

An algorithm to generate two-dimensional critical lattice models using competing anyon condensation

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Contents

1	Pair competitions in Ak, Zp and H3 categories	2
2	Three-condensate competitions in the Haagerup H3 strange correlator	3
3	Condensation Tree and Determining the Amount of Symmetries Preserved	5
4	Cutting Down Phase Space to Avoid First Order Phase Transitions	10
5	Generalised Kramers–Wannier Duality	14
6	Symmetry-Preserving RG of the Strange Correlator	17
	Appendices	21
A	The HGW String-Net Model	21
A.1	Definitions of Frobenius Algebras and Modules	23
A.2	Anyon Condensation in the HGW Model	24
A.3	Lagrangian Algebra of Anyon Condensation	27
B	Categorical data for the examples	28
B.1	The A series	28
B.2	The ZN data	28
B.3	The Modular Tensor Category $\mathcal{Z}(S_3)$	29
B.4	Haagerup Fusion Category H_3	30
C	Critical Strange Correlator of the Ising Category	32
D	The Effect of the Collection of Objects $M_{\mathcal{A}_i} \neq M_{\mathcal{A}_j}$ in the interpolation in a unit cell	33
E	Locating the tricritical point in the A5 phase diagram	34
F	RG flow of the numerical central charge	35
G	More Complicated Boundary Conditions and Enlarging the Unit Cell	39

1 Pair competitions in A_k , $\mathbb{Z}_p$ and H_3 categories

For two Frobenius algebras $\mathcal{A}_i$ and $\mathcal{A}_j$ sharing a common module M and module action ρ_M , the pairwise competition state

$$\langle C_{ij} | = \langle \hat{\mathcal{A}}_i | + \langle \hat{\mathcal{A}}_j |$$

is the transition point predicted by Eq. (10). This section records the representative two-condensate examples summarised in main-text Table 1. The corresponding algebra and module data are collected in Appendix B.

For $\mathcal{F} = A_{k+1}$, the simple objects are $\{0, 1, \dots, k\}$. For $k \geq 2$, the two Frobenius algebras

$$\mathcal{A}_0 = 0, \quad \mathcal{A}_1 = 0 \oplus 2$$

share the module given by the simple object 1. Equation (10) then reproduces the critical A -series integrable lattice models, which flow to the A -series minimal CFTs with

$$C = 1 - \frac{6}{(k+1)(k+2)}.$$

In these examples the middle edge of the unit cell can be 0 or 2, while the common module is 1. Hence the unit cell is two-dimensional before overall normalisation and effectively one-dimensional after normalisation. This reduction of the effective local degrees of freedom is the mechanism discussed in Section 4 for avoiding generic first-order transitions. These critical points are also the usual Kramers–Wannier self-dual points, here reinterpreted as equal-weight competitions of normalised non-commuting condensates.

For $\mathcal{F} = \mathbb{Z}_p$, with p prime, the simple objects are the group elements $\{0, \dots, p-1\}$. The two Frobenius algebras

$$\mathcal{A}_e = 0, \quad \mathcal{A}_m = \bigoplus_{i=0}^{p-1} i$$

describe pure electric and pure magnetic condensates in the $\mathbb{Z}(\mathbb{Z}_p)$ topological order. They share the module $M = \mathcal{A}_m$, since $\mathcal{A}_m$ is both its own module and a module of $\mathcal{A}_e$. Equation (10) gives the p -state Potts model at the Kramers–Wannier self-dual point. The cases $p = 2$ and $p = 3$ are continuous and reproduce the critical Ising and critical three-state Potts models. For prime $p \geq 5$, the same self-dual points are first-order transitions. The $p = 5$ case is weakly first order and related to a complex CFT [1, 2]; in our RG flow the numerical central charge slowly decays from $C \approx 1.1$, consistent with the real part of the complex value $C \approx 1.138 \pm 0.021i$. The corresponding flows are shown in Supplementary Figure 18.

For the Haagerup category H_3 , we use the algebra and module data in Appendix B.4. The pair competition between

$$\mathcal{A}_0 = \mathbf{1}, \quad \mathcal{A}_1 = \mathbf{1} \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho$$

uses the common module object M_H in Eq. (2). This construction is likely to produce the Haagerup anyon chain proposed in Ref. [3]. Initial fits of the entanglement entropy

suggested a central charge close to 2, similar to Ref. [3]. However, this point has not stabilised as a fixed point under our RG algorithm at increased bond dimensions, and its status as a continuous transition requires further scrutiny.

A second important pair competition uses

$$\mathcal{A}_0 = \mathbf{1}, \quad \mathcal{A}_2 = \mathbf{1} \oplus \alpha \oplus \alpha^2.$$

Here one can choose the common module to be $\mathcal{A}_2$ itself. The transfer matrix generated by Eq. (10) then yields two decoupled direct sums of critical three-state Potts models, matching the model considered in Ref. [4].

Other pair competitions in H_3 do not satisfy the sufficient condition in Section 4 for continuous transitions and appear first-order in our RG calculations. In particular, the pair competitions between $\mathcal{A}_0$ and each of the three mutually conjugate algebras $\mathcal{A}_{4,5,6}$ show slowly decaying numerical central charges from $C \approx 1.3$, suggesting weak first-order behavior analogous to the five-state Potts model. These RG flows are shown in Supplementary Figure 20.

2 Three-condensate competitions in the Haagerup H_3 strange correlator

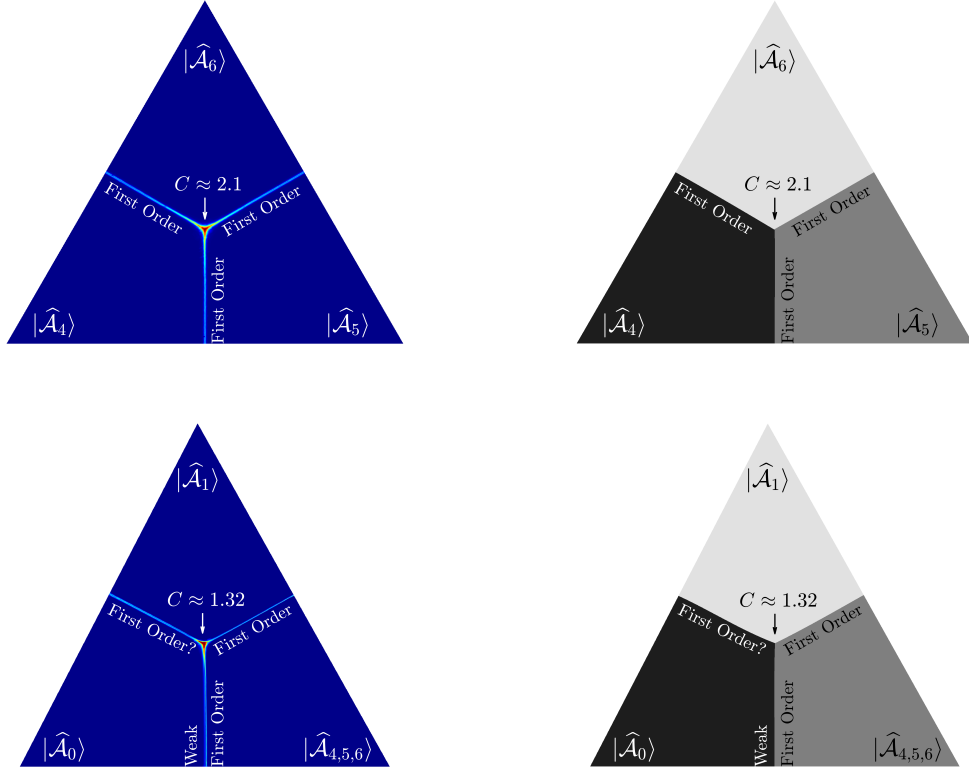
This section gives the details behind the ternary Haagerup phase diagrams summarised in the main text. All examples use the Frobenius algebras listed in Eq. (1) and the common underlying module object M_H in Eq. (2), though the corresponding module actions can differ.

For the A_k ternary phase diagrams, all bipartite phase transitions are second order and converge within only a few RG steps. In this case, using the fully converged numerical central charge to characterise the phase diagram is well justified. In contrast, for H_3 strange correlators, most bipartite phase transitions are first order and typically require substantially more RG steps to reach convergence. In the main text, we therefore present the phase diagram of tripartite algebra competition in H_3 using the non-converged numerical central charge obtained at finite RG depth.

A more stringent characterisation of the phase diagram is obtained by colouring each point according to the Frobenius algebra to which the RG tensor converges. In Supplementary Figure 1, we present these two types of phase diagrams side by side for direct comparison. The two constructions are in perfect agreement, thereby justifying the use of the non-converged numerical central charge as a reliable indicator of phase boundaries in the H_3 strange correlators.

Competitions among $\mathcal{A}_{4,5,6}$

The algebras $\mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6$ are related by conjugation with the invertible objects $\{\mathbf{1}, \alpha, \alpha^2\}$, so the ternary phase diagram has a manifest $\mathbb{Z}_3$ permutation symmetry. Since the three module actions differ, the pairwise transitions among $\mathcal{A}_{4,5,6}$ are not expected to be generically continuous, consistent with the numerical RG results. The equal-weight tricritical point nevertheless appears to flow to a nontrivial conformal fixed point with effective central



Supplementary Figure 1: Comparison of two types of phase diagrams for H_3 strange correlators. Left panels are greyscaled by the non-converged numerical central charge at finite RG depth, while right panels are coloured according to the Frobenius algebras to which the RG tensor converges. The two representations are in perfect agreement, indicating that the non-converged numerical central charge provides an equally reliable signature of phase transitions.

charge

$$C \approx 2.11.$$

Competitions among $\mathcal{A}_{0,1,(4,5,6)}$

We next include $\mathcal{A}_0$ and $\mathcal{A}_1$ together with one of the three conjugate algebras $\mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6$. The three choices

$$(\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_4), \quad (\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_5), \quad (\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_6)$$

are related by the same $\mathbb{Z}_3$ symmetry and share the phase diagram shown in the main text. Along the tensor-network RG flow, the numerical central charge first shows a transient plateau near $C \approx 2.0$ and then drifts toward a stable plateau near

$$C \approx 1.32.$$

We identify the tricritical point as a Haagerup-symmetric conformal fixed-point candidate.

Competitions among $\mathcal{A}_{0,(4,5,6),(5,6,4)}$

There are three symmetry-related interpolations chosen from $\mathcal{A}_{0,4,5,6}$:

$$(\mathcal{A}_0, \mathcal{A}_4, \mathcal{A}_5), \quad (\mathcal{A}_0, \mathcal{A}_5, \mathcal{A}_6), \quad (\mathcal{A}_0, \mathcal{A}_6, \mathcal{A}_4).$$

Their collective phase diagram is shown in the main text. The numerical central charge initially approaches $C \approx 1.8$ before converging to a stable plateau near

$$C \approx 1.31.$$

We therefore identify the corresponding tricritical point as a second Haagerup-symmetric CFT candidate near $C \simeq 1.3$.

Competitions involving $\mathcal{A}_0$ and $\mathcal{A}_2$

Finally, we consider tripartite competitions involving $\mathcal{A}_0$ and $\mathcal{A}_2$. The pair competition between $\mathcal{A}_0$ and $\mathcal{A}_2$ gives a decoupled direct sum of critical three-state Potts models with $C = 0.8$. The tricritical points associated with $(\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2)$ and $(\mathcal{A}_0, \mathcal{A}_2, \mathcal{A}_{4,5,6})$ also appear to give effective central charges close to 0.8, though with greater numerical instability. The main text shows the phase diagram of $(\mathcal{A}_0, \mathcal{A}_2, \mathcal{A}_{4,5,6})$, whose effective central charge is

$$C \approx 0.805.$$

Across all four ternary diagrams, the numerically extracted phase boundaries agree well with the prediction of main-text Eq. (8): they align with the bisectors of the ternary triangles and meet near tricritical points. The RG-flow diagnostics for the quoted effective central charges are given in Supplementary Figure 20. A more precise determination of central charges and operator spectra for these H_3 CFT candidates is left for future work.

3 Condensation Tree and Determining the Amount of Symmetries Preserved

Anyon condensation is mathematically described by a twist-free, commutative, separable Frobenius algebra (CSFA) in the symTO characterised by the UMTC $\mathcal{Z}(\mathcal{F})$. One twist-free CSFA¹ $\mathbf{A}_1$ can be a sub-algebra of a bigger one $\mathbf{A}_2$, i.e., $\mathbf{A}_1 \subset \mathbf{A}_2$. When this occurs, $\mathbf{A}_2$ can be condensed in two steps: first condensing anyons in $\mathbf{A}_1$, then condensing some other unconfined anyons over the $\mathbf{A}_1$ condensate to obtain the aggregate $\mathbf{A}_2$ condensate. Therefore, the sub-algebra $\mathbf{A}_1$ can be considered as a parent of $\mathbf{A}_2$. Such considerations produce a hierarchy of condensates [5–7] that can be conveniently arranged as a tree, which was dubbed a Hasse diagram [7], where each node corresponds to a condensable algebra, and an edge connects one node to a subalgebra node. Lagrangian algebras are those twist-free CSFAs corresponding to maximal condensates – such that the resultant phase is topologically trivial – that would appear as bottom nodes of the tree. That is,

¹We use a bold letter $\mathbf{A}$ for a CSFA in the output data $\mathcal{Z}(\mathcal{F})$ of the HGW model, compared with the curly letter $\mathcal{A}$ for a Frobenius algebra in an input UFC $\mathcal{F}$.

we start from the trivial twist-free CSFA including only the identity object, which is a sub-algebra to all other algebras, as the top node of the tree and grow the tree downwards.

The key refinement of our condensation tree over Hasse diagrams previously considered [7] is the inclusion of Morita equivalent Frobenius algebras in the input category $\mathcal{F}$ as physically distinguishable condensates. While these algebras correspond to the same set of condensed anyons in the output theory, they produce physically distinguishable gapped phases [8–10], such that non-trivial phase transitions between them can in fact occur, as is evident in the previous discussion. They correspond to distinct symmetry breaking patterns that arise from non-trivial internal degrees of freedom of the condensing anyons[9–11], and such details are made explicit in the HGW model.

Meanwhile, we note that intermediate phases where some non-maximal set of anyons condense in the output theory can be described by sub-category² $\mathcal{K}$ of the input category $\mathcal{F}$ [9]. This is natural since a non-trivial condensate would take the bulk symTO to a child topological order, and the corresponding degrees of freedom on the lattice are expected to be reduced to some subset of $\mathcal{F}$. When the condensate L_i in the symTO is Lagrangian, the corresponding HGW string-net model is projected effectively down to one-dimensional, reflecting the fact that the corresponding condensed phase is topologically trivial. In the condensed phase, each edge is projected to the chosen Frobenius algebra $\mathcal{A}_i$ corresponding to L_i . For instance, if we select the trivial Frobenius algebra $\mathcal{A}_0 = \mathbf{1}$, the resulting child model has only a single physical state, where every edge is labelled by the trivial degree of freedom 1.

We are now ready to introduce the refined condensation tree as follows. Each node is labelled by a subcategory $\mathcal{K}$ of $\mathcal{F}$. Note that the identity object of a subcategory $\mathcal{K}$ is, in general, a composite object in $\mathcal{F}$ unless $\mathcal{K}$ is a full subcategory. The requirement that $\mathcal{K}$ possesses an identity object is precisely the requirement that there exists a separable Frobenius algebra – after all, the defining property of a separable Frobenius algebra is that the composite object $\mathcal{A}$ is strictly associative under fusion, and that “bubbles” (trivial summands) can be freely included or removed. Thus, the identity object of $\mathcal{K}$ is given by a Frobenius algebra $\mathcal{A}$, and the remaining objects in $\mathcal{K}$ are precisely the bimodules over $\mathcal{A}$.

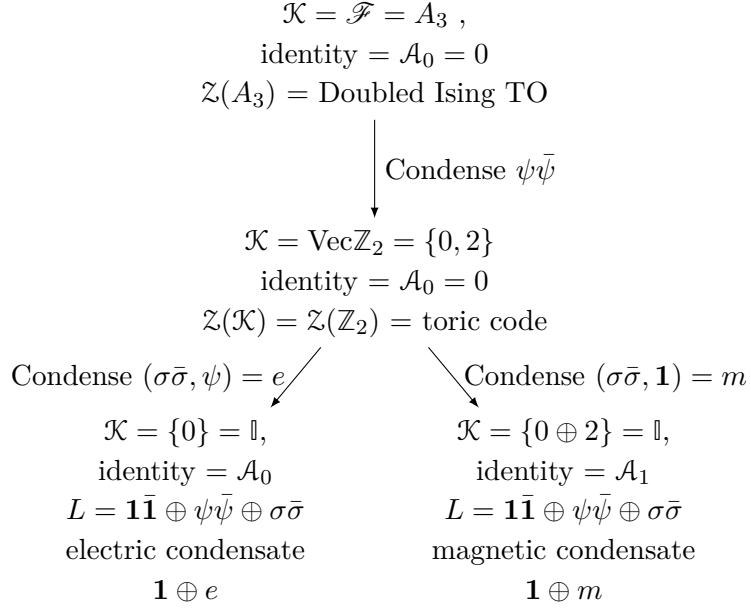
Two nodes are connected by an edge if the identity object of $\mathcal{K}_i$, given by a Frobenius algebra $\mathcal{A}_i$, is a subalgebra of the identity object $\mathcal{A}_j$ of $\mathcal{K}_j$, and $\mathcal{A}_j$ can be obtained from $\mathcal{A}_i$ as a Frobenius algebra in the subcategory $\mathcal{K}_i$.

The above statements are essentially rephrasing the relation between two nodes in the original Hasse diagram – namely that a node is connected to another if the latter phase can be obtained from the former by anyon condensation. The refined condensation tree contains more nodes in general.

