersed in a solution of inverse micelles in an apolar solvent.

- At intermediate field magnitude, $E > E_Q$, another phenomenon competes with electrophoresis and centers the rotor position. The repulsion of Quincke rotors from solid surfaces has already been reported in Ref. [17]. In this article, Pradillo and co-workers show that 100 micron big Quincke spheres overcome both the effects of gravity and of electrophoresis, hovering over solid surfaces under the action of dielectrophoretic forces. To better establish the relevance of dielectrophoretic forces in our geometry, we solved the Maxwell equations and computed the E field around an isolated motor using a Finite Element Method (COMSOL), see (Fig. S10). We find that the rotor experiences a net radial dielectrophoretic force that stabilizes a centered spinning motion [17].

- At high field amplitude, the rotors are not centered anymore, they contact the stator and undergo modulated hula-hoop motion. This additional transition must therefore involve a third interaction that can compete with dielectrophoretic centering forces. The twofold modulation of the orbiting speed hints towards dipolar interactions. Past the Quincke threshold, the conduction charges (the ions) accumulate at the rotor's surface and their spatial distribution forms a dipole, see Ref. [14] and Fig. S3. When the rotor spins, the charge dipole makes a finite angle with the electric field and its magnitude increases with E . The cylindrical stator also features a surface charge with a dipolar symmetry, but the orientation of the dipole remains exactly opposite to the E field [14], see Fig. S3. In principle, increasing E , the magnitude of the dipole-dipole interaction between the rotor and stator can overcome the dielectrophoretic force. In addition, the dipolar force depends on the relative orientation of the dipoles and on the displacement of the rotor. These variations are complex to derive exactly, but the dipolar symmetry implies a twofold modulation of the force with φ , which is consistent with the observed modulation of the precession speed in our experiments.

Going beyond this qualitative reasoning, would require modeling both the electrostatic and frictional interactions between the rotor and stator, taking into account their complex geometries. This complex numerical investigation could help predicting and understanding the variations of the modulation phase with the motor size.

IV. EMERGENCE OF ANTIFERROMAGNETIC ORDER: THEORY

In this section, we investigate the coupled dynamics of Quincke rotors and explain how antiferromagnetic order emerges from transverse frictional interactions. We start by recalling the basic equations governing the motion of a single Quincke rotor within the leaky dielectric approximation [14] actuated by a constant external torque. We then solve analytically the situation of two rotors coupled by frictional torques.

1. Quincke electro-rotation instability: a primer

We first briefly recall how to model the dynamics of a single Quincke rotor, see for instance Ref. [14, 16] for more details. We consider a rotor, an axisymmetric body, immersed in a conducting fluid. Both the solid body and the fluid are weak Ohmic conductors and have a finite electric permittivity. We neglect all electrostatic double layer effects. Within this leaky dielectric approximation, when placed in a constant $\mathbf{E}$, the electric polarization $\mathbf{p}$ and rotation vector $\dot{\boldsymbol{\theta}}$ of the rotor obey two coupled equations:

$$\frac{d\mathbf{p}}{dt} = \dot{\boldsymbol{\theta}} \times (\mathbf{p} - \chi^\infty \mathbf{E}) - \frac{1}{\tau_M} (\mathbf{p} - \chi^0 \mathbf{E}), \quad (\text{S3})$$

$$I \frac{d\dot{\boldsymbol{\theta}}}{dt} = \mathbf{p} \times \mathbf{E} - \gamma \dot{\boldsymbol{\theta}}, \quad (\text{S4})$$

where τ_M is the Maxwell-Wagner time, χ^0 and χ^∞ are respectively the zero-frequency and high frequency susceptibilities, I is the moment of inertia and γ is a friction coefficient. The first equation reflects the conservation of the electric charge at the surface of the particle, and the second is the angular-momentum-conservation equation. In all that follows, the electric field points along the x direction $\mathbf{E} = E\hat{\mathbf{e}}_x$, and the rotation vector along the z direction $\dot{\boldsymbol{\theta}} = \dot{\theta}\hat{\mathbf{e}}_z$. The polarization $\mathbf{p}$ vector lies in the xy plane. It is convenient to recast Eqs. (S3)-(S4) into a dimensionless form by introducing the scaled variables

$$t^* = \frac{t}{\tau_M}, \quad \Pi^x = \frac{\tau_M E}{\gamma} (p_x - \chi^0 E), \quad \Pi^y = -\frac{\tau_M E}{\gamma} p_y, \quad \Omega = \tau_M \dot{\theta} \quad (\text{S5})$$

The dynamics of the Quincke rotor then take the compact form

$$\frac{d\Pi^x}{dt^*} = \Omega \Pi^y - \Pi^x \quad (\text{S6})$$

$$\frac{d\Pi^y}{dt^*} = \Omega(\rho - \Pi^x) - \Pi^y \quad (\text{S7})$$

$$\frac{d\Omega}{dt^*} = \sigma(\Pi^y - \Omega) \quad (\text{S8})$$

$\sigma = \gamma\tau_M/I$ is the ratio of the Maxwell-Wagner time to the angular relaxation time, and $\rho = \tau_M(\chi^\infty - \chi^0)E^2/\gamma$ is the distance to the Quincke threshold. When $\chi^\infty > \chi^0$, a necessary condition for the Quincke instability, we define $E_Q = \sqrt{\gamma/[\tau_M(\chi^\infty - \chi^0)]}$ and $\rho \equiv (E/E_Q)^2$. Eqs. (S6)-(S8) then form the well-known Lorenz system [19]. However, in the experimentally relevant regime, inertia plays a negligible role. The Maxwell-Wagner time τ_M is much larger than the angular relaxation timescale I/γ . In the limit $\sigma \gg 1$, we can therefore further simplify our model and replace the first dynamical equation by the identity $\Pi^y = \Omega$. In this overdamped regime, the dynamical system that rules the dynamics of the Quincke rotors becomes two dimensional:

$$\frac{d\Pi^x}{dt^*} = \Omega^2 - \Pi^x \quad (\text{S9})$$

$$\frac{d\Omega}{dt^*} = \Omega(\rho - 1 - \Pi^x) \quad (\text{S10})$$

$\Omega = \tau_M \dot{\theta}$ quantifies the rotation of the rotor while Π^x is the perturbation of the electric polarization $\mathbf{p}$ along the x direction. The stability analysis of Eq. (S9)-(S10) is straightforward. The quiescent stationary state $(\Pi^x, \Omega) = (0, 0)$ is stable when $\rho < 1$ and becomes unstable when $\rho > 1$. In this regime two stable stationary points exist at $(\Pi^x, \Omega) = (\rho - 1, \pm\sqrt{\rho - 1})$. This is the Quincke instability: when $\rho > 1$, i.e $E > E_Q$, the rotor is unstable to minute rotational perturbations and starts spinning spontaneously along any rotation axis transverse to the $\mathbf{E}$ field.

2. Two coupled rotors

2.1 . How do Quincke spinners respond to torques?

Before addressing the dynamics of coupled motors, we can gain some insight by considering a simpler situation where a single Quincke motor responds to an imposed constant torque. The dynamics then obeys the rescaled overdamped equations:

$$\frac{d\Pi^x}{dt^*} = \Omega\Pi^y - \Pi^x \quad (\text{S11})$$

$$\frac{d\Pi^y}{dt^*} = \Omega(\rho - \Pi^x) - \Pi^y \quad (\text{S12})$$

$$0 = \Pi^y - \Omega + \mathcal{T} \quad (\text{S13})$$

where $\mathcal{T}$ is the rescaled torque. The last equation represents the torque balance condition and it is worth keeping in mind that $\Pi^y \equiv \mathcal{T}_E$ is nothing else but the scaled electric torque, whereas $-\Omega \equiv \mathcal{T}_V$ is the scaled viscous torque. We can use the first two equations to express $\mathcal{T}_E$ as a function of Ω in steady state. It takes the cubic form

$$\mathcal{T}_E = \Omega(\rho + \Omega\mathcal{T} - \Omega^2). \quad (\text{S14})$$

The steady state and their stability are then readily deduced from the torque balance equation. In Fig. S11, we plot the magnitude of $\mathcal{T}_V$ and of $\mathcal{T}_E + \mathcal{T}$ for different values of $\mathcal{T}$. The motor dynamics is then readily inferred by a graphical analysis. When $\mathcal{T} = 0$, when $E > E_Q$, we recover the standard Quincke rotation result. Two opposite values of the rotation rate satisfy the torque balance condition, and the corresponding equilibrium positions are stable. The situation where $\Omega = 0$ corresponds to an unstable fixed point. Applying an external torque bias the motor dynamics. Increasing $\mathcal{T}$ offsets the values of the two possible rotations rates, until the point where only one stable motor rotation is possible, in the direction set by the applied torque. Below E_Q applying an external torque always enable the electric actuation of the motor but solely in the direction imposed by the external torque.

This graphical analysis provides some intuition about the dynamics of motors coupled by frictional forces. In this situation the external torque would be replaced by the interaction torque. When the coupling strength is vanishingly small, we of course expect the dynamics of the two motors to be decoupled. In the opposite limit where friction is very large, the first motor experiences a large torque that promotes rotation only in the direction opposite to the neighboring motor. However at intermediate friction values we can expect configurations where biased counter- and co-rotation states are possible. This qualitative analysis hints towards a scenario where a finite coupling strength is required to achieve antiferromagnetic order, and that past this critical value all other states would be linearly unstable. This scenario is consistent with our experimental observations and confirmed by the detailed theory presented in the next section.

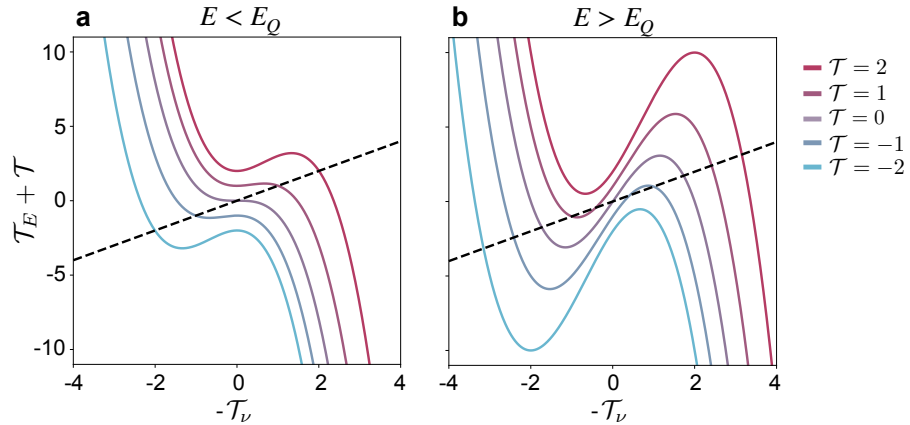


FIG. S11. Plots of the sum of torques $\mathcal{T}_E + \mathcal{T}$ as a function of the viscous torque $\mathcal{T}_V$, for different values of the imposed torque $\mathcal{T}$. The dashed line indicates the condition $\mathcal{T}_E + \mathcal{T} = \mathcal{T}_V$. **a.** Below the Quincke threshold ($\rho = 0.01$), the motor rotates in the direction set by the imposed torque $\mathcal{T}$. **b.** Above the Quincke threshold ($\rho = 4$), a sufficiently strong imposed torque can destabilize the rotation direction opposite to the one it promotes.

