
Order from chaos with adaptive circuits on quantum hardware

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I. DEVICE SPECIFICATIONS

The device connectivity, which represents a heavy-hexagonal lattice on 156 qubits, is shown in Supplementary Fig. 1. The device specifications for the data collection days and the chosen qubits are also detailed in Supplementary Fig. 1.

II. EXPERIMENTAL DETAILS

At each problem size and control probability parameter, 50 different circuit realizations were constructed. Each circuit realization was repeated for 1000 shots. Supplementary Fig. 2 shows the average number of entangling two-qubit operations and measurement-induced conditional resets at various values of the control parameter p . In particular, we considered problem sizes L from 10 to 100 with a step size of 10 and p from 0.2 to 0.85 with a step size of 0.05. The total number of time steps was set to L^2 , and correspondingly, the two-qubit gate and reset count increased with L . Lower values of the control parameter require denser circuits with more entangling operations, with the maximum number of two-qubit gates reached at approximately 8000 gates at $p = 0.2$ for $L = 100$. The maximum number of resets is reached at $p = 0.5$, with $L = 100$, using nearly 5000 resets. At $p_c = 0.5$, the circuit is most resource-intensive, with 5000 resets intermixed with 5000 two-qubit operations. Without transpilation, the number of resets should increase linearly with p , similar to the two-qubit operations that decrease linearly with p . However, a reset operation followed by another one is trivial, so we merge such consecutive resets, shown as the diminishing trend in the number of resets beyond $p = 0.5$ in Supplementary Fig. 2. Supplementary Fig. 2 shows that the circuit duration is maximal at $p = 0.5$ as the duration of reset is substantially longer compared to the two-qubit gate duration.

III. FINITE-SIZE SCALING ANALYSIS

With a given finite-size scaling ansatz, we can numerically fit and extract the critical exponents, such that all curves with different system sizes collapse onto a single universal curve after proper rescaling (see insets in Figs. 2 and 3 of the main text). For example, given a set of data points (observations) $\{\mathcal{O}_i^{(j)}\}$ at a tuning parameter p_i and system size L_i , where $i = \{1, 2, \dots, N\}$ is the number of different parameters, and $j = \{1, 2, \dots, M\}$ is the number of different shots, the generic scaling form is

$$\overline{\mathcal{O}}_i \sim L_i^{-\beta/\nu} f[(p_i - p_c)L_i^{1/\nu}], \quad (1)$$

where $\overline{\mathcal{O}}_i \equiv \frac{1}{M} \sum_{j=1}^M \mathcal{O}_i^{(j)}$. In order to find the optimal set of parameters (β, ν, p_c) , we define the loss function as its reduced χ_ν^2 value to quantify the goodness of the fit,

i.e.,

$$\chi_\nu^2 = \frac{1}{N} \sum_{\iota=1}^N \left(\frac{\tilde{\mathcal{O}}_\iota - \tilde{\mathcal{D}}_\iota}{\tilde{\sigma}_\iota} \right)^2, \quad (2)$$

where ι is the permutation of i to ensure the ascending order of $x_\iota = (p_\iota - p_c)L_\iota^{1/\nu}$, and $\tilde{\mathcal{O}}_\iota = L_\iota^{\beta/\nu} \overline{\mathcal{O}}_\iota$. $\tilde{\mathcal{D}}_\iota$ is the linear interpolation between the nearest two points $(x_{\iota-1}, \tilde{\mathcal{O}}_{\iota-1})$ and $(x_{\iota+1}, \tilde{\mathcal{O}}_{\iota+1})$ on both sides of $(x_\iota, \tilde{\mathcal{O}}_\iota)$, namely,

$$\tilde{\mathcal{D}}_\iota = \frac{x_{\iota+1} - x_\iota}{x_{\iota+1} - x_{\iota-1}} \tilde{\mathcal{O}}_{\iota-1} + \frac{x_\iota - x_{\iota-1}}{x_{\iota+1} - x_{\iota-1}} \tilde{\mathcal{O}}_{\iota+1}, \quad (3)$$

If $(x_\iota, \tilde{\mathcal{O}}_\iota)$ is on the boundary, (i.e., $\iota = 1$ or N), we simply use single side linear interpolation.