Here are a few examples. The simplest example is the symTO corresponding to the doubled Ising model. Recall that there are three objects denoted $\mathbf{1}, \sigma, \psi$ in the input Ising UFC, and there are two connected Frobenius algebras $\mathcal{A}_0 = \mathbf{1}$ and $\mathcal{A}_1 = \mathbf{1} \oplus \psi$. The corresponding refined condensation tree is shown in Supplementary Figure 2.

A more interesting example is the condensation tree for $k = 4$.

²A simple object of $\mathcal{K}$ may be a nonsimple object of $\mathcal{F}$.



Supplementary Figure 2: Condensation tree for the doubled Ising TO. Each link is labelled with the output bulk anyons that condensed. Each node is a subcategory and also the corresponding output condensed phase.

In the refined condensation tree for the symTO $\mathcal{Z}(A_5)$,

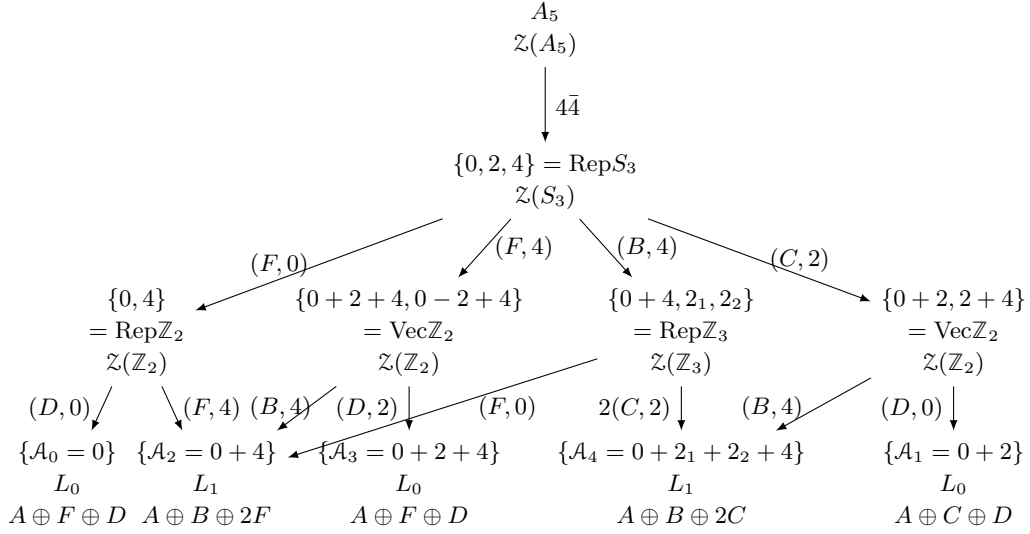
$$L_0 = \oplus_{i=0}^4 i\bar{i}, \quad L_1 = (0 \oplus 4)\overline{(0 \oplus 4)} \oplus 2\bar{2}. \quad (1)$$

At the bottom of Supplementary Figure 3, we express each Lagrangian algebra also as composed of anyons in the intermediate condensed phase $\mathcal{Z}(S_3)$. The latter has 8 different anyons, and they are related to the anyons of $\mathcal{Z}(A_5)$ as shown in the following table.

It is evident from a refined condensation tree that there is a notion of distance between Frobenius algebras. Down the tree, nodes denote symTO with more symmetries spontaneously broken by anyon condensation. The minimal amount of symmetry preserved in the critical state $\langle \Omega \rangle$ obtained from the competing pair of Frobenius algebras—assuming that we managed to choose appropriate modules M to sieve out first-order phase transitions – **is determined by the first common ancestor node of the pair**. The topological excitations carried by the symTO of the ancestor node are the minimal set of symmetries that are preserved by the gapless phase. The resulting CFT may exhibit emergent symmetries beyond those of the ancestor node but cannot have fewer.

Let us revisit the examples we have considered to illustrate these ideas.

1. **Critical theory from the doubled Ising symTO** In the case of the Ising spin model (or the A_3 model), the first common ancestor of $\mathcal{A}_0$ and $\mathcal{A}_1$ is the node corresponding to $\mathcal{Z}(\mathbb{Z}_2)$, as shown in Supplementary Figure 2. Therefore the minimal set of topological symmetries is characterised by the toric code order, which has 3 non-trivial topological excitations corresponding to a $\mathbb{Z}_2$ electric charge, a magnetic



Supplementary Figure 3: The condensation tree for the doubled A_5 symTO. In the second layer of arrows we indicate the intermediate condensing anyons using anyon labels in $\mathcal{Z}(S_3)$ to avoid clutter. In the last layer, we also include the condensate as anyons in the $\mathcal{Z}(S_3)$ phase. As we move down the tree, simple anyons in the partially condensed symTO corresponding to the lower nodes are composites in the parent nodes. Hence, the condensing anyons expressed in terms of $\mathcal{Z}(S_3)$ anyon that label the bottom layer of arrows are only representatives of a group of anyons in $\mathcal{Z}(S_3)$ that are identified in the child $\mathbb{Z}_2$ and $\mathbb{Z}_3$ nodes. The second index in the bracket indicates the microscopic degrees of freedom that take part in the condensation. The meanings of these symbols are explained in the previous section.

flux, and their bound state. The actual symmetry of the Ising CFT is enlarged to the full Ising category, describable by the top node.

2. Critical theories from the A_5 symTO

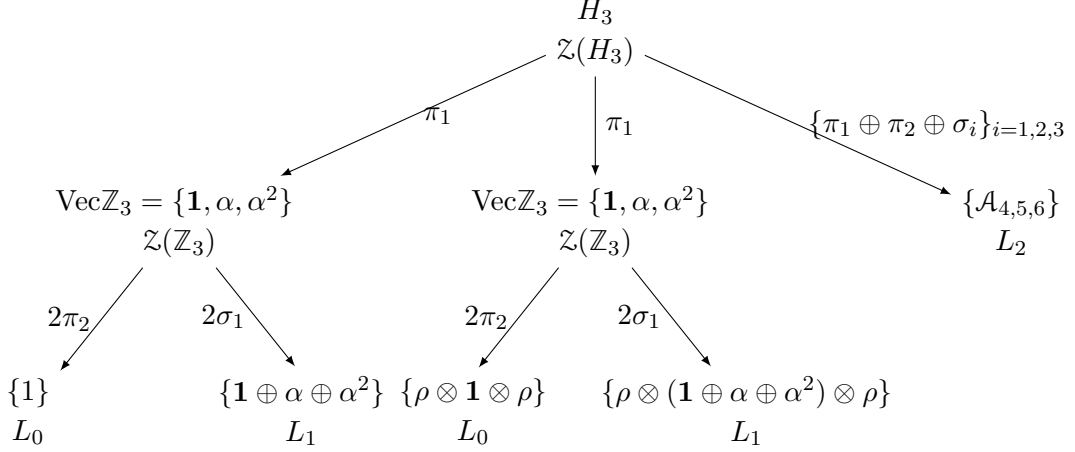
- (a) The first common ancestor of $\mathcal{A}_0$ and $\mathcal{A}_2$ is $\mathcal{Z}(\mathbb{Z}_2)$. Their competition is identical to the case of the Ising spin model and we expect the critical point to be the Ising CFT. Similarly, the competition of $\mathcal{A}_2$ and $\mathcal{A}_3$ is also expected to be Ising.
- (b) The first common ancestor of $\mathcal{A}_2$ and $\mathcal{A}_3$ is $\mathcal{Z}(\mathbb{Z}_3)$. As we discussed above, the electric/magnetic competition in $\mathcal{Z}(\mathbb{Z}_3)$ in a unit cell produces the 3-state Potts model. This is indeed inherited in the competition between $\mathcal{A}_2$ and $\mathcal{A}_3$.
- (c) The competition between $\mathcal{A}_0$ and $\mathcal{A}_3$ should preserve at least symmetries in $\mathcal{Z}(S_3)$, which is their common ancestor. This is indeed confirmed. The critical point constructed by their equilibrium point is given by a free boson CFT with $C = 1$.
- (d) The competition between $\mathcal{A}_0$ and $\mathcal{A}_4$ should also preserve at least the symmetry imposed by $\mathcal{Z}(S_3)$. In fact, this CFT is the minimal model in the A series at

$\mathcal{Z}(\text{Rep}S_3)$ Anyon Type	$\mathcal{Z}(\text{Rep}S_3)$ Dyon Type	$Z(A_5)$ Dyon Types
A	$(A, 0)$	$(0\bar{0}, 0), (4\bar{4}, 0)$
B	$(B, 4)$	$(0\bar{4}, 4), (4\bar{0}, 4)$
C	$(C, 2)$	$(2\bar{2}, 2)$
D	$(D, 0)$	$(1\bar{1}, 0), (3\bar{3}, 0)$
	$(D, 2)$	$(1\bar{1}, 2), (3\bar{3}, 2)$
E	$(E, 2)$	$(1\bar{3}, 2), (3\bar{1}, 2)$
	$(E, 4)$	$(1\bar{3}, 4), (3\bar{1}, 4)$
F	$(F, 0)$	$(2\bar{2}, 0)$
	$(F, 4)$	$(2\bar{2}, 4)$
G	$(G, 4)$	$(0\bar{2}, 2), (4\bar{2}, 2)$
H	$(H, 4)$	$(2\bar{0}, 2), (2\bar{4}, 2)$

Supplementary Table 1: Relation between $\mathcal{Z}(\text{Rep}S_3)$ anyons and $\mathcal{Z}(A_5)$ anyons.

$k = 4$, with $C = 4/5$. Its complete set of topological symmetries is in fact characterised by $\mathcal{Z}(A_5)$.

These examples completely confirm our expectations by means of our refined condensation trees. Similar analysis can be applied to the refined condensation trees of H_3 symTO in [Supplementary Figure 4](#)



Supplementary Figure 4: Condensation tree for the doubled H_3 symTO. The right branch abbreviates the 3 Morita equivalent branches, with the Frobenius algebras involved given above.

4 Cutting Down Phase Space to Avoid First Order Phase Transitions

In principle, as in the golden chain and other cases considered in [12–15], the strange correlator construction imposes (non-invertible) topological symmetries in the 2D theory. Nonetheless, it is generally unclear how to ensure that the symmetry is not broken spontaneously in the quantum spin-chain by $\langle \Omega |$, leading to gapped phases. Our strategy has been to engineer phase transitions between different spontaneous symmetry breaking phases (anyon condensation phases).

As mentioned in the introduction, non-commuting anyons cannot condense simultaneously. When two condensates are at a tie, it is expected that the resultant phase is either gapless or degenerate. This was proved rigorously for the quantum Ising spin chain in 1+1 dimensions [16]. Such degeneracy corresponds to first-order phase transitions from the perspective of the 2D classical statistical lattice model. Usually at a first-order phase transition between two phases, the free energies of the respective phases become equal, and yet the phases remain distinguishable since their derivatives are different, i.e., there are extra degrees of freedom in the partition function that can distinguish these two partition functions, and that is the classical manifestation of degeneracies observed in the 1+1 D quantum model.

In our construction of the lattice model, we focus on a unit cell with 5 edges. As mentioned in the introduction, the “boundary” of the unit cell is chosen to be the shared module M of the condensates, which effectively isolates the internal edge from the other cells. This follows from the non-degenerate algebra action on the module, ensuring that the condensate arising in a different cell does not favor a particular condensate in a given cell. Therefore, we can focus on the physics in an individual cell.

The interpolation in Eq. (6) of the main text at first sight involves a small number of couplings. Under RG, however, all other linear combinations of the fusion channels in the

unit cell could be invoked, i.e., the dimension of *possible* couplings is much greater than the number of gapped phases nearby described by Frobenius algebras. In such cases, there would be plenty of room for first-order phase transitions.

To exclude first-order phase transitions or degeneracies, the key is to reduce the dimension of phase space in a unit cell as much as possible. Naively, in a single unit cell, the dimension of phase space is roughly speaking $|M|^4|X|$, where $|M|$ denotes the number of simple objects in the module M , and $|X|$ the number of objects that could appear in the middle edge. That is an over-count since fusion rules of the input fusion category imposed at each vertex would reduce the dimension of space. A more sensible count is the number of different vertices allowed by the fusion rules. Recall the example of the A-series models associated to the A_{k+1} categories discussed in the main text. To recover the A-series models, the module is chosen to be the object 1, and there are exactly two different possibilities in the unit cell, where the middle edge is either 0 or 2. As explained in the main text, after discarding the overall normalisation, the effective number of degrees of freedom in the unit cell is $D_c = 1$. Meanwhile, within the two dimensional space in the cell, it can accommodate two different condensates, corresponding to the algebra $\mathcal{A}_0 = 0$ and $\mathcal{A}_2 = 0 \oplus 2$. This means that the only coupling in the unit cell describes the relative balance between these two condensates, and that right at the balanced point between them, there are no extra degrees of freedom left in the cell to allow for degeneracy. In such a case, we argue that the balanced point has to describe a continuous phase transition, which is indeed the case for the A-models.

When the module comprises a single object, we thus conclude that the effective dimension D_c in a unit cell is related to the number of possibilities in the middle edge by $D_c + 1$. One can generalise the above considerations to cases where the module $\mathcal{M}$ consists of more than one object. Naively, in that case, the total dimension of space in a unit cell is the total number of distinct vertices allowed by fusion rules such that the outer edges carry labels $x \in \mathcal{M}$. This number is smaller than $|\mathcal{F}|^5$ as restricted by fusion rules, but generically very large. We note, however, that this is a gross over-estimate of the dimension of phase space in the unit cell. Take the A-models discussed above as an example again. One can pick $\mathcal{A}_2 = 0 \oplus 2$ to play the role of the module that colours the outer edges of the unit cell. In this case, it is still true that this module is shared by $\mathcal{A}_0$ and $\mathcal{A}_2$. In fact, one can show that the choice of their common module M is immaterial: Using properties of Frobenius algebras and modules, within one step of symmetric RG (discussed in more detail below), $\langle \Omega(M) |$ constructed from interpolations of normalised Frobenius algebras with module M in each unit cell reduces to the same state $\langle \Omega |$ for all choices of M , up to overall normalisations depending on the quantum dimension of M .

Therefore, despite an apparent increase in the dimension of phase space, there is an “effective” dimension that is much smaller at play. To understand this reduction, we note that a Frobenius algebra in the unit cell would quickly converge to the gapped fixed point state in Eq. (3) of the main text under RG, i.e., a coupling turned on for a given Frobenius algebra is essentially located within *the basin of attraction* under RG to a fixed point. In such a case, new fusion channels are not generated. This implies that: 1) objects that in principle could appear in the middle edge that belong to an algebra *not sharing the module*

are not generated under RG if they were not turned on in the first place. They should not contribute to the effective phase space dimension. 2) For each object $a \in \mathcal{A}_i$ where $\mathcal{A}_i$ shares the module M it is packaged with a unique non-degenerate sum over objects on the outer edges (as expressed in Eq. (3) of the main text). This package is stable under RG and we propose that this package should be considered as contributing a single degree of freedom. In other words, one should count each action of a on the exterior edges as 1

independent dimension, i.e., the linear combination $\sum_{x,y,u,v \in M} \left\langle \begin{array}{c} x \quad y \\ \diagdown \quad \diagup \\ a \\ \diagup \quad \diagdown \\ u \quad v \end{array} \right| [\rho_{M_{\mathcal{A}}}]_{xy}^a ([\rho_{M_{\mathcal{A}}}]_{uv}^a)^*$ for a fixed a where $a \in \mathcal{A}_i$ constitutes an independent degree of freedom in this reduced phase space. $N_{\mathcal{A}}$ is the number of objects in $\mathcal{A}$. For the most generic case, the algebra actions of $\mathcal{A}_i$ and $\mathcal{A}_j$ on M may differ. In such a case, any basis element in each Frobenius algebra $\mathcal{A}_i$ introduces an independent linear combination of the fusion channels $\sum_{x,y,u,v \in M} \left\langle \begin{array}{c} x \quad y \\ \diagdown \quad \diagup \\ a \\ \diagup \quad \diagdown \\ u \quad v \end{array} \right| [\rho_{M_{\mathcal{A}}}]_{xy}^a ([\rho_{M_{\mathcal{A}}}]_{uv}^a)^*$. For a certain basis element $a \in \mathcal{A}_i \cap \mathcal{A}_j$, when it acts identically as an element of $\mathcal{A}_i$ and as an element of $\mathcal{A}_j$ on M , a single linear combination of fusion channels is attached to this element a . We refer to such a shared basis element a as a *perfectly shared element*, and if the common elements of all algebras are perfectly shared on M , we term M a *perfect common module* for these algebras. Therefore, we can count the dimension D_{unit} of the vector space in a unit cell as follows:

$$D_{\text{unit}} = \sum_i N_{\mathcal{A}_i} - \sum_{\text{perfectly shared } a} (N_a^{\text{common}} - 1), \quad (1)$$

where N_a^{common} is the number of algebras $\mathcal{A}_i$ having a perfectly shared element a . For example, if a is perfectly shared by two algebras acting on M , $N_a^{\text{common}} = 2$. In particular, the identity object on any M is always a perfectly shared element of all Frobenius algebras of a UFC. So for n algebras sharing a module, $N_{\text{identity}}^{\text{common}} = n$. Since the overall normalisation of the state $\langle \Omega |$ is physically irrelevant in the strange correlator, $D_{\text{unit}} - 1$ gives the dimension D_c of phase space – the number of independent couplings – introduced in the ansatz, i.e.,

$$D_c = D_{\text{unit}} - 1. \quad (2)$$

For two competing algebras only, the number of degrees of freedom introduced in a unit cell is reduced to

$$D_{\text{unit}}(\mathcal{A}_i, \mathcal{A}_j) = N_{\mathcal{A}_i} + N_{\mathcal{A}_j} - N_{\text{perfectly shared } a}, \quad (3)$$

where $N_{\text{perfectly shared } a}$ directly counts the number of perfectly shared elements $a \in \mathcal{A}_i \cap \mathcal{A}_j$. There is a special case in which the D_{unit} is reduced most drastically. That is to choose a perfect common module M of all competing Frobenius algebras $\mathcal{A}_i$. This can be typically achieved when the competing algebras are ordered as sub-algebras of one another: $\cdots \subset \mathcal{A}_i \subset \mathcal{A}_j \subset \mathcal{A}_k$. Suppose $\mathcal{A}_i \subset \mathcal{A}_j$, they must share a common ancestor in the refined condensation tree, where the identity object of the sub-input-category $\mathcal{K} \subset \mathcal{F}$ is given by the smaller Frobenius algebra $\mathcal{A}_i$. In the said special case,

$$D_{\text{unit}}(\cdots \subset \mathcal{A}_i \subset \mathcal{A}_j \subset \mathcal{A}_k) = N_{\mathcal{A}_k}. \quad (4)$$

That is, the number of degrees of freedom is simply given by the number of objects in the largest algebra among all algebras sharing the same perfect common module M .