2.2 . Coupled dynamics of two Quincke motors: a dynamical system insight

We now consider two Quincke rotors with polarizations $\mathbf{p}_1$ and $\mathbf{p}_2$, and rotation vectors $\dot{\boldsymbol{\theta}}_1$ and $\dot{\boldsymbol{\theta}}_2$. We introduce a frictional torque $-\gamma\mu\dot{\boldsymbol{\theta}}_j$ acting on rotor i , where μ is a positive dimensionless number. This is a minimal ingredient mimicking a “cogwheel” effect that could be of hydrodynamic or frictional origin. This minimal coupling will lead to rotations of the two rotors in opposite directions. In dimensional form, the equations of motion are

$$\begin{aligned}\frac{d\mathbf{p}_1}{dt} &= \dot{\boldsymbol{\theta}}_1 \times (\mathbf{p}_1 - \chi^\infty \mathbf{E}) - \frac{1}{\tau}(\mathbf{p}_1 - \chi^0 \mathbf{E}) \\ \frac{d\mathbf{p}_2}{dt} &= \dot{\boldsymbol{\theta}}_2 \times (\mathbf{p}_2 - \chi^\infty \mathbf{E}) - \frac{1}{\tau}(\mathbf{p}_2 - \chi^0 \mathbf{E}) \\ I \frac{d\dot{\boldsymbol{\theta}}_1}{dt} &= \mathbf{p}_1 \times \mathbf{E} - \gamma(\dot{\boldsymbol{\theta}}_1 + \mu\dot{\boldsymbol{\theta}}_2) \\ I \frac{d\dot{\boldsymbol{\theta}}_2}{dt} &= \mathbf{p}_2 \times \mathbf{E} - \gamma(\dot{\boldsymbol{\theta}}_2 + \mu\dot{\boldsymbol{\theta}}_1)\end{aligned}$$

We can make these equations dimensionless by defining $(\Pi_1^x, \Pi_1^y, \Omega_1)$ and $(\Pi_2^x, \Pi_2^y, \Omega_2)$ according to Eq. (S5). In the limit $\sigma \gg 1$, we can eliminate $\Pi_1^y = \Omega_1 + \mu\Omega_2$ and $\Pi_2^y = \Omega_2 + \mu\Omega_1$. It is useful to recast the dynamical system in terms of the variables $\Pi_\pm^x = \frac{1}{2}(\Pi_1^x \pm \Pi_2^x)$ and $\Omega_\pm = \frac{1}{2}(\Omega_1 \pm \Omega_2)$.

$$\frac{d\Pi_+^x}{dt^*} = (1 + \mu)\Omega_+^2 + (1 - \mu)\Omega_-^2 - \Pi_+^x \quad (\text{S15})$$

$$\frac{d\Pi_-^x}{dt^*} = 2\Omega_+\Omega_- - \Pi_-^x \quad (\text{S16})$$

$$\frac{d\Omega_+}{dt^*} = \frac{1}{1 + \mu} [(\rho - (1 + \mu) - \Pi_+^x)\Omega_+ - \Pi_-^x\Omega_-] \quad (\text{S17})$$

$$\frac{d\Omega_-}{dt^*} = \frac{1}{1 - \mu} [(\rho - (1 - \mu) - \Pi_+^x)\Omega_- - \Omega_+\Pi_-^x] \quad (\text{S18})$$

Solving for the stationary solutions yields :

$$\Pi_+^x = (1 + \mu)\Omega_+^2 + (1 - \mu)\Omega_-^2 \quad (\text{S19})$$

$$\Pi_-^x = 2\Omega_+\Omega_- \quad (\text{S20})$$

$$0 = \Omega_+ [\rho - (1 + \mu)(1 + \Omega_+^2) - (3 - \mu)\Omega_-^2] \quad (\text{S21})$$

$$0 = \Omega_- [\rho - (1 - \mu)(1 + \Omega_-^2) - (3 + \mu)\Omega_+^2] \quad (\text{S22})$$

which are satisfied if $\Omega_\pm = 0$ or if the bracket terms vanish. This yields four types of fixed points. To study the linear stability of a fixed point $(\Pi_+^x, \Pi_-^x, \Omega_+, \Omega_-)$ we need to compute the eigenvalues of the dynamical matrix

$$M = \begin{pmatrix} -1 & 0 & 2(1 + \mu)\Omega_+ & 2(1 - \mu)\Omega_- \\ 0 & -1 & 2\Omega_- & 2\Omega_+ \\ -\frac{\Omega_+}{1 + \mu} & -\frac{\Omega_-}{1 + \mu} & \frac{\rho - \Pi_+^x}{1 + \mu} - 1 & -\frac{\Pi_-^x}{1 + \mu} \\ -\frac{\Omega_-}{1 - \mu} & -\frac{\Omega_+}{1 - \mu} & -\frac{\Pi_-^x}{1 - \mu} & \frac{\rho - \Pi_+^x}{1 - \mu} - 1 \end{pmatrix}. \quad (\text{S23})$$

(i). **Non-rotating rotors.** The simplest solution of Eqs. (S19)-(S20) is $(\Pi_+^x, \Pi_-^x, \Omega_+, \Omega_-) = (0, 0, 0, 0)$, corresponding to $(\dot{\theta}_1, \dot{\theta}_2) = (0, 0)$. The matrix M , Eq. (S23), is then diagonal. The system is stable when $\rho < 1 - \mu$ and unstable when $\rho > 1 - \mu$. Frictional coupling lowers the Quincke threshold.

(ii). **Counter-rotating rotors.** If $\Omega_+ = 0, \Omega_- \neq 0$, the two rotors are counter-rotating. Eqs. (S19)-(S20) then give two fixed points

$$(\Pi_+^x, \Pi_-^x, \Omega_+, \Omega_-) = \left(\rho - (1 - \mu), 0, 0, \pm \sqrt{\frac{\rho - (1 - \mu)}{1 - \mu}} \right)$$

This type of solution exists when $\rho \geq 1 - \mu$, i.e. as soon as the non-rotating fixed point is unstable. This steady state configuration is always stable.

(iii). **Co-rotating rotors.** If $\Omega_- = 0, \Omega_+ \neq 0$, the two rotors are co-rotating. There exists two such fixed points given by

$$(\Pi_+^x, \Pi_-^x, \Omega_+, \Omega_-) = \left(\rho - (1 + \mu), 0, \pm \sqrt{\frac{\rho - (1 + \mu)}{1 + \mu}}, 0 \right),$$

when $\rho \geq 1 + \mu$. The four corresponding eigenvalues are

$$\lambda_{1,\pm} = \frac{1}{2} \left[-1 \pm \sqrt{1 - 8 \frac{\rho - (1 + \mu)}{1 + \mu}} \right], \quad \lambda_{2,\pm} = \frac{1}{2} \left[-\frac{1 - 3\mu}{1 - \mu} \pm \sqrt{\left(\frac{1 - 3\mu}{1 - \mu} \right)^2 - 8 \frac{\rho - (1 + \mu)}{1 - \mu^2}} \right].$$

The fixed points are unstable for $\mu > 1/3$. When $0 < \mu < 1/3$, they are stable if $\rho > (1 + \mu)^2$. When $\mu > 1/3$, the negative eigenvalue is associated with an eigenvector of the form $(\Pi_+^x, \Pi_-^x, \Omega_+, \Omega_-) = (0, a, 0, b)$.

(iv). **Unstable fixed points.** If $\Omega_\pm \neq 0$, Eqs. (S19)-(S20) lead to four fixed points:

$$(\Pi_+^x, \Pi_-^x, \Omega_+, \Omega_-) = \frac{1}{2} \left(\rho - (1 - \mu)^2, \sqrt{[\rho - (1 - \mu)^2][\rho - (1 + \mu)^2]}, \pm \sqrt{\rho - (1 - \mu)^2}, \pm \sqrt{\rho - (1 + \mu)^2} \right)$$

They exist only when $\rho \geq (1 + \mu)^2$. The determinant of M , Eq. (S23), is $\det M = -\frac{2}{1 - \gamma^2} [\rho - (1 - \gamma)^2][\rho - (1 + \gamma)^2] < 0$. This implies that the four eigenvalues of M cannot all have a negative real part. In other words, the four fixed points are always unstable.

The stability analysis of the two coupled motors is summarized in Fig. 2c (main text). When the frictional coupling constant exceeds the critical value $\mu > 1/3$, and above the Quincke threshold ($\rho > 1 - \mu$), the only stable solution corresponds to counter-rotating motors. All other steady states are unstable. This nonlinear dynamics correctly accounts for the emergence of pristine antiferromagnetic order below a finite lattice-spacing value.

In the main text (Figure 2), we exemplify the dynamical states of the two coupled rotors, solving Eqs. (S15), (S16), (S17) and (S18) with $\rho = 1.5$, $\mu = 0.05$ (Low friction) and $\rho = 1.5$, $\mu = 0.35$ (Large friction).

V. IDENTIFICATION OF THE INTERACTIONS RESPONSIBLE FOR PHASE COHERENCE: A MINIMAL MODEL

In this section, we explain the emergence of phase coherence in our metamachines, and single out the classes of interactions that can organize the phases of the interacting motors in space and time.

1. A Minimal Model: coupled phase oscillators

Our goal is not to model in detail all the specifics of our experiments. Rather, we seek to highlight the types of dynamics and interactions that can give rise to a collective phase organization. To this end, we adopt a minimal approach inspired by models developed to study the synchronization of beating cilia in biological systems [2, 4, 9, 15, 23, 24]. These models have the advantage of simplifying the analysis while retaining the fundamental features of motor dynamics.

First, we simplify the geometry of the motors and their precession dynamics by representing each rotor as a point particle following a circular orbit of radius a under the action of a driving torque, see Fig. S12. We then couple the particles by considering, one at a time, all possible interaction torques relevant to our experiments. In the overdamped limit, the torque balance equations govern the rotors' dynamics:

$$0 = -\gamma a \dot{\varphi}_i + \tau_i + \sum_{j \neq i} \tau_{ji} \quad (\text{S24})$$

where γ is a friction coefficient, τ_i is the driving torque, and τ_{ij} is the torque exerted by rotor i on rotor j . In all that follows, we focus on nearest neighbor interactions. We also note $\mathbf{r}_i$ the position of rotor i , and ℓ the lattice spacing.

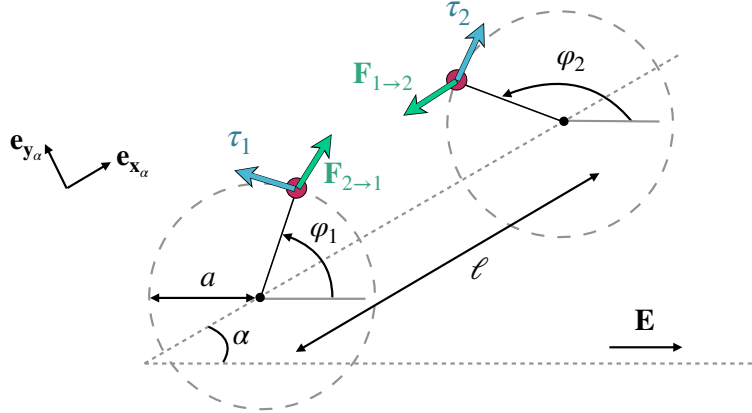


FIG. S12. **A minimal model for interacting motors.** Sketch of the minimal model used to describe the rotor dynamics. The rotors are reduced to point particles orbiting along circular trajectories of radius a . They are driven by constant torques $\tau_{1,2}$ and interact via forces $\mathbf{F}_{j \rightarrow i}$. The forces induce torques τ_{ji} on the particles, possibly accelerating or slowing down their orbital motion to find phase-coherent steady states.

2. Precession at constant speed: Dipolar interactions induce phase coherence

We first disregard the modulation of the precession speed measured for isolated motors and take $\tau_1 = -\tau_2 = \tau_0$. In our experiments the rotors are in principle coupled by three types of interactions: (i) radial repulsive forces: these include both contact interactions, and electrostatic repulsion due to the net charge carried by the rotors. (ii) hydrodynamic interactions. (iii) dipole-dipole interactions: these are induced by the electric dipoles carried by the motors, and which drive the Quincke instability. All three types of interactions could organize the phase dynamics. However, in our system, the sum of the phases $\bar{\varphi}_{\langle k,l \rangle}$ is uniform and equal to π in both lattice directions. Crucially, a uniform $\bar{\varphi}$ corresponds to different kinematics in the horizontal and vertical directions (Fig. S13). We therefore conclude that the interaction responsible for phase organization must be anisotropic.