We note that this packaged counting of degrees of freedom (4) would recover for us $D_c = 1$ in the case of the A-models, even if we switch our choice of the shared module between A_0 and A_2 discussed above (which satisfies $A_0 \subset A_2$), consistent with the fact that the partition has not actually changed.

Now for a choice of perfect common module M shared only by n_M different Frobenius algebras among all competing ones, we note that all other gapped phases corresponding to Frobenius algebras not sharing M would not occur under RG. The RG attraction basins would be restricted to those gapped phases that share this M . The fixed point defined by the algebra not sharing this module is very far in phase space. This is in contrast with an algebra that shares the module in the unit cell. The module function of an algebra, as already mentioned, is only 1 step from an exact RG fixed point. This is an effective method to reduce the possibilities of gapped phases showing up in the reduced phase space.

If $D_c = D_{\text{unit}} - 1$ is larger than the number n_M of algebras $\mathcal{A}_i$ sharing the module M ($1 \leq i \leq n_M$, and n_M is the number of gapped phases allowed in the reduced phase space), there is naively still room for degeneracies and thus room for first-order phase transitions. We thus propose the following *sufficient condition* for continuous phase transitions:

All phase transitions in a phase diagram are forced to be continuous if

$$n_M > D_c. \quad (5)$$

When (5) is satisfied, the number of competing anyon condensates is greater than the number of independent degrees of freedom within a unit cell that can be varied. Therefore, when a pair of condensates is tuned to be commensurate, changing the weight of any of the $D_c - 1$ independent coefficients corresponds to moving in a direction towards another condensate that cannot be condensed at the same time as the pair anyway, leaving no room for degeneracy.

One can check this condition for the A-series minimal models for all k discussed above. There, $D_c = 1$ and $n_M = 2$, and the critical points are all continuous, and flow to the A-series minimal models in the thermodynamic limit. The N -state Potts models can be constructed from the $\mathbb{Z}_N$ string-net model. The critical point is located at the transition point exactly as predicted in Eq. (7) of the main text, from a competition between the Frobenius algebra $\mathcal{A}_e = 0$ corresponding to all electric condensate, and $\mathcal{A}_m = 0 \oplus 1 \oplus \dots \oplus N - 1$ corresponding to all magnetic condensate. In this case, $D_c = N - 1$ and $n_M = 2$ wherever N is prime. It is well known that for $N \geq 5$, the critical point is first order, consistent with the counting here. One can also check with the critical models following from the Haagerup string net model discussed above, and find that indeed they are consistent with the rule here. Note that our condition for continuous phase transition is only sufficient, not necessary, so it is overly strong and could potentially leave out many CFTs. For example in the case for $\mathbb{Z}_N$ where $N = 3$, i.e., the 3-state Potts model, by our counting, the commensurate state of $\mathbb{Z}_3$ could potentially give a first-order phase transition, but it is in fact a well-known continuous transition. This also happens in the Haagerup models, in which continuous transitions are confirmed numerically even though we have not

cut down the phase space sufficiently. This suggests that the present counting of effective degrees of freedom is conservative and may admit refinements.

5 Generalised Kramers–Wannier Duality

The Kramers–Wannier duality is a duality transformation in statistical mechanics that relates the ordered and disordered phases of a system. The phase transition point between these two phases coincides with the self-dual point, where the system often exhibits critical behavior. For example, in the classical Ising model, the KW duality relates the high- and low-temperature phases. The critical temperature is found to coincide with the self-dual point. In $\mathbb{Z}_N$ lattice models, KW duality refers to a Fourier-type transform of couplings [17–19]. In these cases, the KW duality is essentially the electromagnetic duality, which exchanges the electric and magnetic charges of the global $\mathbb{Z}_N$ symmetry of the lattice model. ($N = 2$ for the Ising model.)

In our construction, it turns out that these KW dualities known in the literature can be understood as a transformation that maps the original strange correlator on the square-octagon lattice to a strange correlator on the dual lattice. An advantage of using the square-octagon lattice is that its dual lattice has the same shape: The dual lattice exchanges the squares and octagons of the original lattice. This exchange is achieved by making an F-move on every octagon edge, as shown in Supplementary Figure 5 with the example of A_{k+1} series. The F -move used here is applicable to any trivial Morita class: for any $\mathcal{A} = X \otimes X^*$ with $M = X$, we have

$$\begin{array}{c} \text{Diagram 1: A vertex with four lines. Top-left and bottom-right lines are labeled } X, \text{ top-right and bottom-left lines are labeled } X^*. \text{ A dashed line connects the top-left and bottom-right lines, labeled } 0. \end{array} \xrightarrow{\text{KW}} \begin{array}{c} \text{Diagram 2: A vertex with four lines. Top-left and bottom-right lines are labeled } X, \text{ top-right and bottom-left lines are labeled } X^*. \text{ A solid double line connects the top-left and bottom-right lines.} \end{array} = \sum_{i \in X \otimes X^*} \frac{\sqrt{d_i}}{d_X} \begin{array}{c} \text{Diagram 3: A vertex with four lines. Top-left and bottom-right lines are labeled } X, \text{ top-right and bottom-left lines are labeled } X^*. \text{ A solid double line connects the top-left and bottom-right lines, labeled } i. \end{array} \quad (1)$$

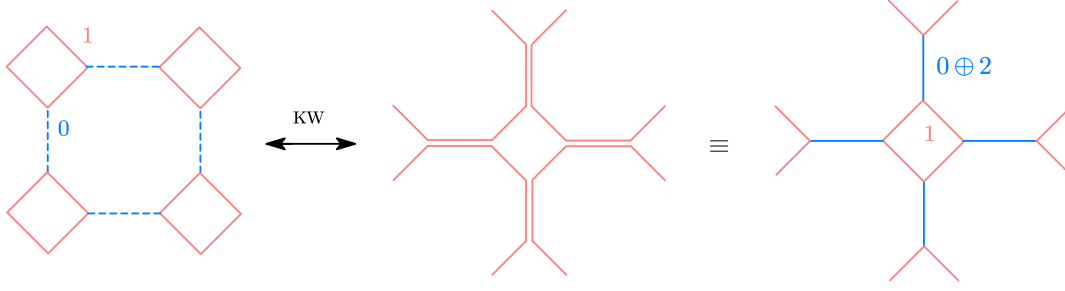
There is thus a self-dual point in the interpolation between $\mathcal{A} = 0$ and $\mathcal{A} = X \otimes X$ that is invariant under KW. In the case of the A -series CFTs, the self-dual point is $\langle \text{Diagram 1} \rangle + \frac{d_1-1}{\sqrt{d_2}} \langle \text{Diagram 2} \rangle$, which exactly matches the critical points obtained by the equal sum of normalised states.

We can also apply (1) to analyze the competition between $\mathcal{A}_0 = \mathbf{1}$ and $\mathcal{A}_1 = \rho \otimes \rho$ in the H_3 with ρ as a perfect common module. The duality transformation maps ρ -squares to ρ -octagons, and the double ρ -lines on an octagon edge is equivalent to $\sum_{i \in \rho \otimes \rho} \frac{\sqrt{d_i}}{d} i$ lines. Then we can verify that the system is self-dual when the octagon edge takes the state

$$|1\rangle + \frac{1}{\sqrt{3d+1}} \left(|1\rangle + \sqrt{d}|\rho\rangle + \sqrt{d}|\alpha\rho\rangle + \sqrt{d}|\alpha^2\rho\rangle \right)$$

This self-dual point again coincides precisely with the point calculated by Eq. (7) of the main text.

This duality can even be applied to compute tricritical points, via a double-layer construction. We consider interpolating among $\mathcal{A}_{0,2,3}$ in A_5 with $M = 2$ as an example.



Supplementary Figure 5: Generalised KW duality for the A_{k+1} category. The common module is 1 and the competing Frobenius algebras are $\mathcal{A}_1 = 0$ and $\mathcal{A}_2 = 0 \oplus 2$. Under KW duality the 1-squares are mapped to 1-octagons, and the positions of squares and octagons are exchanged. The horizontal and vertical octagon edges are exchanged under KW. The double 1-line on the octagon edge is equivalent to algebra $\mathcal{A}_2$ ending on the module $M = 1$. So we see that the two gapped phases given by $\mathcal{A}_1$ and $\mathcal{A}_2$ are KW dual to each other. The self-dual point is obtained by requiring that the unit cell is invariant under an F-move, which exactly matches the critical point obtained by the equal sum of normalised states.

The unit cell is three dimensional with basis $|\bigtimes_2^i\rangle, i \in \{0, 2, 4\}$. The KW duality is performed by doing an F-move on every octagon edge. This operation is represented by:

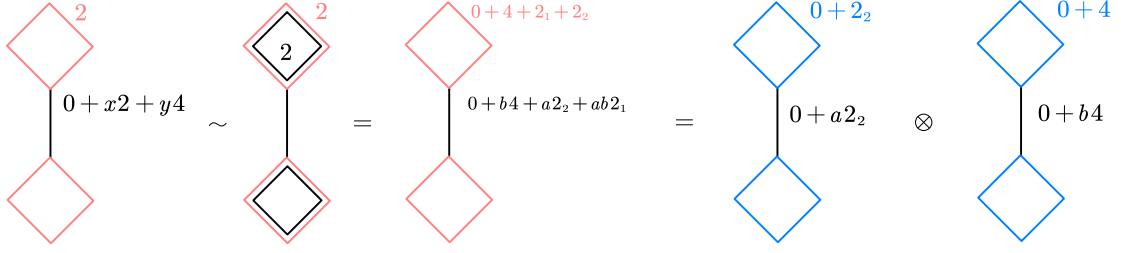
$$\begin{aligned} |\bigtimes_2^0\rangle &\xrightarrow{F} \frac{1}{2} \left(|\bigtimes_2^0\rangle + \sqrt{2} |\bigtimes_2^2\rangle + |\bigtimes_2^4\rangle \right), \\ |\bigtimes_2^2\rangle &\xrightarrow{F} \frac{1}{\sqrt{2}} \left(|\bigtimes_2^0\rangle - |\bigtimes_2^4\rangle \right), \\ |\bigtimes_2^4\rangle &\xrightarrow{F} \frac{1}{2} \left(|\bigtimes_2^0\rangle - \sqrt{2} |\bigtimes_2^2\rangle + |\bigtimes_2^4\rangle \right). \end{aligned}$$

As explained in Supplementary Figure 5, F -moves map the vertical edges to horizontal edges. After doing F -moves, we have implicitly shifted the lattice by a unit cell, shifting our focus to the vertical edges shown on the right-hand side.

In what follows, we denote the basis $|\bigtimes_2^i\rangle$ by $|i\rangle$. Under the normalisation where $|0\rangle$ has a coefficient of 1, the parameter space is two-dimensional, with a generic point represented as $|0\rangle + y|2\rangle + z|4\rangle$. The KW duality acts as a transformation on this parameter space, mapping it onto itself. The KW self-dual points correspond to the fixed points of this transformation. In this case, the self-dual points lie along a line in the parameter space, given by the equation:

$$z = -\sqrt{2}y + 1. \quad (2)$$

The phase boundary between $\mathcal{A}_0$ and $\mathcal{A}_2$ lies entirely on the KW self-dual line and ends at the tricritical point as in Fig. 4a of the main text. The $\mathcal{A}_0$ phase is centered at the point $(y = 0, z = 0)$, and the $\mathcal{A}_2$ phase is centered at the point $(y = \sqrt{2}, z = 1)$. The interpolating line connecting the two points intersects the KW self-dual line at the point



Supplementary Figure 6: Double layer of Ising models.

$(y = \frac{\sqrt{2}}{3}, z = \frac{1}{3})$, which is exactly the critical point constructed by the equilibrium state of $\mathcal{A}_0$ and $\mathcal{A}_2$.

Next, we show how the A_5 model is related to two copies Ising models and derive the critical points and the tricritical point in the phase diagram. The simple object 2 in A_5 is the two dimensional irrep of $\text{Rep}(S_3)$. This fact motivates us to split the simple object 2 into two quasi-objects 2_1 and 2_2 with quantum dimensions equal to 1[9–11]. We make two assumptions about the quasi-objects and show that under these assumptions we can decompose the system into two Ising models and calculate precisely the critical points and the tricritical point.

Assumption 1: as quasiobjects

$$2 = 2_1 \oplus 2_2, \quad 2_2 \otimes 2_2 = 0, \quad 4 \otimes 2_1 = 2_2, \quad 4 \otimes 2_2 = 2_1$$

Assumption 2: as states in local Hilbert space

$$\frac{1}{\sqrt{2}} (|2_1\rangle + |2_2\rangle) = |2\rangle, \quad \frac{1}{\sqrt{2}} (|2_1\rangle - |2_2\rangle) = \frac{1}{\sqrt{2}} (|0\rangle + |4\rangle).$$

For the current strange correlator the module is 2, and up to a global constant we can add a loop labelled by 2 in all the squares, this turns the module to $2 \otimes 2$. Under our assumption, the module is now $2 \otimes 2 = 0 \oplus 2_1 \oplus 2_2 \oplus 4 = (0 \oplus 2_2) \otimes (0 \oplus 4)$. For the octagon edge, we temporarily use a new parametrisation (a and b) and later will show how to map it to the previous one (y and z). The state on the octagon edge is set to be $(|0\rangle + a|2_2\rangle)(|0\rangle + b|4\rangle) = |0\rangle + b|4\rangle + a|2_2\rangle + ab|2_1\rangle$.

In this way, we have rewritten the system into two layers of Ising models, see Supplementary Figure 6. The Ising model has two Frobenius algebras, and we can use the perfect common module σ on the square edge. We already know that the Ising model reaches the critical point when the octagon edge is tuned to $\langle \bigotimes_{\sigma}^1 | + (\sqrt{2} - 1) \langle \bigotimes_{\sigma}^{\psi} |$. In the current double layer system, we identify 0 as $\mathbf{1}_{\text{Ising}}$, and 2_2 or 4 as ψ_{Ising} in each layer. The first layer represents the competition between 0 and 2_2 , while the second layer represents the competition between 0 and 4. The two Ising phase transition lines correspond to the individual critical points for each individual layer. When $a = b = \sqrt{2} - 1$, the two layers are both at the critical point, and the system is self-dual. The system simply stacks two critical Ising models, so the central charge is $C = 1/2 + 1/2 = 1$. In the original parametrisation,

this point corresponds to $y = \frac{\sqrt{2}}{3}$ and $z = \frac{1}{3}$, which is exactly the tricritical point at which the two Ising phase transition lines coalesce.

6 Symmetry-Preserving RG of the Strange Correlator

Our construction of the CFT factory is based on the strange correlator formalism. In this approach, a partition function Z is defined as the overlap between a string-net ground state $|\Psi\rangle$ and a tensor network state $\langle\Omega|$ ³(Supplementary Figure 7):

$$Z = \langle\Omega|\Psi\rangle. \quad (1)$$

Our choice of $\langle\Omega|$ is a tensor network built from local tensors T_{abc}^{ABC} :

$$\langle\Omega| = \sum_{\{A,B,C\}} \prod_{\mu} T_{abc}^{ABC}(\mu) \langle\{a,b,c\}|. \quad (2)$$

The auxiliary indices $A, B, C \in \{1, 2, \dots, \chi\}$ (blue lines) are summed over all triangles, while the physical indices a, b, c (black lines) are labelled by simple objects and contract with the string-net state $|\Psi\rangle$. The orientation of the fusion rules is indicated by arrows. For convenience, we use an orthonormal basis $\{|\{a, b, c\}\rangle\}$ to denote these open legs. The index μ denotes different triangles.

The string-net ground state admits a tensor-network representation involving tetrahedron symbols [15, 20–22]:

$$|\Psi\rangle = \sum_{\{i,j,k\}} \sqrt{d_a} d_i \prod_{\mu} \begin{bmatrix} a & b & c \\ i & j & k \end{bmatrix}(\mu) |\{a, b, c\}\rangle. \quad (3)$$

Here the red indices i, j, k are summed over all simple objects in the category. The tetrahedron symbols are related to 6j-symbols of the UFC by

$$\begin{bmatrix} a & b & c \\ i & j & k \end{bmatrix} = \frac{1}{\sqrt{d_e d_f}} [F_j^{abi}]_{ck} = G_{ji^*k}^{b^*a^*c}.$$

Combining $|\Psi\rangle$ with $\langle\Omega|$, the partition function factorises into local weights:

$$Z = \sum_{\{a,b,c\}, \{A,B,C\}, \{i,j,k\}} \prod_{\mu} z_{abc,ijk}^{ABC}(\mu), \quad (4)$$

where

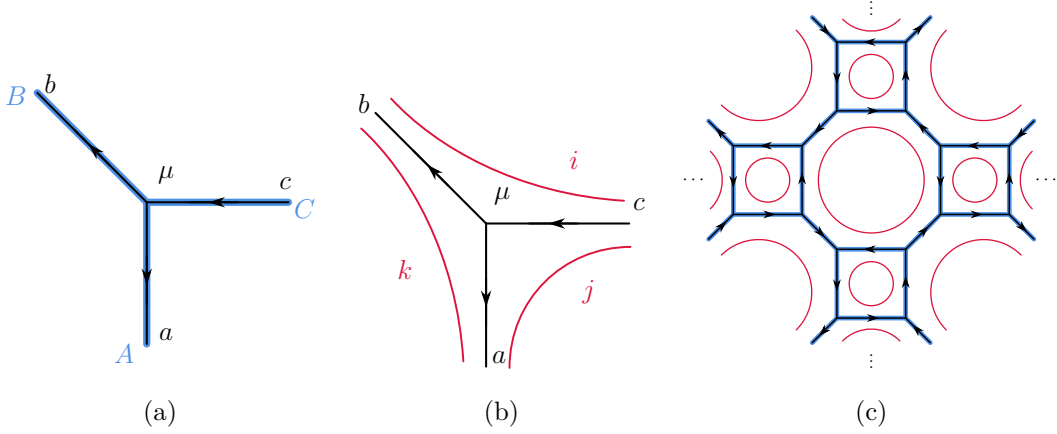
$$z_{abc,ijk}^{ABC}(\mu) = \sqrt[4]{d_i^2 d_j^2 d_k^2 d_a d_b d_c} T_{abc}^{ABC}(\mu) \begin{bmatrix} a & b & c \\ i & j & k \end{bmatrix}(\mu). \quad (5)$$

We adopt a gauge where tetrahedron symbols are both unitary and tetrahedrally symmetric.⁴ Unitarity implies

$$\sum_f d_e d_f \begin{bmatrix} a & b & e \\ c & d & f \end{bmatrix} \begin{bmatrix} a & b & e' \\ c & d & f \end{bmatrix}^* = \delta_{ee'} N_{ab}^e N_{ec}^d, \quad (6)$$

³We restrict to real tensors throughout. With complex tensors, conjugation of neighbouring tensors would be required, leading to a checkerboard arrangement that ensures reality of the partition function.

⁴Generalisation to F -symbols without tetrahedral symmetry is possible, though the RG equations become more involved.



Supplementary Figure 7: The strange correlator $\langle\Omega|\Psi\rangle$. (a) Local tensor T_{abc}^{ABC} of state $\langle\Omega|$. (b) Local tensor of state $|\Psi\rangle$, constructed from tetrahedron symbols and quantum dimensions. (c) Their contraction defines the strange correlator. Blue indices are summed within $\langle\Omega|$, red indices within $|\Psi\rangle$, while black physical indices contract between the two.

while tetrahedral symmetry gives identities such as

$$\begin{bmatrix} a & b & e \\ c & d & f \end{bmatrix} = \begin{bmatrix} a^* & e & b \\ c & f & d \end{bmatrix} = \begin{bmatrix} e & b^* & a \\ f & d & c \end{bmatrix} = \begin{bmatrix} a & f & d \\ c^* & e & b \end{bmatrix}. \quad (7)$$

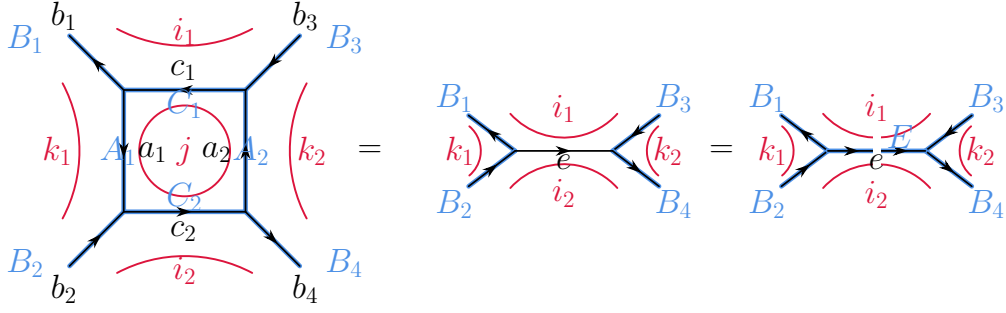
With this symmetry, the pentagon equation takes the form

$$\begin{bmatrix} a & b & e \\ c & d & f \end{bmatrix} \begin{bmatrix} a & b' & e' \\ c' & d & f \end{bmatrix} = \sum_x d_x \begin{bmatrix} a & b' & e' \\ x & e & b \end{bmatrix} \begin{bmatrix} e' & x & e \\ c & d & c' \end{bmatrix} \begin{bmatrix} b' & x & b \\ c & f & c' \end{bmatrix}. \quad (8)$$

The essential virtue of the strange correlator is that deformations of the network $\langle\Omega|$ are precisely the categorical F -moves guaranteed by tetrahedron symbols. Different lattice discretisations are related by these moves, making the construction inherently symmetry-preserving. This feature enables the following RG procedure. In each RG step, four tensors T are contracted into an intermediate tensor M , which is then factorised into two renormalised tensors $\tilde{T}$ (Supplementary Figure 8).

We begin by rewriting the product of four tetrahedron symbols as follows.

$$\begin{aligned} & \sum_j d_j \begin{bmatrix} a_1 & b_1 & c_1 \\ i_1 & j & k_1 \end{bmatrix} \begin{bmatrix} a_1 & b_2 & c_2 \\ i_2 & j & k_1 \end{bmatrix} \begin{bmatrix} a_2 & b_3 & c_1 \\ i_1 & j & k_2 \end{bmatrix} \begin{bmatrix} a_2 & b_4 & c_2 \\ i_2 & j & k_2 \end{bmatrix} \\ &= \sum_j d_j \sum_{e_1} d_{e_1} \begin{bmatrix} a_1 & b_1 & c_1 \\ e_1 & c_2 & b_2 \end{bmatrix} \begin{bmatrix} c_1 & e_1 & c_2 \\ i_2 & j & i_1 \end{bmatrix} \begin{bmatrix} b_1 & e_1 & b_2 \\ i_2 & k_1 & i_1 \end{bmatrix} \sum_{e_2} d_{e_2} \begin{bmatrix} a_2 & b_3 & c_1 \\ e_2 & c_2 & b_4 \end{bmatrix} \begin{bmatrix} c_1 & e_2 & c_2 \\ i_2 & j & i_1 \end{bmatrix} \begin{bmatrix} b_3 & e_2 & b_4 \\ i_2 & k_2 & i_1 \end{bmatrix} \\ &= \left(\sum_j d_j \begin{bmatrix} c_1 & e_1 & c_2 \\ i_2 & j & i_1 \end{bmatrix} \begin{bmatrix} c_1 & e_2 & c_2 \\ i_2 & j & i_1 \end{bmatrix} \right) \sum_{e_1, e_2} d_{e_1} d_{e_2} \begin{bmatrix} a_1 & b_1 & c_1 \\ e_1 & c_2 & b_2 \end{bmatrix} \begin{bmatrix} b_1 & e_1 & b_2 \\ i_2 & k_1 & i_1 \end{bmatrix} \begin{bmatrix} a_2 & b_3 & c_1 \\ e_2 & c_2 & b_4 \end{bmatrix} \begin{bmatrix} b_3 & e_2 & b_4 \\ i_2 & k_2 & i_1 \end{bmatrix} \\ &= \sum_e \sqrt{d_e} \begin{bmatrix} b_1 & e & b_2 \\ i_2 & k_1 & i_1 \end{bmatrix} \begin{bmatrix} b_3 & e & b_4 \\ i_2 & k_2 & i_1 \end{bmatrix} \left(\sqrt{d_e} \begin{bmatrix} a_1 & b_1 & c_1 \\ e & c_2 & b_2 \end{bmatrix} \begin{bmatrix} a_2 & b_3 & c_1 \\ e & c_2 & b_4 \end{bmatrix} \right). \quad (9) \end{aligned}$$



Supplementary Figure 8: RG step: four tensors T are combined into an intermediate tensor M , which is then decomposed into two new tensors $\tilde{T}$ via SVD. Red legs indicate tetrahedron-symbol contractions and are not part of T or M .

Here, the second row is due to (8), and the fourth line is due to (6) and a sum over $\delta_{e_1 e_2}$. We combine the factors in brackets with tensor T to get an intermediate tensor M :

$$M_{b_1 b_2 b_3 b_4 e}^{B_1 B_2 B_3 B_4} = \sum_{a_1, a_2, c_1, c_2}^{A_1, A_2, C_1, C_2} \sqrt{d_{a_1} d_{a_2} d_{c_1} d_{c_2} d_e} \begin{bmatrix} a_1 & b_1 & c_1 \\ e & c_2 & b_2 \end{bmatrix} \begin{bmatrix} a_2 & b_3 & c_1 \\ e & c_2 & b_4 \end{bmatrix} \times T_{a_1 b_1 c_1}^{A_1 B_1 C_1} T_{a_1 b_2 c_2}^{A_1 B_2 C_2} T_{a_2 b_3 c_1}^{A_2 B_3 C_1} T_{a_2 b_4 c_2}^{A_2 B_4 C_2}. \quad (10)$$

A new tensor $\tilde{T}$ is then obtained by applying a singular value decomposition (SVD) to the tensor M ,

$$M_{b_1 b_2 b_3 b_4 e}^{B_1 B_2 B_3 B_4} = \sum_E \tilde{T}_{b_1 e b_2}^{B_1 E B_2} \tilde{T}_{b_3 e b_4}^{B_3 E B_4}. \quad (11)$$

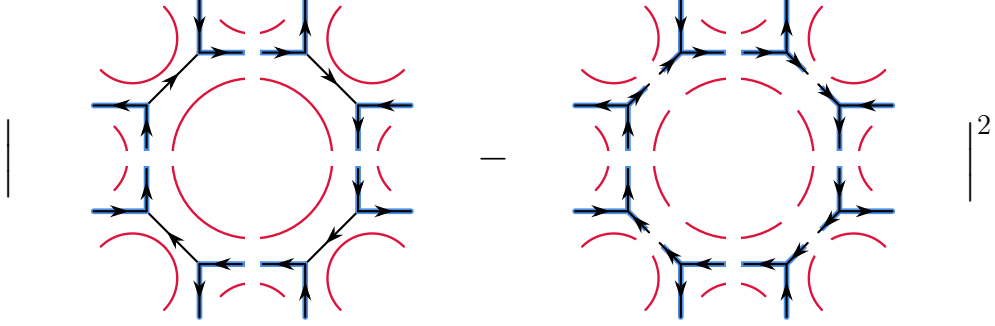
This procedure is followed by a gradient-descent-based loop tensor network renormalization (TNR) [23] to search for an optimal approximation within a given bond dimension χ of the new index E . The loop tensor network used to define the loop error minimised in this step is shown in Supplementary Figure 9.

In total, each RG step is expressed as

$$\sum_e^E \sqrt{d_e} \begin{bmatrix} b_1 & e & b_2 \\ i_2 & k_1 & i_1 \end{bmatrix} \begin{bmatrix} b_3 & e & b_4 \\ i_2 & k_2 & i_1 \end{bmatrix} \tilde{T}_{b_1 e b_2}^{B_1 E B_2} \tilde{T}_{b_3 e b_4}^{B_3 E B_4} = \sum_{a_1, a_2, c_1, c_2, j}^{A_1, A_2, C_1, C_2} \sqrt{d_{a_1} d_{a_2} d_{c_1} d_{c_2} d_j} \begin{bmatrix} a_1 & b_1 & c_1 \\ i_1 & j & k_1 \end{bmatrix} \begin{bmatrix} a_1 & b_2 & c_2 \\ i_2 & j & k_1 \end{bmatrix} \begin{bmatrix} a_2 & b_3 & c_1 \\ i_1 & j & k_2 \end{bmatrix} \begin{bmatrix} a_2 & b_4 & c_2 \\ i_2 & j & k_2 \end{bmatrix} T_{a_1 b_1 c_1}^{A_1 B_1 C_1} T_{a_1 b_2 c_2}^{A_1 B_2 C_2} T_{a_2 b_3 c_1}^{A_2 B_3 C_1} T_{a_2 b_4 c_2}^{A_2 B_4 C_2}. \quad (12)$$

The original form of the strange correlator is preserved after a complete RG step, with each tensor T flowing to $\tilde{T}$. The categorical symmetry, as imposed by the 3D bulk that is carried throughout the RG process, is thus preserved in the RG procedure. The orientated fusions at the new vertices are $b_1 \otimes e \rightarrow b_2$ and $b_3 \otimes e \rightarrow b_4$. One can alternatively obtain $b_2 \otimes e' \rightarrow b_1$ and $b_4 \otimes e' \rightarrow b_3$ by applying a different set of the pentagon equation in (9).

In our setup, a unit cell typically consists of two local T tensors. The initial tensor for the RG flow can be considered as the tensor M in (10) and (11). Consequently, an initial



Supplementary Figure 9: Loop tensor network defining the loop-TNR error. The left network is built from four M tensors together with the underlying tetrahedron symbols. The right network is the corresponding truncated network, built from eight $\tilde{T}$ tensors with the same tetrahedron-symbol structure and with the new index E restricted to bond dimension χ . The minimised objective function is the squared norm of the difference between these two loop tensor networks.

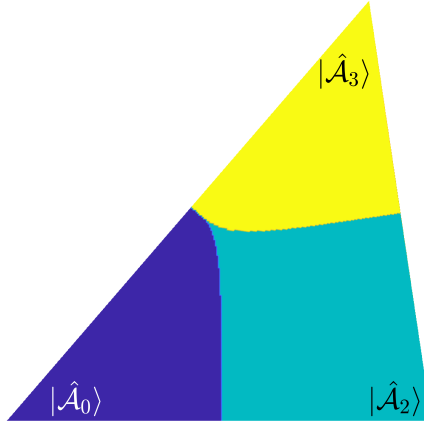
decomposition step of (11) is performed. Explicitly, for unit cells corresponding to Eq. (6) of the main text, where each $\langle \begin{array}{c} \text{blue leg} \\ \text{red tetrahedron} \end{array} \mid$ is weighted by parameter r_i , the initial tensor M is

$$M_{b_1 b_2 b_3 b_4 e} = \sum_i r_i [\rho_{M_{\mathcal{A}_i}}]_{b_1 b_2}^e ([\rho_{M_{\mathcal{A}_i}}]_{b_3 b_4}^e)^*. \quad (13)$$

Here, auxiliary legs B_i are omitted in M . For general initial tensors, B_i and possibly also E account for multiplicities if a simple object b_i or e appears multiple times in $M_{\mathcal{A}}$ or $\mathcal{A}$. In the present cases, the multiplicity is no more than 1, so the initial tensor has trivial auxiliary legs. During the RG procedure, the auxiliary legs' effective bond dimensions can increase, reflecting the growing entanglement among the local tensors.

Computing the Phase Diagram

If the initial tensor corresponds to a single Frobenius algebra, the RG flow converges in one step to its multiplication $f^{\mathcal{A}}$, which is a fixed point tensor. General tensors in gapped phases flow to one of these fixed points, providing an order parameter that distinguishes phases. Phase diagrams obtained this way show excellent accuracy even at minimal bond dimension $\chi = 1$ (Supplementary Figure 10), illustrating the efficiency of the categorical



Supplementary Figure 10: Ternary phase diagram for the interpolation of algebras $\mathcal{A}_{0,2,3}$ under module 2 of the A_5 category. Different colours correspond to different converged fixed-point tensors, computed with $\chi = 1$. This illustrates the role of the fixed-point tensor as an order parameter, highlighting the precision of our RG scheme even at trivial auxiliary bond dimension.

symmetric RG. We also provide this type of diagram for other ternary phases in Section 2 and Appendix F.

At each RG step, a numerical central charge is extracted from the transfer matrices constructed from the coarse-grained tensors, following the procedure of Ref. [24]. At continuous phase transitions, the extracted central charge exhibits a clear plateau over several RG steps before eventually drifting due to finite bond dimension effects. The reported central charge is obtained by averaging over an eight-step-long plateau, while the associated uncertainty δC is estimated as the maximal deviation within this plateau.

For continuous transitions, the numerical central charge can be used to construct phase diagrams, providing complementary and more detailed information beyond fixed-point identification. At first-order transitions the central charge is expected to vanish in the thermodynamic limit. In this case, we instead use the non-converged numerical central charge obtained at shallower RG depth, which still serves as a useful diagnostic for locating phase boundaries. Further technical details and additional examples are presented in Appendices E and F.

Appendices

A The HGW String-Net Model

The HGW model [9, 25, 26] can be defined on a trivalent lattice, where a tail (purple lines in Supplementary Figure 13) is attached to a chosen edge of a plaquette, while the choice is topologically irrelevant. The lattice shape is irrelevant in the string-net model describing

topological orders, but it plays an important role in conformal field theories. In this article, for convenience, we only define the HGW model on the truncated square lattice as shown in Supplementary Figure 13, which facilitates the subsequent renormalization group procedure. The basic configuration is established by labelling each edge and tail with a simple object from the input unitary fusion category $\mathcal{F}$ of the HGW model, subject to the constraint on all vertices that $\delta_{ijk} = 1$ for the three incident edges or tails meeting at this vertex. The Hilbert space $\mathcal{H}$ of the model is spanned by all possible configurations of these labels on the edges and tails. Edges and tails are oriented, but the choice of orientation does not affect the physics.

In the HGW model, anyons reside on the plaquettes of the lattice, and their types are labelled by the simple objects of the Drinfeld center $\mathcal{Z}(\mathcal{F})$ of the input UFC $\mathcal{F}$. A key feature of the HGW construction is that it explicitly manifests the internal gauge degrees of freedom of non-Abelian anyons. Specifically, anyons are realised in the model by *dyons*—a pair consisting of an anyon type J and its internal gauge degree of freedom p , where p is the degree of freedom on the tail of the plaquette where the anyon resides. A given anyon type J , as a simple object in $\mathcal{Z}(\mathcal{F})$, may carry multiple types of internal degree of freedom p , thereby enlarging its internal gauge space. Although these p are gauge degrees of freedom and hence unobservable in the topological phase, they play a central role in the construction of the CFT states, where topological invariance is broken to expose these internal degrees of freedom, which become physical local degrees of freedom that determine the physical phenomena that may be captured by a critical CFT.