Radial forces, in the absence of modulation, are isotropic. They cannot account for the distinct behaviors observed along the principal directions of the lattice. Similarly, hydrodynamic interactions—mediated by a homogeneous

and isotropic solvent—respect rotational symmetry and thus cannot explain the experimentally observed anisotropic dynamics. The only interactions that are explicitly anisotropic are the dipole-dipole interactions. Dipolar interactions are attractive along the E -field direction and repulsive in the perpendicular direction (Fig. S14a). They are therefore the only interactions capable of producing direction-dependent kinematics. They naturally emerge as the leading candidate to explain our experimental observations. To confirm this reasoning, we simulate our model.

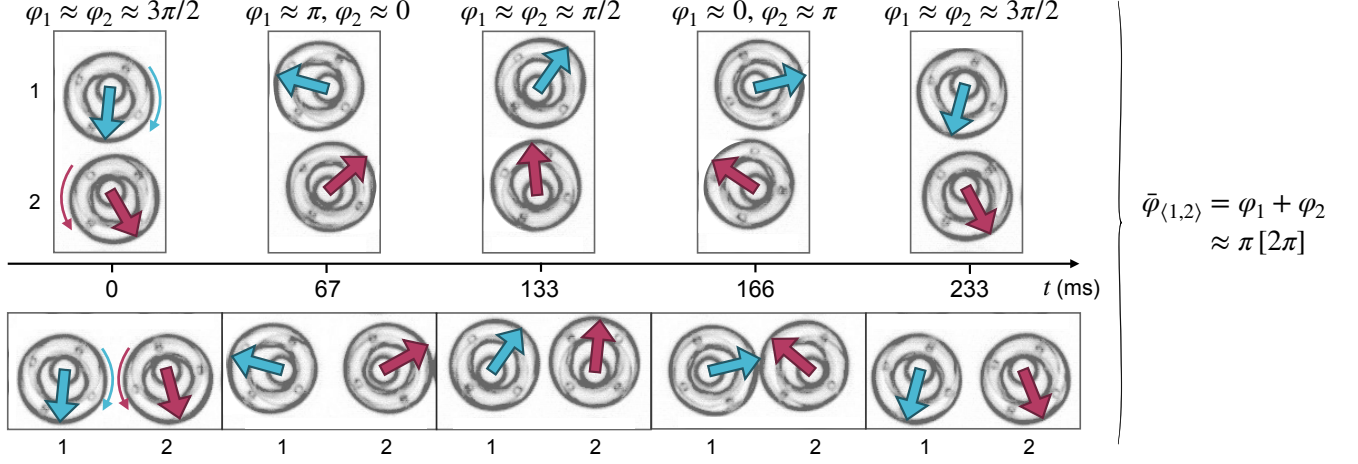


FIG. S13. **Kinematics of a pair of rotors** Two series of snapshots showing the rotor configurations when $\mathbf{r}_{12}$ points in the direction transverse (top row) and parallel (bottom row) to $\mathbf{E}$. In both cases $\bar{\varphi}$ is equal to π but the rotor kinematics are markedly different.

2.1 . Dipolar Interactions

In order to confirm our reasoning based on symmetry considerations, we focus on a simple setting where a pair of pointwise rotors orbit in opposite directions and interact via dipolar interactions. Above the Quincke threshold, the rotor dipoles make a finite angle δ with the electric field $\mathbf{E}$, and the sign of δ sets the rotation direction of the particle [3] (Fig. S14 b). Accordingly, we assume that the first micromotor, rotating counterclockwise, carries a dipole $\mathbf{p}_1$ with angle $\delta > 0$, whereas the second micromotor, rotating clockwise, carries a dipole $\mathbf{p}_2$ with angle $-\delta$, with $|\mathbf{p}_1| = |\mathbf{p}_2|$. The interaction potential between the two particles located at $\mathbf{r}_1$ and $\mathbf{r}_2$ is then given by:

$$U \propto -\frac{1}{r_{21}^3} [3(\mathbf{p}_1 \cdot \hat{\mathbf{r}}_{21})(\mathbf{p}_2 \cdot \hat{\mathbf{r}}_{21}) - \mathbf{p}_1 \cdot \mathbf{p}_2], \quad (\text{S25})$$

where $\mathbf{r}_{21} = \mathbf{r}_2 - \mathbf{r}_1$. Deriving the interaction forces and torques, and choosing units so that $\tau_0/\gamma = 1$ and $a = 1$, we find the equations of motion:

$$\begin{aligned} \dot{\varphi}_1 = 1 - \frac{B}{r_{21}^4} & \left[(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{r}}_{21})\hat{\mathbf{p}}_1 + (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{r}}_{21})\hat{\mathbf{p}}_2 \right. \\ & \left. - 5(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{r}}_{21})(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{r}}_{21})\hat{\mathbf{r}}_{21} + (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2)\hat{\mathbf{r}}_{21} \right] \cdot \hat{\mathbf{e}}_{\varphi_1} \end{aligned} \quad (\text{S26})$$

$$\begin{aligned} \dot{\varphi}_2 = -1 + \frac{B}{r_{21}^4} & \left[(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{r}}_{21})\hat{\mathbf{p}}_1 + (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{r}}_{21})\hat{\mathbf{p}}_2 \right. \\ & \left. - 5(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{r}}_{21})(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{r}}_{21})\hat{\mathbf{r}}_{21} + (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2)\hat{\mathbf{r}}_{21} \right] \cdot \hat{\mathbf{e}}_{\varphi_2}. \end{aligned} \quad (\text{S27})$$

Here, B is a constant that sets the strength of the dipolar interactions. We numerically solve these equations of motion for particle pairs aligned either along the horizontal ($\alpha = 0$) or vertical ($\alpha = \pi/2$) directions. Varying the values of the coupling strength B , the interparticle distance ℓ and dipole angle δ , we systematically find that the dipolar interactions drive the dynamics of the rotor towards a phase-coherent state where $\bar{\varphi} = \pi$, in agreement with our experimental observations (Figs. S14 c–f).

This already suggests that dipole-dipole interactions govern the phase dynamics of the motors.

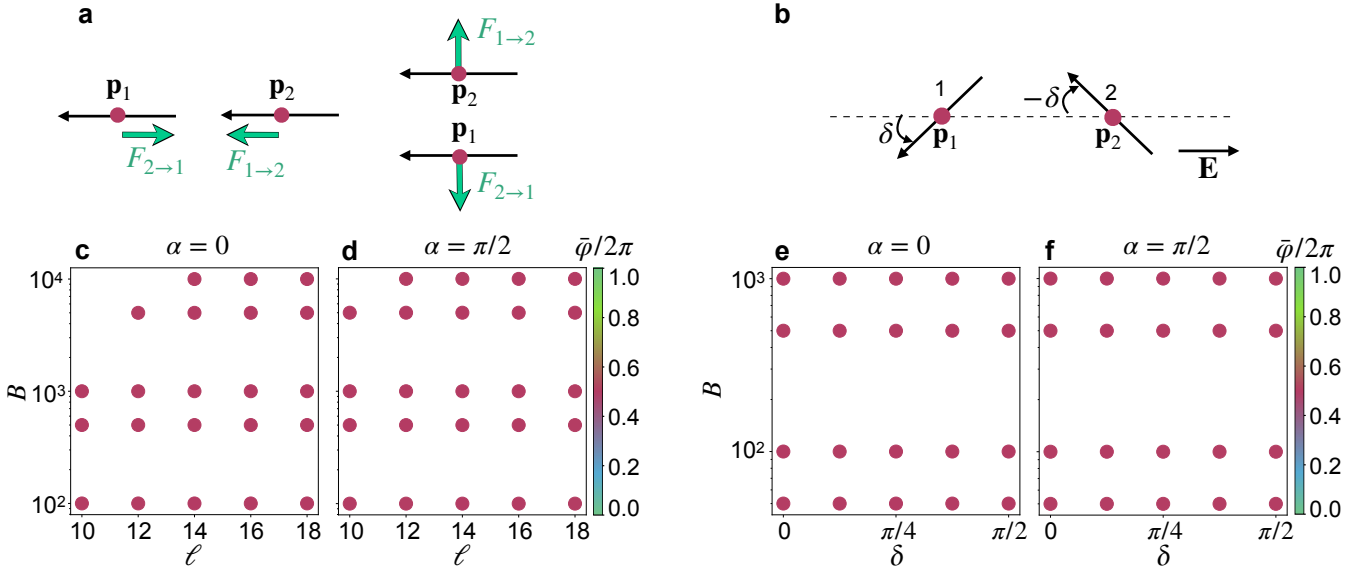


FIG. S14. **Phase coherence induced by dipolar interactions.** **a.** Sketch of the anisotropic interactions between two point dipoles. Two identical dipoles attract when placed on the polarization axis and repel each other in the transverse direction. **b.** Definition of the δ angle. **c-f.** Plot of the phase value $\bar{\varphi}$ computed for a pair of point rotors interacting via dipolar interactions (Eqs. S26 and S27). $\bar{\varphi}$ is averaged over the last 500 time steps in steady state, using four different initial conditions. When $\mathbf{r}_{12}$ points **c.** along the direction of the dipole $\mathbf{p}$ ($\alpha = 0$), **d.** transverse to the dipole $\mathbf{p}$ ($\alpha = \pi/2$) the interactions promote a phase-coherent state with $\bar{\varphi} = \pi$ for all values of B and ℓ . **e-f.** Same observations for **e.** $\alpha = 0$ and **f.** $\alpha = \pi/2$, varying the dipole orientation δ and magnitude B . **c-d.** $\delta = 0.34$ rad, **e-f.** $\ell = 10$.

2.2 . Radial Interactions

In this section we show that different phase-coherence patterns can arise from repulsive interactions between the rotors (Fig. S15a). As discussed earlier, radial interactions are isotropic, and therefore cannot account for the anisotropic dynamics observed experimentally (for unmodulated drives). However, we show below that repulsive forces can induce phase coherence. While they are not dominant in our system, they could play a significant role in other realizations of metamachines. The interaction torque τ_{12} takes generic form $\tau_{12} = f(r_{12})(\mathbf{r}_1 \times \hat{\mathbf{r}}_{21})$, where the magnitude of the force f is a decaying function of the distance r_{21} . In our simulations $f(r_{21}) \propto 1/r_{21}^2$. We find that the equations of motion take the form:

$$\dot{\varphi}_1 = 1 - \frac{C}{r_{21}^3} \left(\frac{\ell}{a} \sin(\alpha - \varphi_1) + \sin(\varphi_2 - \varphi_1) \right) \quad (\text{S28})$$

$$\dot{\varphi}_2 = -1 + \frac{C}{r_{21}^3} \left(\frac{\ell}{a} \sin(\alpha - \varphi_2) + \sin(\varphi_2 - \varphi_1) \right), \quad (\text{S29})$$

where C a positive constant controlling the amplitude of the interactions, and

$$r'_{21} = \left(\left[\frac{\ell}{a} \cos \alpha + (\cos \varphi_2 - \cos \varphi_1) \right]^2 + \left[\frac{\ell}{a} \sin \alpha + (\sin \varphi_2 - \sin \varphi_1) \right]^2 \right)^{1/2}. \quad (\text{S30})$$

We numerically solve these two equations, and find that for all values of C and ℓ/a the rotors self-organize their phases with steady-state values so that $\bar{\varphi} = 0$ for horizontal pairs ($\alpha = 0$) and $\bar{\varphi} = \pi/2$ for vertical pairs ($\alpha = \pi/2$), see Fig. S15b and S15c. We conclude that radial forces can induce phase coherence. However, unlike in our experiments, they would result in identical kinematics for both horizontal and vertical rotor pairs, as illustrated in Fig. S15d.