In the HGW model, anyons are created in pairs by acting with a *creation operator* on the ground state $|\Psi\rangle$. It suffices to define the action of the shortest creation operator $W_E^{J;pq}$, which creates a pair of dyons (J^*, p^*) and (J, q) in the two adjacent plaquettes separated by an edge E :

$$W_E^{J;pq} \begin{array}{c} \uparrow \\ j \end{array} := \sum_{k \in L_{\mathcal{F}}} \sqrt{\frac{d_k}{d_j}} \overline{z_{pqj}^{J;k}} \begin{array}{c} \begin{array}{c} \uparrow j \\ \leftarrow p^* \end{array} \quad \begin{array}{c} \begin{array}{c} \leftarrow q \\ \uparrow k \end{array} \\ j \end{array} \end{array}, \quad (1)$$

where J is a simple object in $\mathcal{Z}(\mathcal{F})$, p, q are internal gauge degrees of freedom on tails of anyons, j is the degree of freedom on edge E , and $z_{pqj}^{J;k}$ are the coefficients called *half-braiding tensors*. All other creation operators in the model can be generated by composing such shortest creation operators.

In a companion paper[9], it is shown that the edges of the lattice carry a flat gauge field valued in $\mathcal{F}$, and the open ends of the tails host anyon excitations coupled with the gauge field via the Gauss law and gauge connection. This finding allows one to recast topological phases back in the Landau-Ginzburg paradigm, though in a more general sense, where phase transitions are triggered by anyon condensation. The HGW model is a convenient framework for constructing any possible kind of anyon condensation in a doubled topological phase and studying the consequent phenomena[9].

$$\begin{aligned}
\begin{array}{c} \text{A} \\ | \\ \text{A} \text{---} \text{A} \end{array} &= \sum_{abc \in L_{\mathcal{A}}} f_{abc}^* \begin{array}{c} \text{c} \\ | \\ \text{a} \text{---} \text{b} \end{array}, & \begin{array}{c} \text{A} \text{---} \text{A} \\ | \\ \text{A} \end{array} &= \sum_{abc \in L_{\mathcal{A}}} f_{cb^*a^*} \begin{array}{c} \text{a} \text{---} \text{b} \\ | \\ \text{c} \end{array}, \quad (2) \\
\begin{array}{c} \text{A} \\ \text{---} \end{array} &\Rightarrow \begin{array}{c} \text{A} \\ \text{---} \end{array},
\end{aligned}$$

A.1 Definitions of Frobenius Algebras and Modules

In this appendix, we quickly review the mathematical definitions of Frobenius algebras and modules. Readers who want to construct their own CFTs via our method can use the definitions of Frobenius algebras and modules in this section.

A unitary Frobenius algebra in $\mathcal{F}$ is a (possibly composite) object in $\mathcal{F}$, i.e.

$$\mathcal{A} = \bigoplus_a n_a a, \quad a \in \mathcal{F}, \quad n_a \in \mathbb{N}, \quad (3)$$

equipped with a product $\mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ and coproduct $\mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$, satisfying certain properties. Here, n_a are non-negative integers describing the multiplicities of object $a \in \mathcal{F}$ participating in $\mathcal{A}$. To avoid clutter, the discussion in this section is limited to $n_a \leq 1$, although it can be readily generalised by introducing extra indices i for each anyon label $a \in \mathcal{A}$ whose $n_a > 1$. For the case where all $n_a \leq 1$, the products and co-products of $\mathcal{A}$ are encoded in a cyclically symmetric function $f^{\mathcal{A}} : L_{\mathcal{A}}^3 \rightarrow \mathbb{C}$, satisfying

$$\sum_{t \in L_{\mathcal{A}}} f_{rst}^{\mathcal{A}} f_{abt}^{\mathcal{A}} G_{abc}^{rst} \sqrt{d_c d_t} = f_{acs}^{\mathcal{A}} f_{rc^*b}^{\mathcal{A}}, \quad (4)$$

$$\sum_{ab \in L_{\mathcal{A}}} f_{abc}^{\mathcal{A}} f_{c^*b^*a^*}^{\mathcal{A}} \sqrt{d_a d_b} = d_{\mathcal{A}} \sqrt{d_c}, \quad (5)$$

$$f_{abc}^{\mathcal{A}} = f_{bca}^{\mathcal{A}}, \quad f_{1ab}^{\mathcal{A}} = \delta_{ab^*}, \quad f_{abc}^{\mathcal{A}} = (f_{c^*b^*a^*}^{\mathcal{A}})^*. \quad (6)$$

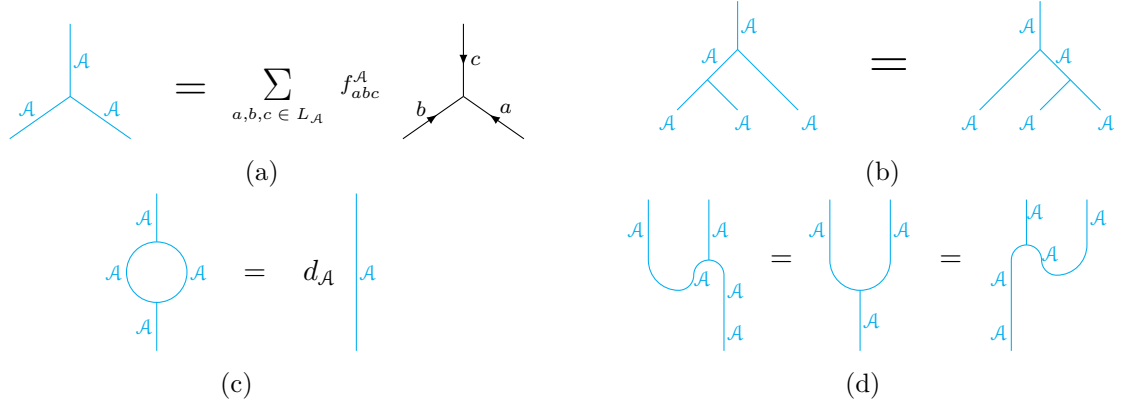
where $L_{\mathcal{A}} = \{a \in \mathcal{F} : n_a = 1\}$ is the set of simple objects appearing in $\mathcal{A}$, and $d_{\mathcal{A}} = \sum_{a \in L_{\mathcal{A}}} d_a$ is the quantum dimension of $\mathcal{A}$. One can express this map in basis form, which is depicted in Supplementary Figure 11.

A right module of Frobenius algebra $\mathcal{A}$ in a UFC $\mathcal{F}$ is a (possibly composite) object

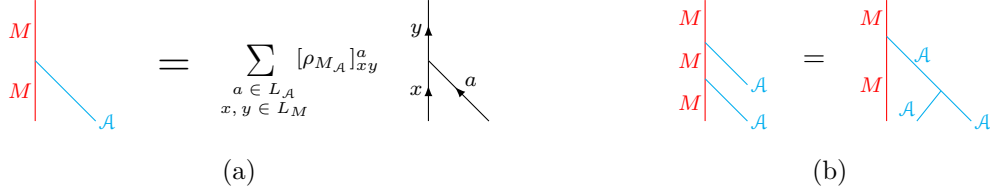
$$M_{\mathcal{A}} := \bigoplus_{x \in L_{\mathcal{F}}} m_x x,$$

equipped with an algebra action on $M_{\mathcal{A}}$: $M_{\mathcal{A}} \otimes \mathcal{A} \rightarrow M_{\mathcal{A}}$. Here, x is a collection of simple objects of $\mathcal{F}$ that appear in the module, and $m_x \in \mathbb{N}$ is the multiplicity of x in M . For simplicity, in this paper, we restrict to the case $m_x \leq 1$, and define $L_{M_{\mathcal{A}}} = \{x \in L_{\mathcal{F}} \mid m_x = 1\}$. The algebra action is recorded in a *module function* $\rho_{M_{\mathcal{A}}} : L_{\mathcal{A}} \otimes L_{M_{\mathcal{A}}}^2 \rightarrow \mathbb{C}$, which satisfies an associativity constraint:

$$[\rho_{M_{\mathcal{A}}}]_{xy}^a [\rho_{M_{\mathcal{A}}}]_{yz}^b G_{xac}^{bz^*by} = f_{abc}^{\mathcal{A}} [\rho_{M_{\mathcal{A}}}]_{xz}^c, \quad [\rho_{M_{\mathcal{A}}}]_{xy}^a = ([\rho_{M_{\mathcal{A}}}]_{x^*y^*}^{a^*})^*. \quad (7)$$



Supplementary Figure 11: (a) Frobenius algebras in basis form. (b) The Associativity: Eq. (4). (c) The unitarity: Eq. (5). (d) Equations (6). Taking different graphical bases yields different equations in Eq. (6). This equation is equivalent to the Frobenius condition[27].



Supplementary Figure 12: (a) The basis representation of right- $\mathcal{A}$ module M . (b) The right-module condition: Eq. (7).

In particular, for $\text{UFC Vec}(G)$ with a finite group G , the modules over a Frobenius algebra $\mathcal{A} = \bigoplus_{g \in H} g$ (with multiplication $f_{fgh}^{\mathcal{A}} = \delta_{e, fgh}$) for a subgroup $H \leq G$ correspond to the representations of H , and the function ρ encodes the representation matrix entries.

One can express the module function in basis form, which is depicted in Supplementary Figure 12: The vertices connecting the edges coloured by $\mathcal{A}$ and the chosen module $M_{\mathcal{A}}$ are weighted by the coefficients $[\rho_{M_{\mathcal{A}}}]_{xy}^a$ that define the action of $\mathcal{A}$ on the module.

A.2 Anyon Condensation in the HGW Model

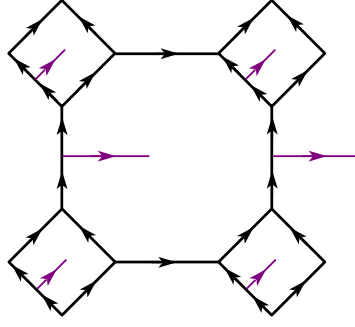
The HGW model provides a finer description of anyon-condensation-induced topological phase transitions.

To describe the procedure of condensing a Lagrangian set L of anyons in the HGW model, we can add to the HGW Hamiltonian a condensation term:

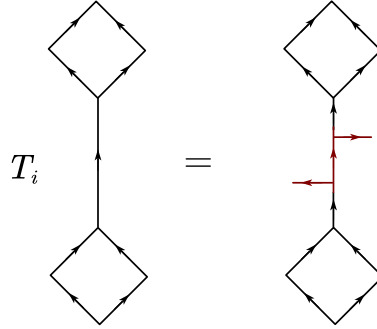
$$H = H_{\mathcal{F}} - \lim_{\Lambda \rightarrow \infty} \sum_E P_E^{\mathcal{A}},$$

where $H_{\mathcal{F}}$ is HGW Hamiltonian of the parent topological phase, and $P_E^{\mathcal{A}}$ is a local projector acting on an edge E :

$$P_E^{\mathcal{A}} = \sum_{J \in L} \sum_{p, q} \pi_J^{p, q} W_E^{J; pq} = {}_E \mathcal{A} \rangle \langle \mathcal{A} |_E,$$



Supplementary Figure 13: The HGW model.



Supplementary Figure 14: Anyon creation operator

where $W_E^{J;pq}$ are creation operators of condensed dyons, $\pi_J^{p,q} \in \mathbb{C}$ are coefficients to make the sum a projector. In the limit $\Lambda \rightarrow \infty$, the projectors P_E^A ensure that the new ground states $|A\rangle_E$ are +1 eigenstates of all P_E^A , which are coherent states filled with arbitrarily many condensed anyons throughout the lattice. This state can be locally represented as:

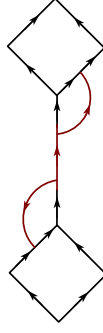
$$|A\rangle_E = \begin{array}{c} | \\ A \end{array} \begin{array}{c} A \\ A \\ A \end{array} = \sum_{j,k,p,q \in L_A} f_{jk^*p^*} f_{kq^*j^*} \begin{array}{c} j \\ p \end{array} \begin{array}{c} q \\ j \end{array},$$

where E is an edge of the lattice, while the violet lines refer to auxiliary tails in plaquettes, which will then be contracted via topological moves and result in only one tail in each plaquette, representing the entanglement between different edges. The global ground state of the trivial topological order is the state where each edge and tail in the square-octagon lattice carries Frobenius algebra object A .

In the case of the Doubled Ising model, there are two Frobenius algebras of the input Ising UFC:

$$\mathcal{A}_1 = 1, \quad [f_1]_{111} = 1, \quad \mathcal{A}_2 = 1 \oplus \psi, \quad [f_2]_{111} = [f_2]_{1\psi\psi} = 1.$$

These two input Frobenius algebras are Morita equivalent because they have the same full



Supplementary Figure 15: Anyon condensation

center⁵ $L = 1\bar{1} \oplus \psi\bar{\psi} \oplus \sigma\bar{\sigma}$. The corresponding projectors are

$$P_E^{A_1} = \frac{W_E^{1\bar{1};1,1} + W_E^{\psi\bar{\psi};1,1} + 2W_E^{\sigma\bar{\sigma};1,1}}{4} = \delta_{j_E,0}, \quad P_E^{A_2} = \frac{W_E^{1\bar{1};1,1} + W_E^{\psi\bar{\psi};1,1} + 2W_E^{\sigma\bar{\sigma};\psi,\psi}}{4}. \quad (8)$$

In our square-octagon lattice as shown in the main text, the anyon operator, the pair creation operator should act as shown in Supplementary Figure 14. The HGW lattice is different from the HGW string net with these extra tails. To make contact with the original version of the strange correlator which follows from the HGW ground state wave-function, we would like to remove these tails. One very natural way of achieving this is to connect these lines to the square, as shown in Supplementary Figure 15, and then removing the extra bubble by F-moves.

We note that the Ising critical state

$$\left| \bigvee_{\text{critical}} \right| = \frac{1}{2} \left((\sqrt{2} + 1) \left| \bigvee_{\sigma}^1 \right| + \left| \bigvee_{\sigma}^{\psi} \right| \right), \quad (9)$$

is the eigenstate of $P_E^{A_1} + P_E^{A_2}$ with the largest eigenvalue.

This story can be repeated for all pair competitions in which $\mathcal{A}_i \subset \mathcal{A}_j$ in other fusion categories. The predicted phase transition for each pair competition is

$$\left| \bigvee_{(\mathcal{A}_i, M_{\mathcal{A}_i}), (\mathcal{A}_j, M_{\mathcal{A}_j})} \right|_{\text{critical}} = \langle \hat{\mathcal{A}}_i |_{M_{\mathcal{A}_i}} + \langle \hat{\mathcal{A}}_j |_{M_{\mathcal{A}_j}} \quad (10)$$

One can show that this is always the eigenstate with the largest eigenvalue of the sum of projection operators, analogously defined as the Ising case above, irrespective of the choice of the common module between $\mathcal{A}_{i,j}$.

⁵Given a Frobenius algebra $\mathcal{A}$ in a fusion category $\mathcal{F}$, its full center $\mathcal{Z}(\mathcal{A})$ is the universal Frobenius algebra in the Drinfeld center $\mathcal{Z}(\mathcal{F})$ equipped with a canonical morphism $\xi : \mathcal{Z}(\mathcal{A}) \rightarrow \mathcal{A}$, such that the $\mathcal{A}$ -multiplication of two objects respectively from $\mathcal{Z}(\mathcal{A})$ and $\mathcal{A}$ (via morphism ξ) commutes with exchanging these two objects using the half-braiding. Physically, it says the condensed anyons in $\mathcal{Z}(\mathcal{A})$ behave like “trivial” anyons in the condensed string-net model with only degree of freedom $\mathcal{A}$.

A.3 Lagrangian Algebra of Anyon Condensation

Now, after anyon condensation via a specific projector P_E^A , the resulting model is a trivial topological phase with a one-dimensional Hilbert space spanned by $|\mathcal{A}\rangle_E$. The projector P_E^A is defined as

$$P_E^A = |\mathcal{A}\rangle_{EE} \langle \mathcal{A}| = \sum_{J \in L} \sum_{p,q} \pi_J^{p,q} W_E^{J;pq}.$$

Solving this equation, the coefficients $\pi_J^{p,q}$ encapsulate all information about the Lagrangian algebra L associated with the anyon condensation.

More explicitly, those anyon types J with $\pi_J^{pq} \neq 0$ are precisely the condensed anyons appearing in the Lagrangian algebra. It can be proven[9] that all condensed anyons are bosonic and braid trivially with each other.

There may exist multiple dyonic sectors of the same anyon type J that condense simultaneously; these are called the condensed sectors of J . The number n_{Cond}^J of condensed sectors for anyon J is referred to as the *multiplicity* of condensed anyon J during anyon condensation. Collect the condensation coefficients π_J^{pq} for a fixed anyon type J into a matrix:

$$\Pi^J = \begin{pmatrix} \pi_J^{p_1 p_1} & \pi_J^{p_1 p_2} & \dots \\ \pi_J^{p_2 p_1} & \pi_J^{p_2 p_2} & \dots \\ \vdots & \vdots & \ddots \\ \pi_J^{p_{n_J^p} p_1} & \pi_J^{p_{n_J^p} p_2} & \dots \end{pmatrix}.$$

Then,

$$n_{\text{Cond}}^J = \text{rank}(\Pi^J).$$

The corresponding Lagrangian algebra of the modular tensor category $\mathcal{Z}(\mathcal{F})$ is thus

$$L = \bigoplus_{J \in L_{\text{Cond}}} n_{\text{Cond}}^J J.$$

We can further diagonalise the condensation matrix using a unitary transformation U_J :

$$U_J \Pi^J U_J^\dagger = \text{diag}\{\pi_J^1, \pi_J^2, \dots, \pi_J^{n_{\text{Cond}}^J}, 0, 0, \dots, 0\},$$

where $\pi_J^i \in \mathbb{C}$ and $1 \leq i \leq n_{\text{Cond}}^J$. The i -th condensed sector of anyon J is then given by

$$|J, i\rangle_{\text{Cond}} = \sum_p \frac{1}{d_p} [U_J]_{ip} |J, p\rangle,$$

and its corresponding creation operator is

$$\tilde{W}_E^{J_i} = \sum_{p,q} \frac{1}{d_p d_q} [U_J^\dagger]_{J_i p} W_E^{J;pq} [U_J]_{q J_i}, \quad (11)$$

where the sums run over the T -charges of anyon J . The condensation projector P_E^A can now be written as a sum of these normalised creation operators:

$$P_E^A = \frac{1}{d_A} \sum_{\text{Condensed } J} \sum_{i=1}^{n_{\text{Cond}}^J} \tilde{W}_E^{J_i},$$

which faithfully realise the algebra basis elements of the Lagrangian algebra L . In particular, the operator product expansion coefficients of these creation operators encode the algebra multiplication structure of L :

$$W_E^{J_i} W_E^{K_j} = \sum_{M_k} f_{J_i K_j}^{M_k} W_E^{M_k}.$$

B Categorical data for the examples

B.1 The A_{k+1} series

We collect here the categorical data for the A_{k+1} fusion category used in the examples. The category A_{k+1} contains $k+1$ simple objects labelled $0, 1, 2, \dots, k$, with fusion rules given by

$$a \otimes b = |a - b| \oplus (|a - b| + 2) \oplus \dots \oplus \min\{a + b, 2k - a - b\}.$$

The quantum dimension of a simple object a is

$$d_a = \frac{\sin\left(\frac{(a+1)\pi}{k+2}\right)}{\sin\left(\frac{\pi}{k+2}\right)}.$$

For $k \geq 2$, the first few quantum dimensions are $d_0 = 1$, $d_1 = 2 \cos \frac{\pi}{k+2}$, and $d_2 = d_1^2 - 1$. In particular, the Ising UFC corresponds to the special case $k = 2$, where the simple objects 0, 1, and 2 map to 1, σ , and ψ , respectively.

The string-net model with input A_{k+1} UFC outputs the $\mathcal{Z}(A_{k+1})$ UMTC, which contains $(k+1)^2$ anyon types, labelled by a pair $a\bar{b}$, where $0 \leq a, b \leq k$. The dyon types of anyon $a\bar{b}$ are labelled by a pair $(a\bar{b}, p)$, where p is the internal degree of freedom, satisfying

$$0 \leq p \leq k, \quad |a - b| \leq p \leq \min(a + b, 2k - a - b), \quad 2|(a + b - p)|.$$

For all $k \geq 2$, one can check that $\mathcal{A}_0 = 0$ and $\mathcal{A}_1 = 0 \oplus 2$ are two Frobenius algebras in A_{k+1} UFC. Specifically,

$$\mathcal{A}_0 = 0, \quad [f_0]_{000} = 1, \quad d_{\mathcal{A}_0} = 1,$$

$$\mathcal{A}_1 = 0 \oplus 2, \quad [f_1]_{000} = [f_1]_{022} = 1, \quad [f_1]_{222} = \sqrt{\sqrt{d_2} - \frac{1}{\sqrt{d_2}}}, \quad d_{\mathcal{A}_1} = d_1^2 = d_2 + 1.$$

Both algebras admit a common right-module $M = 1$ containing only one simple object, with module function

$$[\rho_{\mathcal{A}_0}]_{11}^0 = 1, \quad [\rho_{\mathcal{A}_1}]_{11}^0 = 1, \quad [\rho_{\mathcal{A}_1}]_{11}^2 = \sqrt[4]{d_2}.$$

B.2 The $\mathbb{Z}_N$ data

The Abelian fusion category $\mathbf{Vec} \mathbb{Z}_N$, $N \geq 2$ is the representation category of the cyclic group $\mathbb{Z}_N$. Basically, we can treat $\mathbb{Z}_N$ as the input category in the string-net model. Its

simple objects are the group elements of $\mathbb{Z}_N$, and are conveniently labelled by $0, 1, \dots, N-1$. The fusion rule is inherited from the group multiplication of $\mathbb{Z}_N$:

$$i \otimes j = i + j \pmod{N}, \quad 0 \leq i, j < N,$$

and every simple object has quantum dimension $d_i = 1$. The output topological order is a $\mathbb{Z}_N$ quantum double.

The case of $N = 2$ as an example is in fact the $\mathbb{Z}_2$ toric code model, and for generic N they are simply the $\mathbb{Z}_N$ quantum double/ Dijkgraaf-Witten models.

Fusion category $\mathbf{Vec}(\mathbb{Z}_N)$ contains at least two connected special symmetric Frobenius algebras:

$$\begin{aligned} \mathcal{A}_e &= 0, & f_{000}^{\mathcal{A}_e} &= 1, & d_{\mathcal{A}_e} &= 1, \\ \mathcal{A}_m &= \bigoplus_{a=0}^{N-1} a, & f_{abc}^{\mathcal{A}_m} &= \delta_{a \otimes b, c}, & d_{\mathcal{A}_m} &= N. \end{aligned}$$

These two Frobenius algebras correspond to two Lagrangian condensates. Namely, they correspond to the all-electric condensate and all-magnetic condensate respectively. Clearly, $\mathcal{A}_e \subset \mathcal{A}_m = \mathbb{Z}_N$, and their simplest common module is $M = \mathbb{Z}_N$:

$$M_{\mathcal{A}_e} = M_{\mathcal{A}_m} = M = \bigoplus_{x=0}^{N-1} x, \quad d_M = N,$$

whose module functions are

$$[\rho_{M_{\mathcal{A}_e}}]_{xy}^0 = \delta_{x,y}, \quad [\rho_{M_{\mathcal{A}_m}}]_{xy}^a = \delta_{a \otimes x, y}, \quad x, y, a \in \mathbb{Z}_N.$$

Here $\delta_{p,q}$ denotes the Kronecker delta. In fact, each subgroup of $\mathbb{Z}_N$ forms a Frobenius algebra in $\mathbb{Z}_N$, and they clearly are sub-algebra of the largest Frobenius algebra $\mathcal{A}_m = \mathbb{Z}_N$ itself. Therefore, M can always be chosen as a perfect common module shared by all these Frobenius algebra.

B.3 The Modular Tensor Category $\mathcal{Z}(S_3)$

We would like to review here some basic data of the $\mathcal{Z}(S_3)$ category. The symmetry group $S_3 = \langle x, y | x^3 = y^2 = e, xy = x^2y \rangle$. The anyons of $\mathcal{Z}(S_3)$ are labelled by a pair (W, γ_W) , where W is a conjugacy class of the group S_3 , and γ_W an irreducible representation of the centralizer of W . There are eight kinds of anyons in $\mathcal{Z}(S_3)$, conventionally labelled by letters A through H . A summary of all the anyons is listed below.

	A	B	C	D	E	F	G	H
conjugacy class W	$\{e\}$			$\{y, xy, x^2y\}$		$\{x, x^2\}$		
centralizer $\cong$	S_3			$\mathbb{Z}_2$		$\mathbb{Z}_3$		
irrep γ_W of centralizer	$\mathbf{1}$	sign	π	$\mathbf{1}$	$-\mathbf{1}$	$\mathbf{1}$	ω	ω^*
$\dim(\gamma_W)$	1	1	2	1	1	1	1	1
quantum dimension $d = W \times \dim(\gamma_W)$	1	1	2	3	3	2	2	2
twist θ	1	1	1	1	-1	1	$e^{2\pi i/3}$	$e^{-2\pi i/3}$

Their fusion rules are given by

$\otimes$	A	B	C	D	E	F	G	H
A	A	B	C	D	E	F	G	H
B	B	A	C	E	D	F	G	H
C	C	C	$A \oplus B \oplus C$	$D \oplus E$	$D \oplus E$	$G \oplus H$	$F \oplus H$	$F \oplus G$
D	D	E	$D \oplus E$	$A \oplus C \oplus F \oplus G \oplus H$	$B \oplus C \oplus F \oplus G \oplus H$	$D \oplus E$	$D \oplus E$	$D \oplus E$
E	E	D	$D \oplus E$	$B \oplus C \oplus F \oplus G \oplus H$	$A \oplus C \oplus F \oplus G \oplus H$	$D \oplus E$	$D \oplus E$	$D \oplus E$
F	F	F	$G \oplus H$	$D \oplus E$	$D \oplus E$	$A \oplus B \oplus F$	$C \oplus H$	$C \oplus G$
G	G	G	$F \oplus H$	$D \oplus E$	$D \oplus E$	$C \oplus H$	$A \oplus B \oplus G$	$C \oplus F$
H	H	H	$F \oplus G$	$D \oplus E$	$D \oplus E$	$C \oplus G$	$C \oplus F$	$A \oplus B \oplus H$

There are 4 distinct Lagrangian algebras in $\mathcal{Z}(S_3)$, labelled by the 4 different subgroups of S_3 . The Lagrangian algebras corresponding to each subgroup are listed in the following table.

subgroup K	Lagrangian algebra L
$\mathbb{1}$	$A \oplus B \oplus 2C$
$\mathbb{Z}_2$	$A \oplus C \oplus D$
$\mathbb{Z}_3$	$A \oplus B \oplus 2F$
S_3	$A \oplus D \oplus F$

B.4 Haagerup Fusion Category H_3

The *Haagerup* fusion category is a notably special category. It contains six types of simple objects, labelled by

$$1, \quad \alpha, \quad \alpha^2, \quad \rho, \quad \alpha\rho, \quad \alpha^2\rho,$$

with the following quantum dimensions

$$d_1 = d_\alpha = d_{\alpha^2} = 1, \quad d_\rho = d_{\alpha\rho} = d_{\alpha^2\rho} = \frac{3 + \sqrt{13}}{2}.$$

The fusion rules are

1	α	α^2	ρ	$\alpha\rho$	$\alpha^2\rho$
α	α^2	1	$\alpha\rho$	$\alpha^2\rho$	ρ
α^2	1	α	$\alpha^2\rho$	ρ	$\alpha\rho$
ρ	$\alpha^2\rho$	$\alpha\rho$	$1 \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho$	$\alpha^2 \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho$	$\alpha \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho$
$\alpha\rho$	ρ	$\alpha^2\rho$	$\alpha \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho$	$1 \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho$	$\alpha^2 \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho$
$\alpha^2\rho$	$\alpha\rho$	ρ	$\alpha^2 \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho$	$\alpha \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho$	$1 \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho$

The seven Frobenius algebras used in the examples are

$$\begin{aligned}
\mathcal{A}_0 &= \mathbf{1}, & \mathcal{A}_1 &= \rho \otimes \mathcal{A}_0 \otimes \rho = \mathbf{1} \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho, \\
\mathcal{A}_2 &= \mathbf{1} \oplus \alpha \oplus \alpha^2, & \mathcal{A}_3 &= \rho \otimes \mathcal{A}_2 \otimes \rho = (\mathbf{1} \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho) \otimes (\mathbf{1} \oplus \alpha \oplus \alpha^2), \\
\mathcal{A}_4 &= \mathbf{1} \oplus \rho \oplus \alpha\rho, & \mathcal{A}_5 &= \alpha \otimes \mathcal{A}_4 \otimes \alpha^2 = \mathbf{1} \oplus \alpha\rho \oplus \alpha^2\rho, \\
\mathcal{A}_6 &= \alpha^2 \otimes \mathcal{A}_4 \otimes \alpha = \mathbf{1} \oplus \rho \oplus \alpha^2\rho.
\end{aligned} \tag{1}$$

The algebras $\mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6$ are related by conjugation with the invertible objects and hence are permuted by the $\mathbb{Z}_3$ symmetry.

Several of these algebras share the same underlying module object

$$M_H = \rho \oplus \alpha\rho \oplus \alpha^2\rho. \tag{2}$$

This common module object appears in several of the examples. Importantly, however, the module actions associated with different algebras need not agree. Thus two interpolations can have the same underlying module object while still activating different microscopic couplings through different module actions.

Frobenius Algebras and Modules

The Haagerup category has seven Frobenius algebras divided into three Morita classes. The Haagerup fusion rules are not commutative, so the left and right modules are slightly different. We only consider the right modules here. The module function tensor component $[\rho_M^A]_{xy}^a$ represents how the algebra object $a \in \mathcal{A}$ fuses from the right with the module object $x \in M$ and transforms it to $y \in M$, i.e., $y \in x \otimes a$. The three minimal algebras along with their modules are listed below:

1. The trivial Frobenius algebra $\mathcal{A}_0 = \mathbf{1}$, such that $f_1^{\mathbf{1}\mathbf{1}} = 1$. It has six *right* modules: $M_x = x, \rho_{xx}^{\mathbf{1}} = \mathbf{1}$, where x is a Haagerup simple object.
2. The $\mathbb{Z}_3$ Frobenius algebra $\mathcal{A}_2 = \mathbf{1} \oplus \alpha \oplus \alpha^2$, such that $f_a^{bc} = N_a^{bc}$, the fusion rules. It has two simple modules:

$$\begin{aligned}
M_1 &= \mathbf{1} \oplus \alpha \oplus \alpha^2, & [\rho_1]_{xy}^a &= N_y^{xa}, \\
M_\rho &= \rho \oplus \alpha\rho \oplus \alpha^2\rho, & [\rho_\rho]_{xy}^a &= N_y^{xa}(-1)^{\delta_{a,\alpha^2\delta_{x,\alpha\rho\delta_{y,\alpha^2\rho}}}},
\end{aligned}$$

3. The special Frobenius algebra $\mathcal{A}_4 = \mathbf{1} \oplus \rho \oplus \alpha\rho$. The algebra multiplication f_a^{bc} is too cumbersome to write here. It has four simple right modules:

$$\begin{aligned}
M_{\mathbf{1}} &= \mathbf{1} \oplus \rho \oplus \alpha\rho, & [\rho\mathbf{1}]_{xy}^a &= f_y^{xa}. \\
M_{\alpha} &= \alpha \otimes M_{\mathbf{1}} = \alpha \oplus \alpha\rho \oplus \alpha^2\rho, & [\rho\alpha]_{xy}^a &= \frac{f_y^{xa}}{[F_y^{\alpha, \alpha^2 \otimes x, a}]_{x, \alpha^2 \otimes y}}. \\
M_{\alpha^2} &= \alpha^2 \otimes M_{\mathbf{1}} = \alpha^2 \oplus \rho \oplus \alpha^2\rho, & [\rho\alpha^2]_{xy}^a &= \frac{f_y^{xa}}{[F_y^{\alpha^2, \alpha \otimes x, a}]_{x, \alpha \otimes y}}, \\
M_{\rho} &= \rho \oplus \alpha^2\rho \oplus \alpha\rho.
\end{aligned}$$

The module function of M_{ρ} is somewhat messy and will not be presented in the text.

The other connected Frobenius algebras can be obtained by conjugating these minimal algebras with some simple objects. The modules of the other non-minimal Frobenius algebras can be easily calculated from those of the minimal ones. We take $\mathcal{A}_5 = \alpha \otimes \mathcal{A}_4 \alpha^2$ as an illustrating example. For M_i any right- $\mathcal{A}_4$ module, $M_i \otimes \alpha^2$ must be a right- $\mathcal{A}_5$ module. The module function of $M_i \otimes \alpha^2$ (on $\mathcal{A}_5$) differs from M_i (on $\mathcal{A}_4$) only by some F-symbols, which can be determined immediately. We summarise all the algebras and their right-modules below.

$\mathcal{A}$	Right- $\mathcal{A}$ modules
$\mathbf{1}$	Every simple object is an independent module
$\rho \otimes \rho$	$i \otimes \rho$ where i is any simple object
$\mathbf{1} \oplus \alpha \oplus \alpha^2$	$\mathbf{1} \oplus \alpha \oplus \alpha^2, \rho \otimes (\mathbf{1} \oplus \alpha \oplus \alpha^2)$
$\rho \otimes (\mathbf{1} \oplus \alpha \oplus \alpha^2) \otimes \rho$	$(\mathbf{1} \oplus \alpha \oplus \alpha^2) \otimes \rho, \rho \otimes (\mathbf{1} \oplus \alpha \oplus \alpha^2) \otimes \rho$
$\mathbf{1} \oplus \rho \oplus \alpha\rho$	$\mathbf{1} \oplus \rho \oplus \alpha\rho, \alpha \otimes (\mathbf{1} \oplus \rho \oplus \alpha\rho), \alpha^2 \otimes (\mathbf{1} \oplus \rho \oplus \alpha\rho), \rho \oplus \alpha\rho \oplus \alpha^2\rho$
$\alpha \otimes (\mathbf{1} \oplus \rho \oplus \alpha\rho) \otimes \alpha^2$	$M \otimes \alpha^2$ where M is any right module of $\mathbf{1} \oplus \rho \oplus \alpha\rho$
$\alpha^2 \otimes (\mathbf{1} \oplus \rho \oplus \alpha\rho) \otimes \alpha$	$M \otimes \alpha$ where M is any right module of $\mathbf{1} \oplus \rho \oplus \alpha\rho$

C Critical Strange Correlator of the Ising Category

As an example, we explicitly write down the local transfer matrix for competing Ising anyons. The predicted critical unit cell (9) reads

$$M_{\sigma\sigma\sigma\sigma 1} = (\sqrt{2} + 1)/2, M_{\sigma\sigma\sigma\sigma\psi} = 1/2,$$

which gives rise to the local transfer matrix for a unit cell

$$z_{b_1 b_2 b_3 b_4 e, i_1 i_2 k_1 k_2} = \sqrt[4]{d_e^2 d_{b_1} d_{b_2} d_{b_3} d_{b_4} d_{i_1} d_{i_2} d_{k_1} d_{k_2}} M_{b_1 b_2 b_3 b_4 e} \begin{bmatrix} b_1 & e & b_2 \\ i_2 & k_1 & i_1 \end{bmatrix} \begin{bmatrix} b_3 & e & b_4 \\ i_2 & k_2 & i_1 \end{bmatrix} \quad (1)$$

We omit $b_1 = b_2 = b_3 = b_4 = \sigma$ and sum over the inner index e to

$$z_{\sigma\sigma 11} = \sqrt{2} + 1, z_{\sigma\sigma\psi\psi} = \sqrt{2} + 1, z_{\sigma\sigma 1\psi} = 1, z_{\sigma\sigma\psi 1} = 1, \quad (2a)$$

$$z_{11\sigma\sigma} = \sqrt{2} + 1, z_{\psi\psi\sigma\sigma} = \sqrt{2} + 1, z_{1\psi\sigma\sigma} = 1, z_{\psi 1\sigma\sigma} = 1. \quad (2b)$$

If we interpret 1 and ψ as spin up and down, each line in (2) gives one independent copy of the local weights $e^{2\beta} = \sqrt{2} + 1$ for a classical Ising model with the critical inverse temperature $\beta = \frac{1}{2}\ln(\sqrt{2} + 1)$. The object σ acts as a placeholder to keep these two copies disentangled.