2.3 . Hydrodynamic interactions

Guided by earlier works on hydrodynamic synchronization in driven colloid lattices, we have systematically explored the effect of the far-field flow on phase coherence [2, 4, 9, 15, 23, 24]. Using the same approximations as in the previous section, we find that neither the Stokeslet nor the rotlet singularities can induce any form of phase coherence when

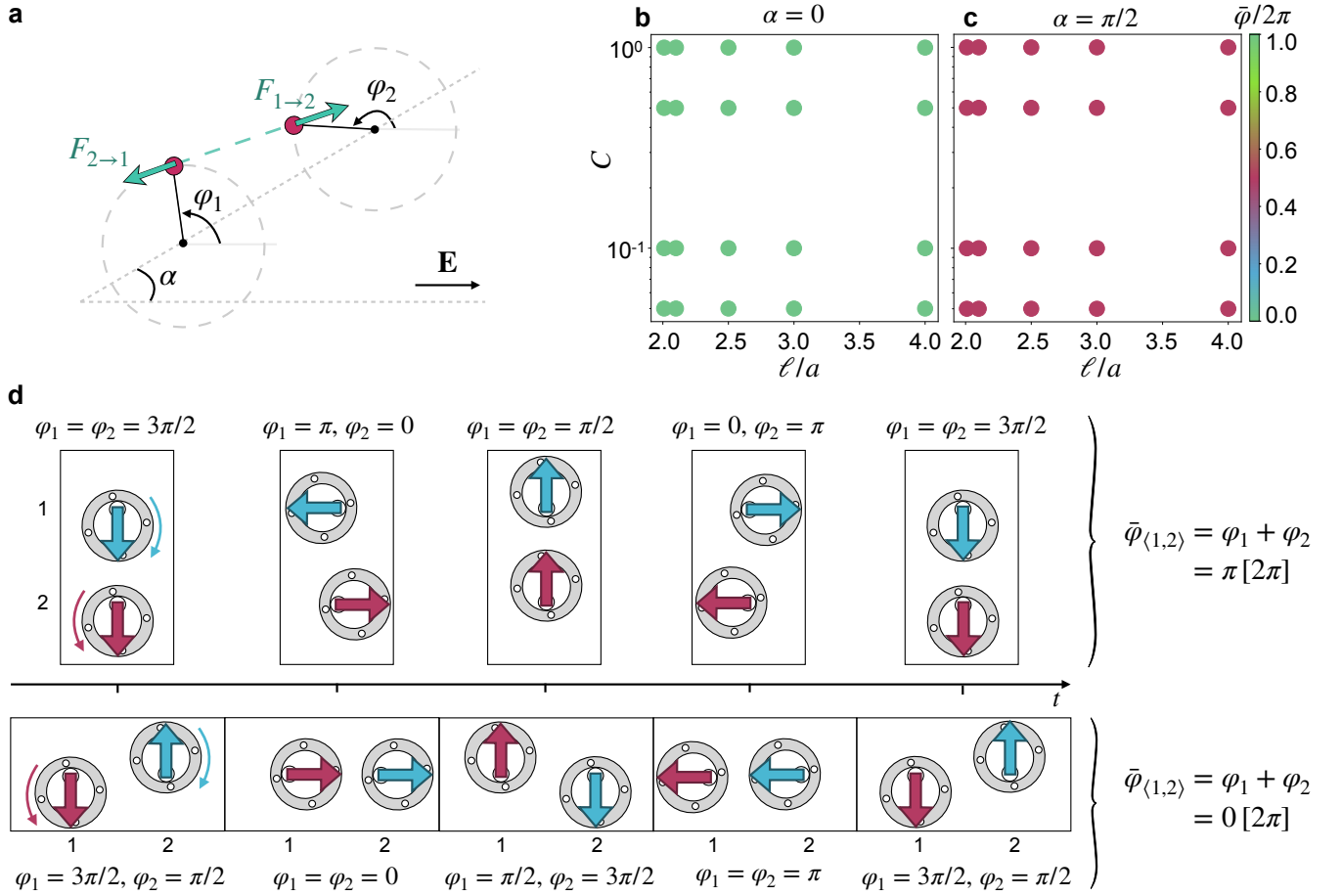


FIG. S15. **Phase coherence induced by repulsive radial interactions.** **a.** Geometry of the interacting particles and of the repulsive forces $\mathbf{F}_{i \rightarrow j}$ leading to the interaction torques τ_{ij} . **b, c.** Phase diagrams showing the steady-state value of $\bar{\varphi}$ as a function of ℓ/a and C . Panel **b.** corresponds to $\alpha = 0$ (horizontal configuration), while panel **c.** corresponds to $\alpha = \pi/2$ (vertical configuration). For each diagram, the value of $\bar{\varphi}$ is averaged over the last 500 data points, using four different initial conditions. **d.** Sketch of the rotor configurations that would result from repulsive interactions, for a horizontal ($\bar{\varphi} = 0$) and a vertical pair ($\bar{\varphi} = \pi$). In both cases, the rotor kinematics are identical.

the driving torque is constant [24]. We have also used a minimal model for near-field hydrodynamics interactions, that again showed no synchronisation phenomena when the driving torques are constant. We do not report these negative results here, but detail the effect of near-field forces below, when we address the role of drive-modulations on hydrodynamic interactions (Section 3).

3. Twofold-modulated drive: impact of hydrodynamic interactions on phase coherence

Earlier studies have shown that hydrodynamic interactions can effectively synchronize the phase of driven colloids and beating microcilia when their oscillation pattern is modulated [2, 4, 9, 15, 23, 24]. In section III 2 we have shown that the fluctuations of the rotation speed has a two-fold symmetry. In order to determine whether this modulated dynamics can impact the phase coherence of our motors we resort to our numerical model. We address the effect of a twofold modulation of the driving torque τ_0 considering the effect of all possible interactions separately.

In short, in the following sections, we show that the fluid flows induced by the modulated precession of our motors could self-organize their phases. However, comparisons between our numerical results and our experiments performed on different rotor sizes will rule out hydrodynamic interactions as the primary source of phase ordering. The conclusion of our detailed analysis remains unchanged: the phase coherence of our metamachines should be chiefly controlled by the dipolar interactions between the Quincke motors.

Solving the dynamical equations for φ_1 and φ_2 when the forces are either radial or dipolar, we find no effect of the

drive modulation on the phase coherence. We therefore do not report these negative results and focus on hydrodynamic interactions only (rotlet, Stokeslet and near-field flows). We investigate the impact of drive modulations on one type of hydrodynamic interaction at a time.

3.1 . Twofold modulation of the driving torques

We consider twofold modulation of the driving torques informed by the characterization of the single-rotor dynamics reported in Section III 2. The modulation of the mean torque τ_0 is characterized by its amplitude A_2 and phase χ . For a rotor pair:

$$\tau_1 = \tau_0 [1 - A_2 \cos(2\varphi_1 + \chi_1)] \quad (\text{S31})$$

$$\tau_2 = -\tau_0 [1 - A_2 \cos(2\varphi_2 + \chi_2)] \quad (\text{S32})$$

where the modulation phases are opposite: $\chi_1 = -\chi_2$. In our experiments we found that the sign of χ can change as the motor size increases. For small rotors ($d = 86 \mu\text{m}$) $\chi_1 = 0.41 \pm 0.04$ rad, for large rotors ($d = 144 \mu\text{m}$) $\chi_1 = -0.37 \pm 0.12$ rad. In all that follows, we set $A_2 = 0.25$ and $|\chi_1| = 0.44$ rad. We checked that none of our conclusions depend on this arbitrary choice.

3.2 . Hydrodynamic interactions: Rotlet flows

As the rotors spin and orbit, in the far field they are modeled by two essential singularities: a rotlet and a Stokeslet. We inspect the role of these two flows separately starting with the rotlet. Within a standard far-field approximation ($\ell/a \gg 1$) the rotlet flow points in the direction transverse to $\mathbf{r}_{12}$ (Fig. S16a). At leading orders, the interaction torque τ_{21} takes the form $\tau_{21}(\varphi_1, \varphi_2, \alpha) = (\tau_R^0 + \tau_R^1 \delta r) \cos(\alpha - \varphi_1)$, with

$$\delta r = |\mathbf{r}_{12} - \ell| = [(\cos \varphi_2 - \cos \varphi_1) \cos(\alpha) + (\sin \varphi_2 - \sin \varphi_1) \sin(\alpha)], \quad (\text{S33})$$

The resulting equations of motion hence thus read:

$$\begin{aligned} \dot{\varphi}_1 &= 1 - A_2 \cos(2\varphi_1 + \chi_1) \\ &+ \left\{ \tau_R^0 + \tau_R^1 [(\cos \varphi_2 - \cos \varphi_1) \cos(\alpha) + (\sin \varphi_2 - \sin \varphi_1) \sin(\alpha)] \right\} \cos(\alpha - \varphi_1) \end{aligned} \quad (\text{S34})$$

$$\begin{aligned} \dot{\varphi}_2 &= -1 + A_2 \cos(2\varphi_2 + \chi_2) \\ &+ \left\{ \tau_R^0 + \tau_R^1 [(\cos \varphi_2 - \cos \varphi_1) \cos(\alpha) + (\sin \varphi_2 - \sin \varphi_1) \sin(\alpha)] \right\} \cos(\alpha - \varphi_2) \end{aligned} \quad (\text{S35})$$

We solve these equations numerically and compute the time average of $\bar{\varphi}$ in steady state. Figures. S16b-e show that the twofold modulation of the driving torques results in phase coherence. However, unlike in our experiments, we find that the asymptotic value of $\bar{\varphi}$ strongly depends on the sign of the $\chi_1 = -\chi_2$. This essential difference rules out the rotlet-mediated interactions as the cause of phase organization in our metamachines.

3.3 . Hydrodynamic interactions: Stokeslet flows

We now examine the role of Stokeslet flows (Fig. S17a) that have been extensively studied in previous works. Adapting the model detailed in Ref. [2] in the so-called rigid limit, we find the equations of motion for the two phase variables:

$$\begin{aligned} \dot{\varphi}_1 &= 1 - A_2 \cos(2\varphi_1 + 2\alpha + \chi_1) \\ &+ \frac{3d}{16\ell} (-1 + A_2 \cos(2\varphi_2 + 2\alpha + \chi_2)) (3 \cos(\varphi_1 - \varphi_2) - \cos(\varphi_1 + \varphi_2)) \end{aligned} \quad (\text{S36})$$

$$\begin{aligned} \dot{\varphi}_2 &= -1 + A_2 \cos(2\varphi_2 + 2\alpha + \chi_2) \\ &+ \frac{3d}{16\ell} (1 - A_2 \cos(2\varphi_1 + 2\alpha + \chi_1)) (3 \cos(\varphi_1 - \varphi_2) - \cos(\varphi_1 + \varphi_2)). \end{aligned} \quad (\text{S37})$$

As before, we solve these coupled nonlinear equations numerically and compute the steady-state value of $\bar{\varphi}$. Our conclusions are identical to those drawn in the previous section (rotlet flows). Stokeslet flows can stabilize a phase-coherent dynamics under modulated driving (Figs. S17b-c). We also studied in details the influence of the Stokeslet