We thus conclude that the strange correlator with the critical unit cell (9) is a partition function of two copies of critical Ising.

D The Effect of the Collection of Objects $M_{\mathcal{A}_i} \neq M_{\mathcal{A}_j}$ in the interpolation in a unit cell

We emphasised in the main text that to deconstruct the competition of condensates in the global lattice into competitions within the unit cells, it is crucial to colour the slanted edges in the unit cell by module $M_{\mathcal{A}_{i,j}}$ such that these modules contain exactly the same set of objects, even though generically module functions could differ.

Here, we illustrate with examples that when $M_{\mathcal{A}_i}$ and $M_{\mathcal{A}_j}$ contain different objects, condensates in neighbouring unit cells would be entangled non-trivially. The equilibrium constructed within one unit cell in (10) would be disrupted by its neighbours.

As an illustration, let us consider the Ising model. We again consider the competition between $\mathcal{A}_1 = \mathbf{1}$ and $\mathcal{A}_2 = \mathbf{1} \oplus \psi$ with however the following interpolation

$$\langle \bigvee_{(\mathcal{A}_1, \sigma), (\mathcal{A}_2, 1 \oplus \psi)} \rangle = \langle \hat{\mathcal{A}}_1 |_{\sigma} + r \langle \hat{\mathcal{A}}_2 |_{1 \oplus \psi}, \quad (1)$$

where we have chosen $M_{\mathcal{A}_1} = \sigma$ and $M_{\mathcal{A}_2} = \mathbf{1} \oplus \psi$. Now consider assembling the global $|\Omega\rangle$ from this unit cell, as indicated in Fig. 1 of the main text. It is clear that for two neighbouring unit cells connected by a slanted edge in which the module runs, when $\mathcal{A}_1$ appears in one unit cell, all of its neighbours must also be $\mathcal{A}_1$, and as a result all unit cells are forced into $\mathcal{A}_1$. Similarly when $\mathcal{A}_2$ appears in one unit cell, all unit cells in the global state are simultaneously $\mathcal{A}_2$. Each of these corresponds to a global RG fixed point state of a globally condensed phase.

There is thus complete correlation between all unit cells, and the equilibrium intended for a single unit cell in equation (10) did not correspond to the phase transition point as r is varied. Instead, it is found that the critical coupling occurs at $r_c = 2^{1/4} = \sqrt{\frac{d_{M_{\mathcal{A}_2}}}{d_{M_{\mathcal{A}_1}}}}$. One can readily check that this is equivalent to the equal-weight summation of the $\mathcal{A}_1$ and $\mathcal{A}_2$ condensate, with normalisation defined by the *global lattice* instead of the unit cell. Also, given that the entire global lattice now plays the role of the unit cell, the dimension of phase space is huge and, not surprisingly, the phase transition point is first-order.

When the module objects are common between $M_{\mathcal{A}_i}$ and $M_{\mathcal{A}_j}$, one can readily show that every module object in one unit cell to be fed to the next neighbouring cell would be

acted upon by every member of the condensate in the linear combination of condensates in the next cell. Therefore each unit cell forms a common background to all the condensates appearing in its neighbour cell. As a result, the equilibrium achieved in a unit cell is not disrupted by its neighbour. The unit cells are essentially factorised as desired.

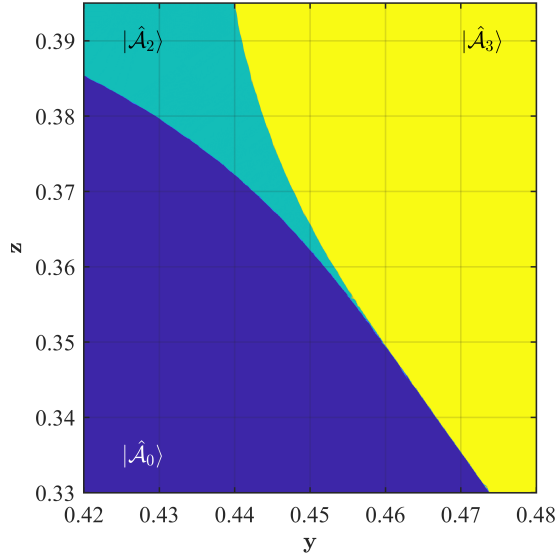
E Locating the tricritical point in the A_5 phase diagram

In the A_5 phase diagram, there appears to be a discrepancy between the analytically predicted and numerically observed locations of the tricritical point. We now resolve this issue.

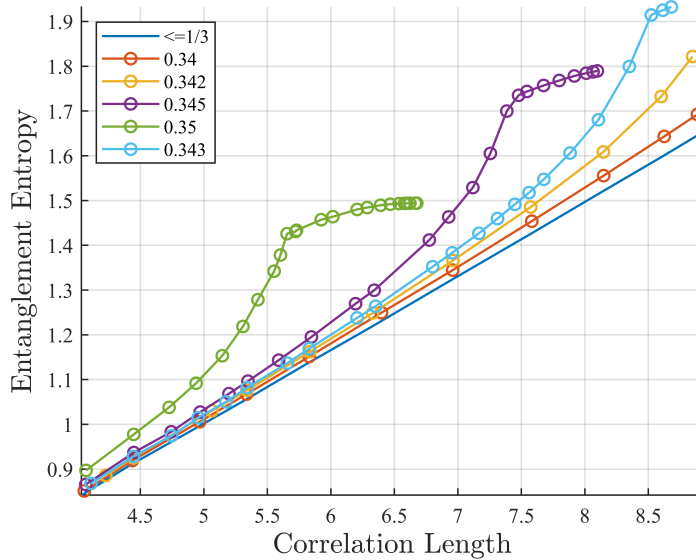
Analytically, the partition function along the KW duality line can be mapped to the duality line of the well-known Ashkin–Teller (AT) model. The $c = 1$ critical segment of the AT model corresponds to the KW dual line in our model for $0 \leq z \leq \frac{1}{3}$, with $z = \frac{1}{3}$ marking the tricritical point [28]. Beyond this point, the critical line bifurcates into two distinct Ising-critical branches.

A magnified phase diagram is shown in Supplementary Figure 16. The separation of the two Ising critical lines becomes visible around $z \approx 0.35$, below which they appear to merge into a single critical line. The distance between the two branches under our parametrisation is of order 10^{-4} at $z = 0.35$.

Here, we also present the entanglement scaling method [29] to confirm our RG calculations. In Supplementary Figure 17, we plot the entanglement entropy versus correlation length along the KW duality line $z = -\sqrt{2}y + 1$. For $z \leq \frac{1}{3}$, all data fall on the same $c = 1$ line. For larger z , however, the plots exhibit varying slopes. At small bond dimension, corresponding to less coarse-grained system sizes, the data initially suggest a CFT with



Supplementary Figure 16: Close-up view of the A_5 phase diagram near the tricritical point, coloured according to the converged tensor.



Supplementary Figure 17: Entanglement scaling for different values of z along the KW dual line of the $\mathcal{A}_5$ phase diagram.

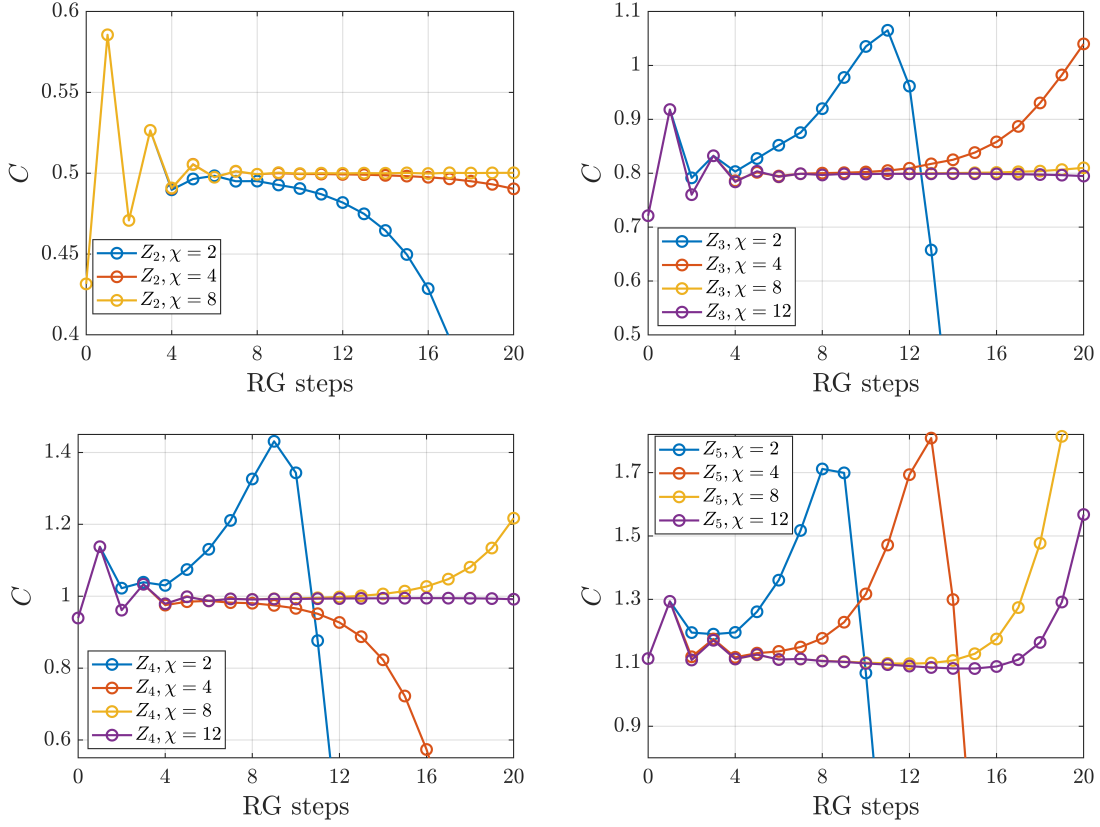
effective central charge $c > 1$. As the bond dimension increases, the scaling crosses over to a line with central charge $c \approx 0.5$ before eventually saturating, indicating a gapped phase. Slightly above $z = \frac{1}{3}$, such as at $z = 0.342$, the data form an almost straight line with apparent $c > 1$. We believe for sufficiently large bond dimensions a crossover to the $c = 0.5$ line would appear, followed by saturation. Nevertheless, these observations indicate that parts of the seemingly $c \approx 1$ region actually arise from two closely spaced Ising CFT lines.

F RG flow of the numerical central charge

In this section, we describe how the numerical central charge flows under the tensor RG, and how the associated numerical uncertainties are estimated.

At an ideal second-order critical point, the RG flow of the tensor network is expected to converge to a critical fixed-point tensor. In practice, however, the tensors are truncated to finite bond dimensions, which inevitably introduces relevant perturbations and drives the RG flow away from the critical fixed point at sufficiently large scales. The numerical central charge is extracted from the lowest eigenvalue of the transfer matrix constructed from the normalised RG tensors. Numerically, one observes that at criticality the RG flow of the estimated central charge develops a plateau, where the value remains approximately constant for several RG steps before eventually drifting away due to truncation effects.

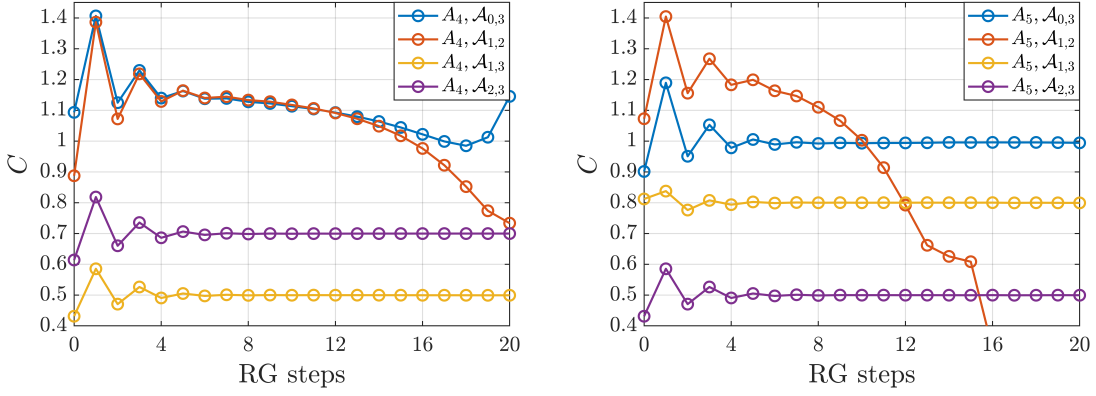
Supplementary Figure 18 shows the RG flow of the numerical central charge C for the critical N -state Potts models with $N = 2, \dots, 5$. In all cases, a clear plateau is observed around the exact known central charge. For each value of N , we present results obtained with several bond dimensions χ , corresponding to the truncation of the auxiliary legs A, B, C of the RG tensor T_{abc}^{ABC} . Increasing χ systematically extends the length of the



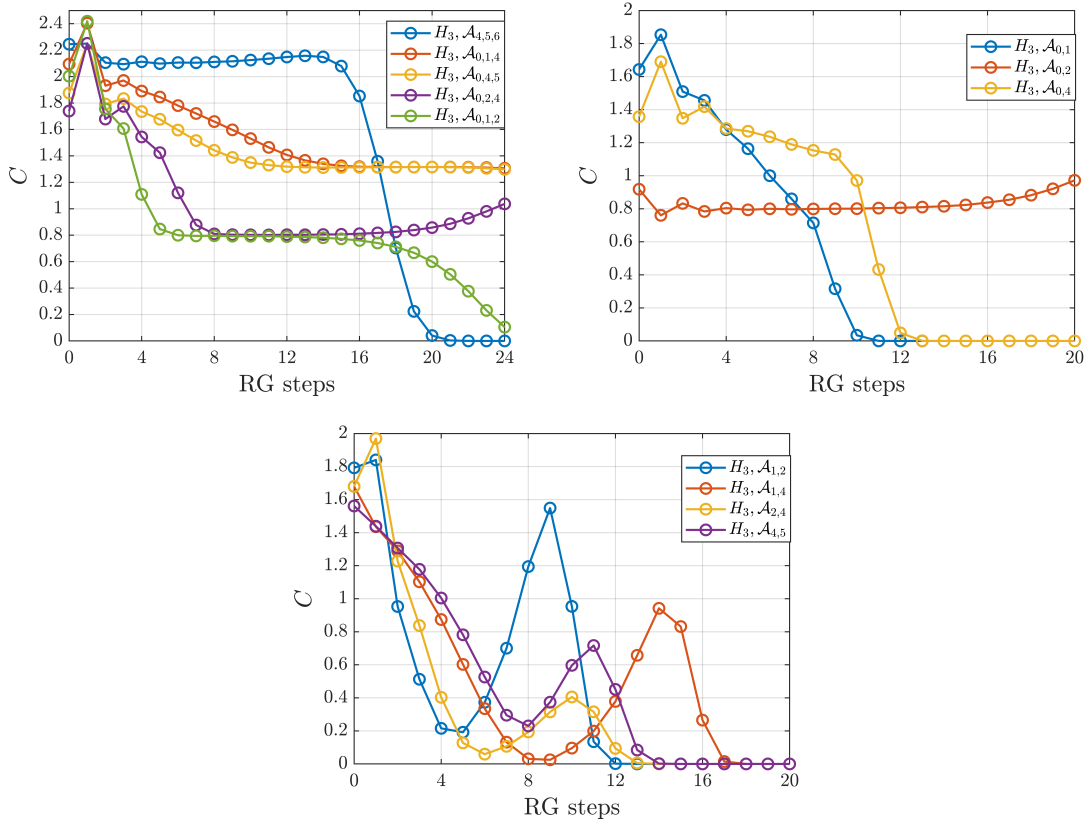
Supplementary Figure 18: RG flow of the numerical central charge C for the critical Z_N Potts model, computed using the categorical-symmetric loop TNR algorithm with different bond dimensions χ . A clear plateau develops around the exact known central charges for $Z_{2,3,4}$. Increasing χ systematically extends the length and stability of the plateau, while small χ leads to an early drift and strong deviations. For Z_5 , C is slowly decaying even for large χ , which signals a weak first-order phase transition.

plateau. Moreover, as the complexity of the critical theory increases from Z_2 to Z_4 , larger bond dimensions are required to maintain a plateau of comparable length. Concretely, for the Z_2 model, $\chi = 8$ already yields a plateau lasting more than 20 RG steps, whereas the plateau length is reduced to approximately 18 steps for Z_3 and 12 steps for Z_4 .