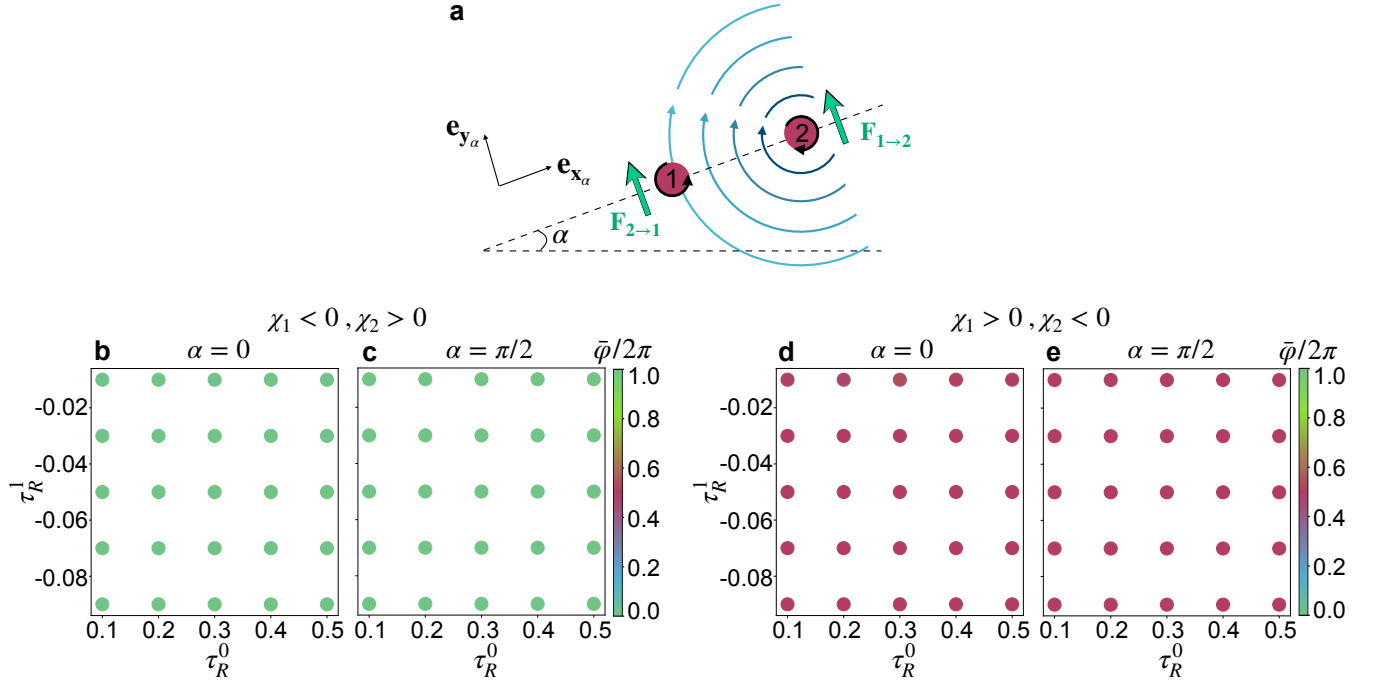


FIG. S16. **Results of Rotlet-type hydrodynamic interactions with modulated driving forces.** **a.** Sketch of the rotlet flows and forces $\mathbf{F}_{i \rightarrow j}$ leading to the interaction torques τ_{ij} . **b-e.** Phase diagrams of the steady-state value of $\bar{\varphi}$ as a function of f_R^0 and f_R^1 . Panels **b** and **d** correspond to $\alpha = 0$, and panels **c** and **e** to $\alpha = \pi/2$. In each diagram, the value of $\bar{\varphi}$ is averaged over 500 time points and four different initial conditions.

strength and found that it merely affects the convergence time towards the phase-coherent states. In all cases, we find that unlike in our experiment, the value of $\bar{\varphi}$ strongly depends on the sign of $\chi_1 = -\chi_2$. We therefore draw the same conclusions: Stokeslet-mediated interactions alone cannot explain the phase organization observed in our metamachines.

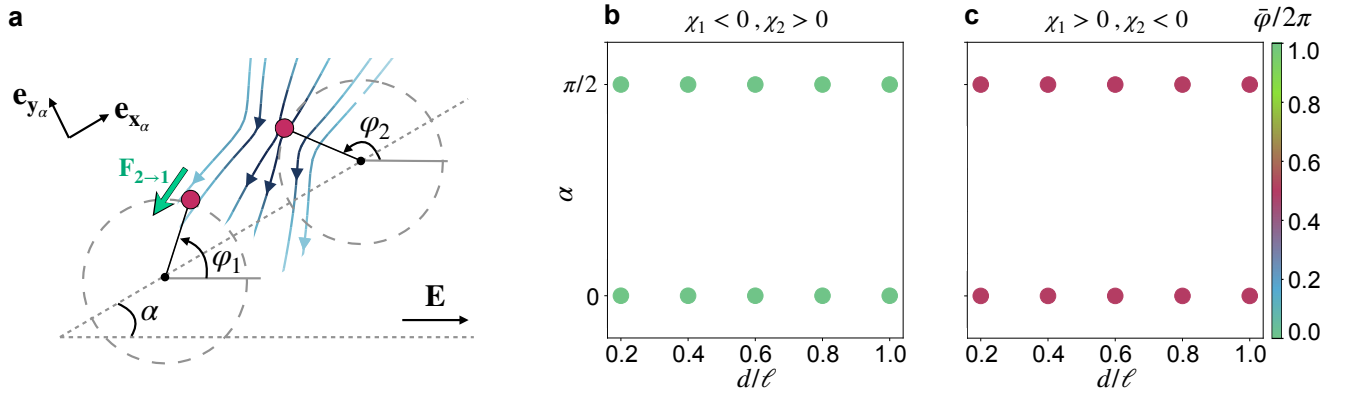


FIG. S17. **Results of Stokeslet-type hydrodynamic interactions with modulated driving forces.** **a.** Sketch of the Stokeslet flows and forces $\mathbf{F}_{i \rightarrow j}$ leading to the interaction torques τ_{ij} . **b-c.** Phase diagrams of the steady-state value of $\bar{\varphi}$ as a function of d/ℓ and α . In each diagram, the value of $\bar{\varphi}$ is averaged over 500 time points and four different initial conditions.

Since our experimental results indicate $\bar{\varphi} = \pi$ regardless of motor size, we conclude that although hydrodynamic interactions can induce phase coherence in the presence of modulation, they can only play a secondary role in the observed phase organization.

3.4 . Near-field Hydrodynamic interactions

In addition to the effect of the far field flows (rotlet and stokeslet), we also consider the effect of near-field hydrodynamics interactions. An accurate model of these lubrication interactions goes beyond the scope of our research. However, we can gain some valuable insight based on a minimal model where the near-field lubrication forces are proportional to the velocity difference of the two rotors, with a drag coefficient ξ that decays rapidly with the inter-rotor distance [8]. The resulting interactions torques take the form $\tau_{21} = \xi(r_{21})(\dot{\varphi}_2 \mathbf{e}_{\varphi_2} - \dot{\varphi}_1 \mathbf{e}_{\varphi_1}) \cdot \mathbf{e}_{\varphi_1}$, where $\mathbf{e}_{\varphi_i}$ is the tangent vector of the trajectory for the i -th rotor. We take arbitrarily $\xi(r_{21}) \propto 1/r_{21}$. The equation of motion for the two phase variable are then given by:

$$\dot{\varphi}_1 (1 + \xi(r_{21})) = 1 - A_2 \cos(2\varphi_1 + \chi_1) + \xi(r_{21})\dot{\varphi}_2 \cos(\varphi_2 - \varphi_1) \quad (\text{S38})$$

$$\dot{\varphi}_2 (1 + \xi(r_{21})) = -1 + A_2 \cos(2\varphi_2 - \chi_2) + \xi(r_{21})\dot{\varphi}_1 \cos(\varphi_2 - \varphi_1) \quad (\text{S39})$$

Then writing these equations for the variables $\bar{\varphi} = \varphi_1 + \varphi_2$ and $\delta\varphi = \varphi_2 - \varphi_1$, we obtain:

$$\dot{\bar{\varphi}} [1 + D/r'_{21} - \cos(\delta\varphi)D/r'_{21}] = -2A_2 \sin(\bar{\varphi} + (\chi_1 + \chi_2)/2) \sin(\delta\varphi + (\chi_2 - \chi_1)/2) \quad (\text{S40})$$

$$\dot{\delta\varphi} [1 + D/r'_{21} + \cos(\delta\varphi)D/r'_{21}] = -2 + 2A_2 \cos(\bar{\varphi} + (\chi_1 + \chi_2)/2) \cos(\delta\varphi + (\chi_2 - \chi_1)/2) \quad (\text{S41})$$

where r'_{21} is defined in Eq.S30 and D is a positive constant that control the strength of the interactions.

As before, we solve these equations numerically and compute the steady-state value of $\bar{\varphi}$. Our conclusions are identical to those drawn for far-field interactions (rotlet and stokeslet). In all cases, we find that unlike in our experiment, the value of $\bar{\varphi}$ strongly depends on the sign of χ_1 and χ_2 (Figs. S18a-d). We therefore draw the same conclusions: our simplified model of near-filed hydrodynamics interactions alone cannot explain the phase organization observed in our metamachines.

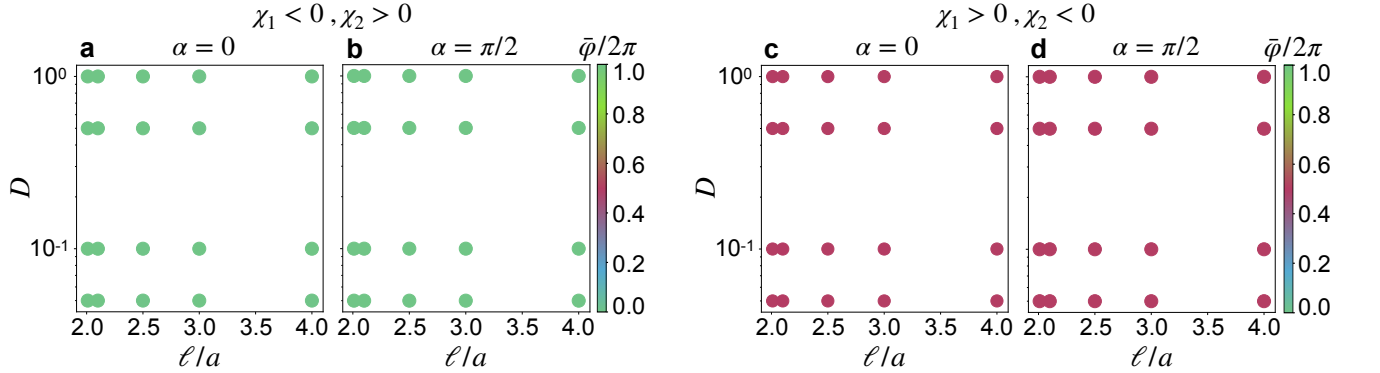


FIG. S18. **Results of Near-field hydrodynamic interactions with modulated driving forces.** Phase diagrams of the steady-state value of $\bar{\varphi}$ as a function of a/ℓ and D . Panels **a** and **c** correspond to $\alpha = 0$, and panels **b** and **d** to $\alpha = \pi/2$. In each diagram, the value of $\bar{\varphi}$ is averaged over 500 time points and four different initial conditions.

4. How to shape the phase-coherent states of phase-oscillator lattices

We close this section with a more general discussion on the phase coherent states accessible to metamachines. We focus on coupled motors whose directions of rotation self-organize into an ordered antiferromagnetic state, as in our experiments. Beyond the specifics of our system, a broader question is: what type of phase-coherent state can be achieved when phase oscillators interact. Having microscopic systems in mind we have investigated the self-organization that can result from longitudinal repulsive forces, transverse forces (Rotlets), long-range hydrodynamic interactions (Stokeslet) and dipolar interactions. We have systematically considered the cases of constant and twofold-modulated drives. We summarize our main findings in the table Fig. S19. We essentially find two types of phase coherent states. (i) States where $\bar{\varphi}$ is homogeneous. They correspond to different phase kinematics along the principal directions of the lattice. They require either anisotropic drives or interactions. In all cases we find that the only possible phase coherent states correspond to $\bar{\varphi} = 0$ or $\bar{\varphi} = \pi$ thereby resulting in the kinematics illustrated in Figure S13. (ii)

States where $\bar{\varphi}$ takes two different values along the x and y axes. In all our simulations we find that $\bar{\varphi}$ measured along the x and y axes can only take the values $(0, \pi)$. This leads to the isotropic phase kinematics depicted in Figure S15.