There is no widely accepted and reliable method to quantify the numerical uncertainty in central charges computed using TNR-based approaches. Here, we estimate the uncertainty δC by considering the maximal deviation of the central charge within a selected plateau. Specifically, we identify a window of eight consecutive RG steps exhibiting the smallest fluctuations prior to the onset of the RG drift (thus excluding the trivial plateau at 0 values), and define δC as the maximal variation of C within this window. The choice of eight RG steps reflects a compromise between statistical stability and the limitations imposed by our current computational resources. The numerical results are summarised in Supplementary Table 2. Overall, increasing the bond dimension χ systematically improves



Supplementary Figure 19: RG flow of the numerical central charge C for selected critical points of the A_4 and A_5 models.



Supplementary Figure 20: RG flow of the numerical central charge C for selected critical and tricritical points of the H_3 model. Different flows exhibit qualitatively distinct behaviors, including (slow) decay of (weak) first-order transitions and plateaus associated with continuous criticality.

the numerical quality of the TNR results, leading to longer and more stable plateaus and a clear reduction of short-scale fluctuations, as reflected by the decreasing values of δC . In most cases, this improvement is accompanied by a better agreement with the exact central charge. Nevertheless, in a few instances (notably when increasing χ from 8 to 12 in the Z_3 models), the estimates obtained at the largest accessible bond dimension do not exhibit a more accurate central charge, despite the reduced plateau fluctuations. We attribute this behavior to residual systematic effects.

As a side remark, for the 5-state Potts model, it is known that the critical point lies in the vicinity of a complex conformal field theory with central charge $C \approx 1.138 \pm 0.021 i$ [30, 31]. Consistently, the numerical central charge shown in Supplementary Figure 18 exhibits a slow decay from values $C \approx 1.1$ before eventually flowing away from the fixed-point regime. We interpret this behavior as a numerical signature of a nearby complex fixed point: The initial RG steps probe the real part of the complex central charge, while the subsequent drift reflects the instability associated with the imaginary component. Note that in such cases, the mean value of the “plateau” with the least deviation may not be the best estimation for the real part of the complex central charge.

The RG flows of some representative critical A_4 and A_5 models are shown in Supplementary Figure 19. Notice that, for the pair competitions $(\mathcal{A}_1, \mathcal{A}_2)$ and $(\mathcal{A}_0, \mathcal{A}_3)$ in A_4 and $(\mathcal{A}_1, \mathcal{A}_2)$ in A_5 , the RG flow of the numerical central charge does not have a stable plateau, so they are identified as first-order phase transitions. However, it decays slowly rather than dropping steeply. Since this shows a similar behavior to the weak first-order phase transition of the 5-state Potts model, we propose that it is also a weak first-order phase transition related to a complex CFT.

The RG flows of some representative H_3 models are shown in Supplementary Figure 20. The critical points between $\mathcal{A}_{0,4}$ appear to correspond to weak first-order phase transitions, with the real part of the central charge approximately 1.4. The tricritical points at $\mathcal{A}_{0,1,4}$ exhibit a transient plateau $C \approx 2.0$, before drifting toward a stable second plateau at $C \approx 1.32$. We identify the tricritical point as a conformal fixed point with effective central charge $C \approx 1.32$. The tricritical points at $\mathcal{A}_{0,4,5}$ initially approach $C \approx 1.9$, before converging to a stable plateau $C \approx 1.31$.

For relatively simple critical points, such as the $Z_{2,3,4}$ models discussed above, the TNR algorithm readily produces central-charge plateaus that remain stable over more than 20 RG steps. In contrast, for the Haagerup model, a substantially larger fraction of numerical resources is allocated to the categorical degrees of freedom, namely the fusion-space legs a, b, c of the tensor T_{abc}^{ABC} . This significantly increases the effective numerical cost and restricts the accessible bond dimension to $\chi \leq 5$ in our simulations. As a consequence, the RG flows exhibit stronger finite- χ effects, including increased numerical fluctuations and shorter plateaus, which limit the attainable precision of the extracted central charges. In addition, the tricritical point is itself located numerically, contributing further to the overall uncertainty. We leave possible algorithmic improvements and more accurate determinations of the central charge for future work.

Supplementary Table 2: Numerical central charge and uncertainty for $Z_{2,3,4,5}$

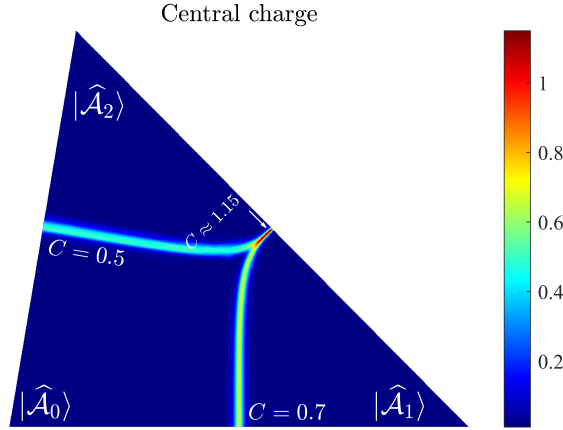
Category	χ	C	δC
Z_2	2	0.493	0.011
	4	0.4992	0.0018
	8	0.50005	0.00044
Z_3	2	0.85	0.13
	4	0.802	0.015
	8	0.7998	0.0027
	12	0.79886	0.00073
Z_4	2	1.07	0.27
	4	0.975	0.036
	8	0.994	0.011
	12	0.9942	0.0012
Z_5	2	1.27	0.4
	4	1.15	0.11
	8	1.103	0.014
	12	1.105	0.013

G More Complicated Boundary Conditions and Enlarging the Unit Cell

A salient question is how to generalise our construction to other lattices. Our choice of square lattice is to ensure that there is an equal number of plaquettes and vertices, making the equal weight summation of competing condensates a natural location of criticality. This is because we need not to distinguish fluxons and chargeons, which supposedly reside on vertices and in plaquettes respectively. For more generic lattices, we suspect that the equal mixture of competing condensates may not always generate a critical state globally. The correct relative weights may depend on the ratio of plaquettes and vertices in the given lattice. This certainly deserves future exploration.

Another important question is how to explore the effect of enlarging the unit cell.

In this paper, we have chosen a minimal unit cell, and explored the phase space of theories by varying couplings within the unit cell, which is obviously only a small corner of possible theories that realise the given symmetries. Hence, two paramount questions are



Supplementary Figure 21: The ternary phase diagram of interpolating the Frobenius algebras $\mathcal{A}_0 = 0, \mathcal{A}_1 = 0 \oplus 2, \mathcal{A}_2 = 0 \oplus 3$ with module $1 \oplus 2$ in category $\mathcal{A}_4$. They are all subalgebras of $\mathcal{A}_3 = 0 \oplus 1 \oplus 2 \oplus 3$. Each vertex is marked by one of the three normalised states $\mathcal{A}_i$. The three phases are separated by the critical lines with central charge $c > 0$.

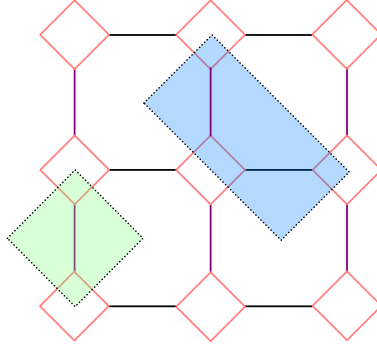
1) what the possible systems that can be realised within this restricted phase space are, and 2) how to systematically enlarge the space of theories that one explores.

To the first question, we observe that when the phase subspace gets smaller, the central charge of the resultant critical point also appears to be smaller. For example, the A -series were realised by picking a module that essentially reduces the phase space in the unit cell to 2-dimensions. In that case, the central charge of the resultant critical point seems to be smaller than that with the same choice of unit cell but a large phase space. The number of degrees of freedom involved in the unit cell seems to be related to the central charge of any realised critical point. We suspect that the “canonical” CFT for a given symTO proposed in [32] can always be realised from models constructed in the unit cell.

This observation can be qualitatively explained by the generalised KW duality. As one increases the dimension of the subspace, the degrees of freedom often fall into a decoupled multi-layer system, a direct generalisation of the double-layer structure described in the text. The CFT at the critical point of the multi-layer system is a tensor product of the CFTs of each critical layer; hence, one expects the central charge to be larger.

As the chosen unit cell’s size increases, one expects that the strange correlator should be able to describe more complicated CFTs carrying the symmetries with larger central charges. We considered a larger unit cell as shown in Supplementary Figure 22.

We observe that generalised KW duality can effectively search for critical points here. The KW duality exchanges squares with octagons and swaps horizontal edges with vertical ones. On enlarging the unit cell to accommodate the anisotropic $\langle \Omega |$, the n -dimensional phase space is promoted to a $2n$ -dimensional phase space. In general, the critical self-dual manifold in the n -dimensional phase diagram is promoted to a critical self-dual manifold with higher dimensions in the $2n$ -dimensional phase diagram. The extra dimensions of the



Supplementary Figure 22: Example of a larger unit cell. The blue region is the unit cell of the anisotropic state $\langle \Omega |$, of which the horizontal and vertical edges are in general different. If $\langle \Omega |$ is isotropic, then the unit cell is the green region, which is discussed extensively in the text above.

critical manifold are often irrelevant to the CFT. We have checked this in the case of Ising, Fibonacci, and A_k categories and the results will be presented elsewhere.

Larger unit cells can be converted to an entangled $\langle \Omega |$ state and computed by the same numerical RG process. We have not yet looked into a more general ansatz to systematically produce critical models for $\langle \Omega |$ that carry entanglement. This and other important questions will be discussed elsewhere.

References

- [1] D. B. Kaplan, J. W. Lee, D. T. Son, and M. A. Stephanov, “Conformality lost,” *Int. J. Mod. Phys. A* **25** (2010) 422–432.
- [2] A. F. Faedo, C. Hoyos, D. Mateos, and J. G. Subils, “Holographic Complex Conformal Field Theories,” *Phys. Rev. Lett.* **124** no. 16, (2020) 161601, [arXiv:1909.04008 \[hep-th\]](#).
- [3] T.-C. Huang, Y.-H. Lin, K. Ohmori, Y. Tachikawa, and M. Tezuka, “Numerical Evidence for a Haagerup Conformal Field Theory,” *Phys. Rev. Lett.* **128** no. 23, (2022) 231603, [arXiv:2110.03008 \[cond-mat.stat-mech\]](#).
- [4] L. E. Bottini and S. Schafer-Nameki, “Construction of a Gapless Phase with Haagerup Symmetry,” *Phys. Rev. Lett.* **134** no. 19, (2025) 191602, [arXiv:2410.19040 \[hep-th\]](#).
- [5] W. Ji and X.-G. Wen, “Categorical symmetry and noninvertible anomaly in symmetry-breaking and topological phase transitions,” *Phys. Rev. Res.* **2** no. 3, (2020) 033417, [arXiv:1912.13492 \[cond-mat.str-el\]](#).
- [6] A. Chatterjee and X.-G. Wen, “Symmetry as a shadow of topological order and a derivation of topological holographic principle,” *Phys. Rev. B* **107** no. 15, (2023) 155136, [arXiv:2203.03596 \[cond-mat.str-el\]](#).
- [7] L. Bhardwaj, D. Pajer, S. Schafer-Nameki, and A. Warman, “Hasse Diagrams for Gapless SPT and SSB Phases with Non-Invertible Symmetries,” [arXiv:2403.00905 \[cond-mat.str-el\]](#).
- [8] L. Chen, K. Ji, H. Zhang, C. Shen, R. Wang, X. Zeng, and L.-Y. Hung, “CFTD from TQFTD+1 via Holographic Tensor Network, and Precision Discretization of CFT2,” *Phys. Rev. X* **14** no. 4, (2024) 041033, [arXiv:2210.12127 \[hep-th\]](#).
- [9] Y. Zhao and Y. Wan, “Nonabelian anyon condensation in 2+1d topological orders: A string-net model realization,” *JHEP* **05** (2025) 156, [arXiv:2409.05852 \[cond-mat.str-el\]](#).
- [10] Y. Zhao and Y. Wan, “Landau-ginzburg paradigm of topological phases,” [arXiv:2506.05319 \[cond-mat.str-el\]](#).
- [11] Y. Zhao, H. Wang, Y. Hu, and Y. Wan, “Symmetry fractionalized (irrationalized) fusion rules and two domain-wall verlinde formulae,” *Journal of High Energy Physics* **2024** no. 4, (2024) 115.
- [12] A. Feiguin, S. Trebst, A. W. W. Ludwig, M. Troyer, A. Kitaev, Z. Wang, and M. H. Freedman, “Interacting anyons in topological quantum liquids: The golden chain,” *Phys. Rev. Lett.* **98** (2007) 160409, [arXiv:cond-mat/0612341](#).
- [13] R. Vanhove, M. Bal, D. J. Williamson, N. Bultinck, J. Haegeman, and F. Verstraete, “Mapping topological to conformal field theories through strange correlators,” *Phys. Rev. Lett.* **121** no. 17, (2018) 177203, [arXiv:1801.05959 \[quant-ph\]](#).
- [14] D. Aasen, R. S. K. Mong, and P. Fendley, “Topological Defects on the Lattice I: The Ising model,” *J. Phys. A* **49** no. 35, (2016) 354001, [arXiv:1601.07185 \[cond-mat.stat-mech\]](#).
- [15] D. Aasen, P. Fendley, and R. S. K. Mong, “Topological Defects on the Lattice: Dualities and Degeneracies,” [arXiv:2008.08598 \[cond-mat.stat-mech\]](#).
- [16] M. Levin, “Constraints on order and disorder parameters in quantum spin chains,” *Commun. Math. Phys.* **378** no. 2, (2020) 1081–1106, [arXiv:1903.09028 \[cond-mat.str-el\]](#).

- [17] F. C. Alcaraz and R. Koberle, “The Phases of Two-dimensional Spin and Four-dimensional Gauge Systems With $Z(N)$ Symmetry,” *J. Phys. A* **14** (1981) 1169.
- [18] H. Wang, Y. Li, Y. Hu, and Y. Wan, “Electric-magnetic duality in the quantum double models of topological orders with gapped boundaries,” *Journal of High Energy Physics* **2020** no. 2, (2020) 1–43.
- [19] Y. Hu and Y. Wan, “Electric-magnetic duality in twisted quantum double model of topological orders,” *Journal of High Energy Physics* **2020** no. 11, (2020) 1–39.
- [20] O. Buerschaper, M. Aguado, and G. Vidal, “Explicit tensor network representation for the ground states of string-net models,” *Phys. Rev. B* **79** (Feb, 2009) 085119.
<https://link.aps.org/doi/10.1103/PhysRevB.79.085119>.
- [21] Z.-C. Gu, M. Levin, B. Swingle, and X.-G. Wen, “Tensor-product representations for string-net condensed states,” *Phys. Rev. B* **79** (Feb, 2009) 085118.
<https://link.aps.org/doi/10.1103/PhysRevB.79.085118>.
- [22] R. Vanhove, L. Lootens, H.-H. Tu, and F. Verstraete, “Topological aspects of the critical three-state Potts model,” *J. Phys. A* **55** no. 23, (2022) 235002, [arXiv:2107.11177](https://arxiv.org/abs/2107.11177) [[math-ph](#)].
- [23] S. Yang, Z.-C. Gu, and X.-G. Wen, “Loop optimization for tensor network renormalization,” *Physical Review Letters* **118** no. 11, (Mar., 2017) .
<http://dx.doi.org/10.1103/PhysRevLett.118.110504>.
- [24] Z.-C. Gu and X.-G. Wen, “Tensor-entanglement-filtering renormalization approach and symmetry-protected topological order,” *Physical Review B—Condensed Matter and Materials Physics* **80** no. 15, (2009) 155131.
- [25] Y. Hu, N. Geer, and Y.-S. Wu, “Full dyon excitation spectrum in extended Levin-Wen models,” *Phys. Rev. B* **97** no. 19, (2018) 195154, [arXiv:1502.03433](https://arxiv.org/abs/1502.03433) [[cond-mat.str-el](#)].
- [26] Y. Zhao and Y. Wan, “Noninvertible gauge symmetry in $(2+1)$ d topological orders: A string-net model realization,” *arXiv preprint arXiv:2408.02664* (2024) .
- [27] Y. Hu, Z.-X. Luo, R. Pankovich, Y. Wan, and Y.-S. Wu, “Boundary Hamiltonian theory for gapped topological phases on an open surface,” *JHEP* **01** (2018) 134, [arXiv:1706.03329](https://arxiv.org/abs/1706.03329) [[cond-mat.str-el](#)].
- [28] Y. Aoun, M. Dober, and A. Glazman, “Phase Diagram of the Ashkin–Teller Model,” *Commun. Math. Phys.* **405** no. 2, (2024) 37.
- [29] L. Vanderstraeten, J. Haegeman, and F. Verstraete, “Tangent-space methods for uniform matrix product states,” *SciPost Phys. Lect. Notes* **7** (2019) 1, [arXiv:1810.07006](https://arxiv.org/abs/1810.07006).
- [30] V. Gorbenko, S. Rychkov, and B. Zan, “Walking, Weak first-order transitions, and Complex CFTs,” *JHEP* **10** (2018) 108, [arXiv:1807.11512](https://arxiv.org/abs/1807.11512) [[hep-th](#)].
- [31] V. Gorbenko, S. Rychkov, and B. Zan, “Walking, Weak first-order transitions, and Complex CFTs II. Two-dimensional Potts model at $Q > 4$,” *SciPost Phys.* **5** no. 5, (2018) 050, [arXiv:1808.04380](https://arxiv.org/abs/1808.04380) [[hep-th](#)].
- [32] L. Kong and H. Zheng, “A mathematical theory of gapless edges of 2d topological orders. Part I,” *JHEP* **02** (2020) 150, [arXiv:1905.04924](https://arxiv.org/abs/1905.04924) [[cond-mat.str-el](#)].