Our results suggest the possibility of directing the self-organization of metachines into desired phase-coherent states by tuning the strength and symmetry of the dominant interactions.

	No modulation: $\tau_i = \pm \tau_0$		Modulation: $\tau_i = \pm \tau_0 (1 - A_2 \cos(2\varphi + \chi_i))$			
			$\chi_1 < 0, \chi_2 > 0$		$\chi_1 > 0, \chi_2 < 0$	
	$\alpha = 0$	$\alpha = \pi/2$	$\alpha = 0$	$\alpha = \pi/2$	$\alpha = 0$	$\alpha = \pi/2$
Dipolar interactions	π	π	π	π	π	π
Repulsive interactions	0	π	0	π	0	π
Hydrodynamics interactions			0	0	π	π

FIG. S19. **Summary table of the effect of different interactions on phase coherence.** This table summarizes, for each interaction type and parameter set considered, the steady-state value of $\bar{\varphi}$ reached in the stationary regime.

5. Synchronization without contact interactions

We report in Fig. S20 an additional experimental observation that discards contact interactions as the primary source of phase coherence. When the lattice spacing is too large to yield pristine AF order, ordered domains remains (see Figure 2 in the main text). In those domains, we find that the motors precession remains phase coherent even though the rotor never contact. Contact interactions are therefore not the dominant interactions leading to phase coherence in our experiments.

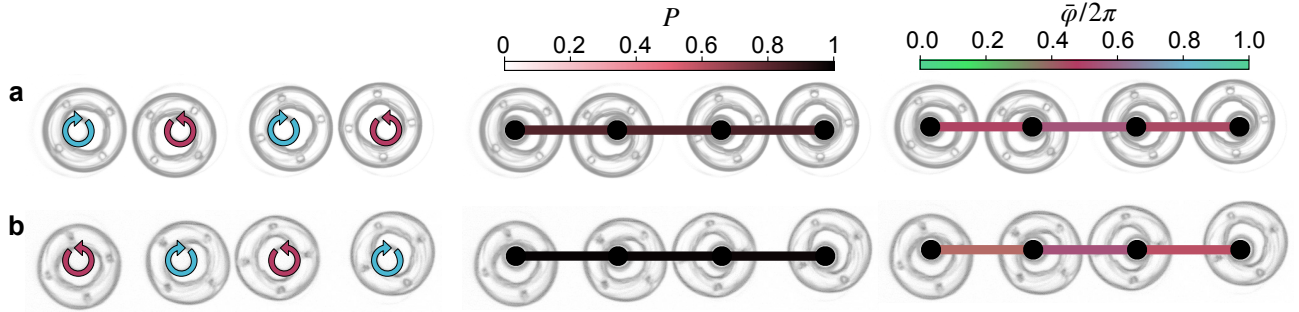


FIG. S20. **Phase coherence without physical contacts.** **a.** Close up on four large rotors in an AF domain. The lattice spacing ℓ is larger than the radius of the precession orbit R so that the rotors never contact. However the corresponding three links support a high level of phase coherence (middle panel), and the corresponding phase sum $\bar{\varphi}$ remains equal to π as found deep in the ordered phase (right panel). Large rotors, $E = 0.75 \text{ V}/\mu\text{m}$. **b.** Same observations in an experiment with small motors, $E = 1.08 \text{ V}/\mu\text{m}$.

6. Oscillations of the phase sum in steady state

Figure 3b in the main text shows that $\bar{\varphi}(t)$ oscillates around the mean value $\bar{\varphi}(t) = \pi$. These oscillations at the same frequency as the average rotation speed ω_0 do not require the rotation speed to be modulated. To see this,

we simulate the dynamics of a pair of rotors with constant natural frequencies $\pm\omega_0$, and interacting through dipolar interactions. Figure S21 shows that in steady state the phase sum oscillate at ω_0 around the mean value $\bar{\varphi} = \pi$.

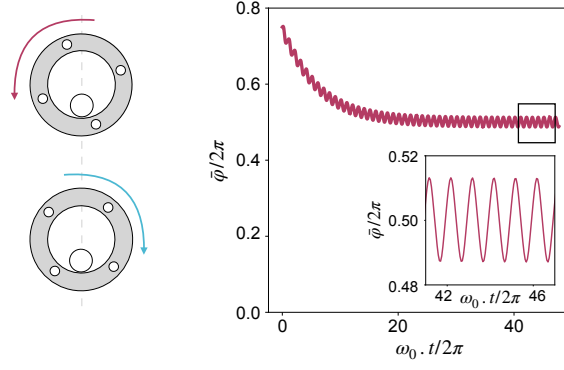


FIG. S21. **Oscillations of the phase sum.** Numerical simulations of two rotors with constant natural frequencies $\pm\omega_0$ and coupled by dipolar forces. In steady state the phase sum oscillates around a constant value (π). $B = 500$, $\ell = 10$.

VI. SYNCHRONIZATION TRANSITION AND PHASE WAVES: THE ROLE OF DISORDER

In this section, we explain the origin of the phase waves observed in the ordered state of the metamachines.

1. Phase coherence in homogeneous motor lattices

To guide our numerical investigation, we first inspect the role of the lattice spacing ℓ and dipole strength B on the self-organization of 10×10 lattices (Figure S22). Informed by our previous analysis we focus on a minimal model where pointwise oscillators are only coupled via dipolar interactions. We impose a perfect antiferromagnetic order for the rotation directions. We neglect the modulation of the rotation speed, and assume that all rotor dipoles point in the same direction, opposite to the electric field $\mathbf{E}$. This simplifying assumption is justified because both the modulation and dipole orientation have a negligible impact on dipolar interactions. Solving Eqs. S24 within a nearest neighbor approximation, we find that over a broad range of parameters the system reaches an ordered steady state with a high degree of phase coherence where $P \simeq 1$ and $\bar{\varphi} = \pi$ when averaged over all the edges. The only noticeable difference between our simulations is the length τ of the transient, which increases with ℓ and decreases with B , as expected. We also note that when the dipolar forces are much stronger than the individual drives, the rotors can freeze their dynamics (Figure S22d).

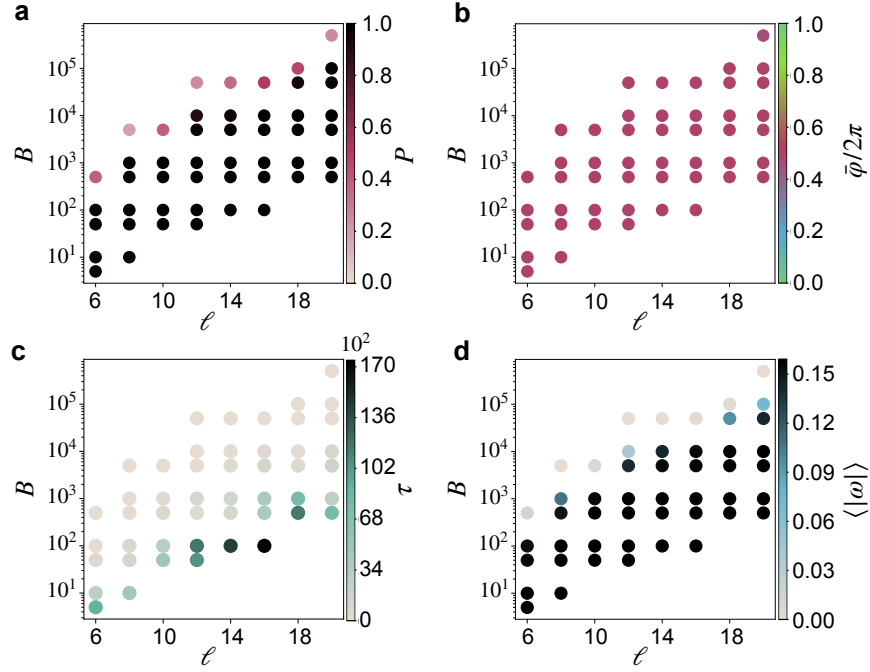


FIG. S22. **Determination of the simulation parameters** **a.** Value of the order parameter $|P|$ for different values of B and ℓ . **b.** Phase difference $\bar{\varphi}$ for different values of B and ℓ . **c.** Convergence time τ for different values of B and ℓ . τ is defined as the time required for P to reach within $\pm 5\%$ of its final value. **d.** Average absolute motor speed $\langle |\omega| \rangle$ for different values of B and ℓ . All data correspond to steady state measurements averaged over five independent realizations.

We performed simulations with open, fixed and periodic boundary conditions (Fig. S23). We find that the value of the phase sum $\bar{\varphi}$ always peak on the value $\bar{\varphi} = \pi$ regardless of the type of boundary conditions in the phase-coherent state. To illustrate the nearly pristine organization of the phase sum, we show the results of simulations with periodic boundary conditions in Fig. S23d using the modified phase variable φ^* ($B = 500$ and $\ell = 10$). φ^* is uniform over the whole lattice at all times: no phase wave propagates in homogeneous motor arrays. (Fig. S23d).

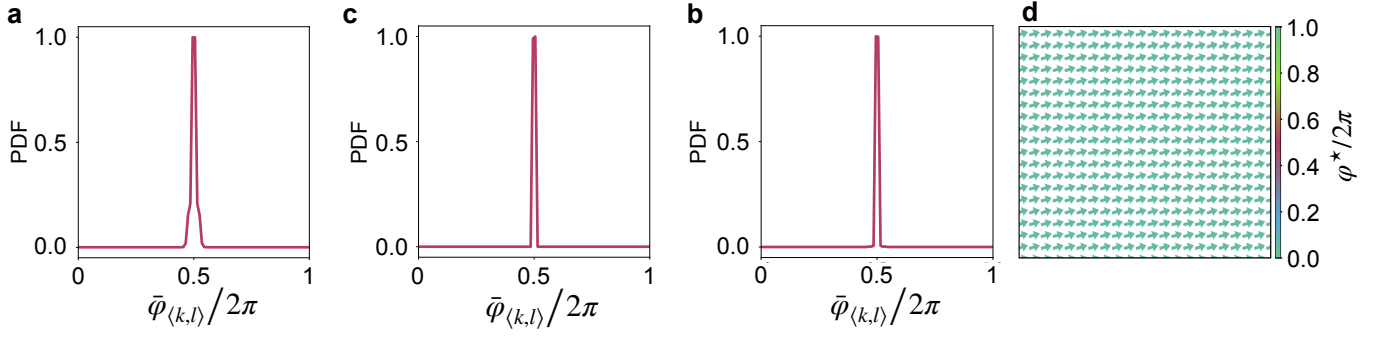


FIG. S23. **Boundary conditions do not alter phase coherence** We plot the histogram of the phase sums $\bar{\varphi}$ measured on each link of the lattice in simulations corresponding to different boundary conditions. Phase coherence is not altered by the type of boundary conditions and $\langle \bar{\varphi} \rangle = \pi$ for **a**. Fixed boundary conditions, **b**. free boundary conditions and **c**. Periodic boundary conditions. **d**. Instantaneous map of the φ^* field in a simulation with periodic boundary conditions. Simulation parameters: $B = 500$, $\ell = 10$. The PDFs are obtained by concatenating the values of $\bar{\varphi}$ measured on all adjacent pairs of motors over more than 3000 time steps in a lattice of 48x44 motors.

2. Disorder in rotation speeds as a source of waves

To assess the impact of disorder, we introduce noise in the driving torques of the motors. The equations of motion take the same form:

$$0 = -\gamma a \dot{\varphi}_i + \tau_i + \sum_{j \neq i} \tau_{ji}, \quad (\text{S42})$$

but the τ_i are now quenched random variables. The sign of τ_i alternates on each site to account for antiferromagnetic ordering, but their amplitudes follow a uniform distribution centered at 1 with width σ . We retain the same parameters as in the previous section: $B = 500$ and $\ell = 10$. However, we now perform simulations with open boundary conditions

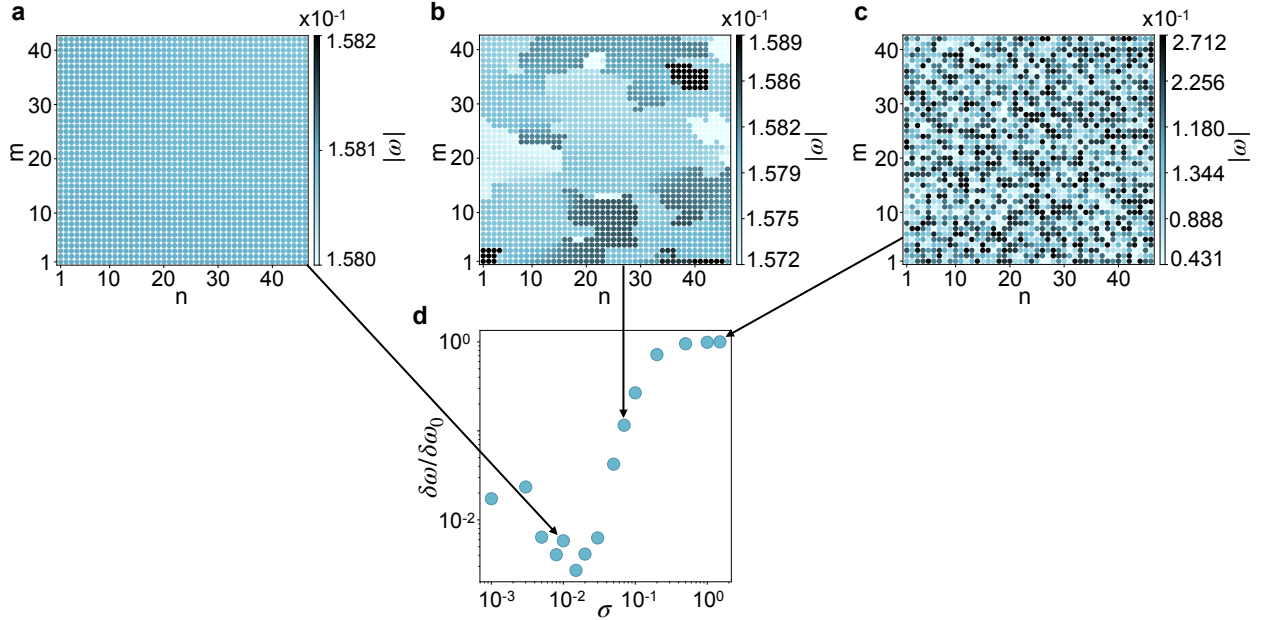


FIG. S24. **Phase-locking transition.** Maps of the absolute rotation speeds $|\omega|$ for **a**. $\sigma = 0.01$, **b**. $\sigma = 0.07$, and **c**. $\sigma = 1.5$. **d**. Standard deviation $\delta\omega$ of the absolute motor rotation speeds, normalized by the standard deviation $\delta\omega_0$ of a uniform distribution of width σ ($\delta\omega_0 = \sigma/\sqrt{12}$), as a function of σ .

2.1 . A Kuramoto-like Phase-locking Transition

When varying σ , we observe a phase-locking transition à la Kuramoto [1, 10, 11]. For low values of σ ($\sigma < 0.03$), the oscillators phase-lock. Despite differences in the driving torques, dipolar interactions organize their angular speeds, leading to a homogeneous state where all rotors orbit at a common rotation rate (Fig. S24a). Increasing the noise strength above $\sigma \simeq 0.1$ all forms of phase-locking are lost and $\delta\omega$ matches the standard deviation of the imposed torques (Fig. S24d). Interactions are not strong enough to phase lock the rotors, leading to uncorrelated disorder in the rotation speeds (Fig. S24c). At intermediate values of σ , ($0.03 < \sigma < 0.1$), the motors are not globally phase-locked but their angular speeds correlates over finite-size domains reminiscent of those seen in our experiments (Fig. S24b). This phenomenology is the antiferromagnetic analogue of the (ferromagnetic) Kuramoto synchronization transition, see Ref. [1] for a comprehensive review.

2.2 . Relationship between angular-speed disorder and phase coherence

In Figs. S25a and S25b, we plot the map of the angular speed next to the map of the phase coherence order parameter P for both our simulations and experiments. Consistently, we find that the phase coherence and angular speed domains coincide. As expected from Kuramoto physics, their size decreases when disorder increases (see Fig. 4d in the main text). This reduction in domain size leads to a gradual decline in the global order parameter P (Figs. 4c and 4e in the main text).

A remarkable consequence of the spatial heterogeneity in the angular speed is the emergence of phase waves propagating freely in all the phase-coherent domains (Fig. S25c).

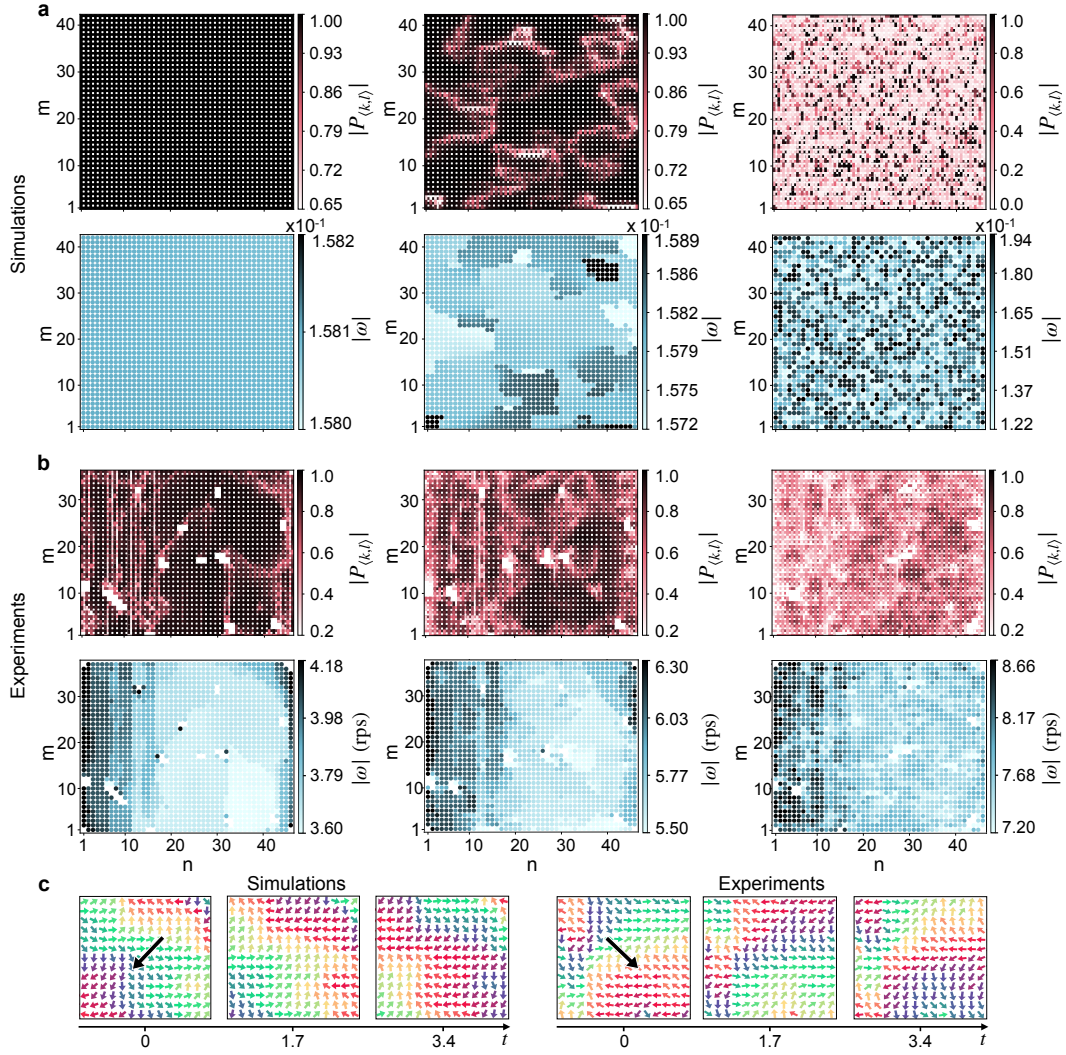


FIG. S25. **Speed disorder and phase coherence.** **a–b.** Maps of the local order parameter $|P_{\langle k,l \rangle}|$ (top) and of the absolute rotation speeds $|\omega|$ (bottom). **a.** Simulations for different noise strengths: $\sigma = 0.01$ (left), $\sigma = 0.07$ (middle), and $\sigma = 0.5$ (right). **b.** Experimental results at increasing field strengths: $E = 1.2 E_Q$ (left), $E = 1.5 E_Q$ (middle), and $E = 1.8 E_Q$ (right); with $E_Q = 0.52 \text{ V}/\mu\text{m}$. **c.** Phase maps φ^* at different times for simulations (left) and experiments (right), showing a wave propagating in the direction indicated by the black arrow.

3. The boundary conditions do not impact the wave spectra

In order to assess whether the phase-wave patterns depends on the boundary conditions at the edges of the motor lattices, we performed three series of numerical simulations changing the type of boundary conditions and keeping all the other parameters identical. Figure VI 2.2.2 shows that the wave spectra are identical for open, rigid and periodic boundary conditions.

4. Wave Generation Mechanism

To elucidate the mechanism behind wave generation, we perform simplified one-dimensional simulations with dipolar interactions and periodic boundary conditions (Lattice size: 100×6). We initialize the system in a perfectly coherent state and induce two distinct velocity domains by imposing two different driving torques in the halves of the lattice (Fig. S27a and c). Given the amplitude of the torque mismatch ($\sigma = 0.1$), the Kuramoto transition cannot operate, leaving two domains where the angular speeds are distinct, thereby preventing the self-organization of a homogeneous phase coherent state (Note that we use the Greek letter σ for the torque mismatch, but we stress that there is no

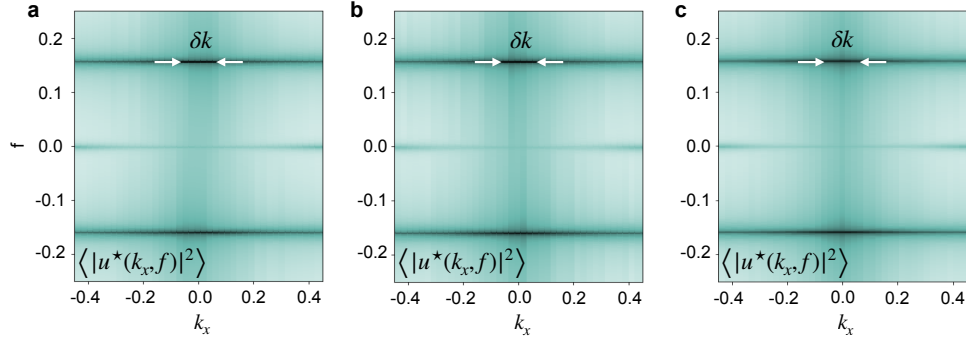


FIG. S26. **Impact of boundary conditions on the wave spectra.** Same wave spectra as in Figure 4 (main text) but for three types of boundary conditions **a.** Open boundary conditions, as in the main text. **b.** Fixed boundary conditions. The rotors are fixed at the edge of the lattice. **c.** Periodic boundary conditions.

quenched disorder).

To quantify the deviations from the homogeneous situation where φ^* is uniform, we compute the phase difference $\delta\varphi_h^*$ on all horizontal edges of the lattice. We find that the phase difference adopts a linear spatial profile in the two domains (Fig. S27d). This results in quadratic spatial variations of φ^* across the domains, with sharp phase variations near the boundaries (Fig. S27e). Since φ^* evolves periodically in time, the spatial gradients manifest as waves propagating freely across the velocity domains (Fig. S27b).

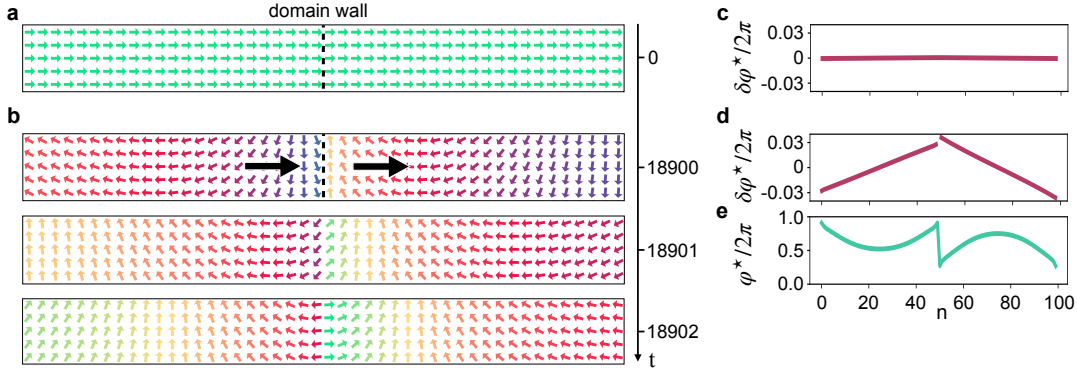


FIG. S27. **Wave generation at domain walls.** **a, b.** Phase maps φ^* as a function of time. **c, d.** Spatial profile of the horizontal phase difference $\delta\varphi_h^*$. **e.** Phase φ^* plotted as a function of n . **a, c.** The system is initialized with a perfect phase coherence with a constant $\bar{\varphi}_h^*$ equal to 0. **b.** After some time, waves appear in the system. They are associated with **d.** a linear profile of $\delta\varphi_h^*$ which induces **e.** a quadratic profile for the phase φ^* . $\sigma = 0.1$, $B = 500$ et $\ell = 10$.

5. Long wave-length dynamics of $\delta\varphi^*$: emergence of phase waves from domain walls

To gain more insight into the propagation of phase waves from the boundaries of the ordered domains, we provide a long-wave-length approximation of the minimal model developed in Section V 1. Within this approximation, we will show that the modified phase difference $\delta\varphi^*$ is a slow variable whose dynamics is diffusive. We will then use this central results to account for our experimental and numerical observations.

We consider an infinite chain of rotors aligned along the electric field, neglect the modulation of the rotation speed and focus on dipolar interactions. For the sake of simplicity, we consider the limit $\ell \gg a$ where the dipolar forces are predominantly longitudinal, and linearize the equations of motion Eqs. (S26) and (S27) as in Eqs. (S28) and (S29). Given these simplifying hypotheses, the equations of motion take the compact form

$$\dot{\varphi}_n = (-1)^n + \tau_D^1 \sin \varphi_n (\cos \varphi_{n+1} + \cos \varphi_{n-1} - 2 \cos \varphi_n), \quad (\text{S43})$$

where the coupling torque τ_D^1 is a positive constant that is set by the dipole strength and lattice spacing. Assuming even-numbered rotors rotate counterclockwise and odd-numbered ones clockwise. The dynamics of the modified phase

field can then be expressed as:

$$\partial_t \varphi_n^* = 1 - \tau_D^1 \sin \varphi_n^* (\cos \varphi_{n+1}^* + \cos \varphi_{n-1}^* + 2 \cos \varphi_n^*) \quad (\text{S44})$$

and that of the phase difference $\delta \varphi_n^* = \varphi_{n+1}^* - \varphi_n^*$:

$$\begin{aligned} \partial_t \delta \varphi_n^* = -\frac{\tau_D^1}{2} & [-\sin(\delta \varphi_{n-1}^*) - 2 \sin(2\varphi_n^*) - \sin(\varphi_{n-1}^* + \varphi_n^*) \\ & + 2 \sin(\delta \varphi_n^*) + 2 \sin(2\varphi_{n+1}^*) - \sin(\delta \varphi_{n+1}^*) + \sin(\varphi_{n+1}^* + \varphi_{n+2}^*)] \end{aligned} \quad (\text{S45})$$

Assuming a high level of phase coherence, where $\delta \varphi^*$ remains small, we can linearize the equation:

$$\partial_t \delta \varphi_n^* = -\frac{\tau_D^1}{2} (-\delta \varphi_{n-1}^* + 2\delta \varphi_n^* - \delta \varphi_{n+1}^*) \quad (\text{S46})$$

$$-\frac{\tau_D^1}{2} [-2 \sin(2\varphi_n^*) - \sin(\varphi_{n-1}^* + \varphi_n^*) + 2 \sin(2\varphi_{n+1}^*) + \sin(\varphi_{n+1}^* + \varphi_{n+2}^*)] \quad (\text{S47})$$

Taking the continuum limit and assuming that $\delta \varphi^*$ and φ^* vary slowly in space, we find:

$$\partial_t \delta \varphi^*(x, t) = D \Delta \delta \varphi^*(x, t) - 4\tau_D^1 \delta \varphi^*(x, t) \cos[2\varphi^*(x, t)] \quad (\text{S48})$$

with $D = \frac{\tau_D^1}{2} \ell^2$. We further simplify the equation by approximating $\varphi^*(x, t)$ by its leading-order term $\varphi^*(x, t) \simeq \omega t$. A spatial Fourier transform then yields:

$$\partial_t \delta \varphi_q^* = -Dq^2 \delta \varphi_q^* - 4\tau_D^1 \delta \varphi_q^* \cos(2\omega t) \quad (\text{S49})$$

which we can readily integrate to find how each Fourier mode evolves in time:

$$\delta \varphi_q^* = B \exp \left(-Dq^2 t - \frac{4\tau_D^1}{2\omega} \sin(2\omega t) \right) \quad (\text{S50})$$

In the limit of small qs , the term $\exp(-Dq^2 t)$ varies slowly on the timescale of one period. We can therefore average the dynamics over a rotation period T . We then find

$$\langle \delta \varphi_q^* \rangle_T = B' \exp(-Dq^2 t) \quad (\text{S51})$$

where $B' = \left\langle \exp \left(-\frac{4\tau_D^1}{2\omega} \sin(2\omega t) \right) \right\rangle_T$. In conclusion, in the long-wavelength limit, the time-averaged phase difference $\langle \delta \varphi_h^* \rangle_T$ obeys a diffusion equation:

$$\partial_t \langle \delta \varphi_h^* \rangle_T = D \Delta \langle \delta \varphi_h^* \rangle_T \quad (\text{S52})$$

This diffusive dynamics explains the linear evolution of $\delta \varphi^*$ in a domain as discussed in Section VI 4. Indeed, the Green's function of the Laplacian in one dimension is a linear function. In steady state, linear variations of the modified phase difference are therefore the only possible solutions of the dynamics. This asymptotic behavior hence translates into quadratic variations of the modified phase away from one-dimensional domain walls in excellent agreement with our numerical simulations.

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- [1] J. A. Acebrón, L. L. Bonilla, C. J. Pérez Vicente, F. Ritort, and R. Spigler. The kuramoto model: A simple paradigm for synchronization phenomena. *Rev. Mod. Phys.*, 77:137–185, Apr 2005.
 - [2] S. Box, L. Debono, D. Phillips, and S. Simpson. Transitional behavior in hydrodynamically coupled oscillators. *Physical Review E*, 91(2):022916, 2015.

- [3] A. Bricard, J.-B. Caussin, N. Desreumaux, O. Dauchot, and D. Bartolo. Emergence of macroscopic directed motion in populations of motile colloids. *Nature*, 503(7474):95–98, nov 2013.
- [4] D. R. Brumley, M. Polin, T. J. Pedley, and R. E. Goldstein. Metachronal waves in the flagellar beating of volvox and their hydrodynamic origin. *Journal of the Royal Society Interface*, 12(108):20141358, 2015.
- [5] S. Ceron, G. Gardi, K. Petersen, and M. Sitti. Programmable self-organization of heterogeneous microrobot collectives. *Proceedings of the National Academy of Sciences*, 120(24):e2221913120, 2023.
- [6] Y.-C. Chao, S. Gokhale, L. Lin, A. Hastewell, A. Bacanu, Y. Chen, J. Li, J. Liu, H. Lee, J. Dunkel, et al. Selective excitation of work-generating cycles in nonreciprocal living solids. *arXiv preprint arXiv:2410.18017*, 2024.
- [7] M. F. Hsu, E. R. Dufresne, and D. A. Weitz. Charge stabilization in nonpolar solvents. *Langmuir*, 21(11):4881–4887, 2005.
- [8] S. Kim and S. J. Karrila. *Microhydrodynamics: principles and selected applications*. Butterworth-Heinemann, 2013.
- [9] J. Kotar, L. Debono, N. Bruot, S. Box, D. Phillips, S. Simpson, S. Hanna, and P. Cicuta. Optimal hydrodynamic synchronization of colloidal rotors. *Physical review letters*, 111(22):228103, 2013.
- [10] Y. Kuramoto. Self-entrainment of a population of coupled non-linear oscillators. In *International symposium on mathematical problems in theoretical physics: January 23–29, 1975, kyoto university, kyoto/Japan*, pages 420–422. Springer, 1975.
- [11] Y. Kuramoto and Y. Kuramoto. *Chemical turbulence*. Springer, 1984.
- [12] K. A. Materials. Omnicoat. 2020.
- [13] K. A. Materials. Su-8 permanent negative epoxy photoresist. 2020.
- [14] J. Melcher and G. Taylor. Electrohydrodynamics: a review of the role of interfacial shear stresses. *Annual review of fluid mechanics*, 1(1):111–146, 1969.
- [15] F. Meng, R. R. Bennett, N. Uchida, and R. Golestanian. Conditions for metachronal coordination in arrays of model cilia. *Proceedings of the National Academy of Sciences*, 118(32):e2102828118, 2021.
- [16] F. Peters, L. Lobry, and E. Lemaire. Experimental observation of Lorenz chaos in the Quincke rotor dynamics. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 15(1):013102, Mar. 2005.
- [17] G. E. Pradillo, H. Karani, and P. M. Vlahovska. Quincke rotor dynamics in confinement: rolling and hovering. *Soft matter*, 15(32):6564–6570, 2019.
- [18] G. Quincke. Ueber rotationen im constanten electrischen felde. *Annalen der Physik*, 295(11):417–486, 1896.
- [19] S. Smale, M. W. Hirsch, and R. L. Devaney. *Differential Equations, Dynamical Systems, and an Introduction to Chaos*. Elsevier Science, 2003.
- [20] G. N. Smith, J. E. Hallett, and J. Eastoe. Celebrating soft matter’s 10th anniversary: Influencing the charge of poly (methyl methacrylate) latexes in nonpolar solvents. *Soft Matter*, 11(41):8029–8041, 2015.
- [21] V. Soni, E. S. Bililign, S. Magkiriadou, S. Sacanna, D. Bartolo, M. J. Shelley, and W. T. Irvine. The odd free surface flows of a colloidal chiral fluid. *Nature physics*, 15(11):1188–1194, 2019.
- [22] T. H. Tan, A. Mietke, J. Li, Y. Chen, H. Higinbotham, P. J. Foster, S. Gokhale, J. Dunkel, and N. Fakhri. Odd dynamics of living chiral crystals. *Nature*, 607(7918):287–293, 2022.
- [23] N. Uchida and R. Golestanian. Synchronization and collective dynamics in a carpet of microfluidic rotors. *Phys. Rev. Lett.*, 104:178103, Apr 2010.
- [24] N. Uchida and R. Golestanian. Generic conditions for hydrodynamic synchronization. *Phys. Rev. Lett.*, 106:058104, Feb 2011.
- [25] B. Zhang, H. Yuan, A. Sokolov, M. O. de la Cruz, and A. Snezhko. Polar state reversal in active fluids. *Nature Physics*, 18(2):154–159, 2022.