The denominator in Eq. (2) $\tilde{\sigma}_\iota$ is the corresponding standard deviation of the interpolated $\tilde{\mathcal{D}}_\iota$, defined as

$$\tilde{\sigma}_\iota^2 = \sigma_\iota^2 + \left(\frac{x_{\iota+1} - x_\iota}{x_{\iota+1} - x_{\iota-1}} \sigma_{\iota-1} \right)^2 + \left(\frac{x_\iota - x_{\iota-1}}{x_{\iota+1} - x_{\iota-1}} \sigma_{\iota+1} \right)^2, \quad (4)$$

where $\sigma_\iota^2 = \frac{1}{M} L^{-\beta/\nu} \sum_{j=1}^M (\mathcal{O}_\iota^{(j)} - \overline{\mathcal{O}}_\iota)^2$.

We minimize the loss function in Eq. (2) using the Levenberg-Marquardt algorithm [1, 2] provided by the Python package `lmfit` [3] to obtain the optimal value of β, ν, p_c along with their error bars.

For the temporal scaling in Fig. 3(a,b) of the main text, we can use the same procedure to collapse Eq. (1) but for the scaling function $\sim f(t/L^z)$ (and its generalization to finite scaling dimension). The corresponding error bar can thus be estimated by $\delta z = \nu^{-2} \delta \nu$.

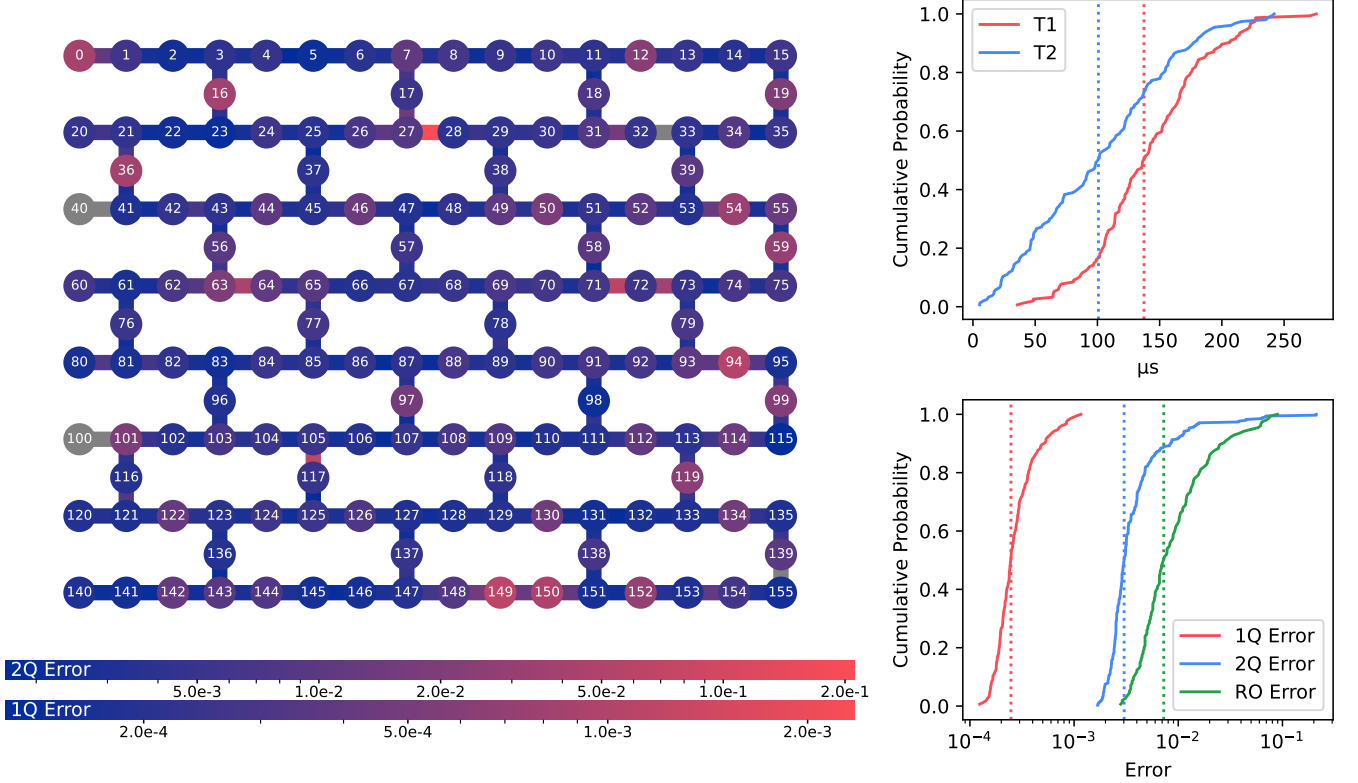
IV. NOISE ANALYSIS FOR THE MAGNETIZATION

In this section, we discuss the noise effect on the steady state of the magnetization, as shown in Fig. 2a of the main text. We model noise as a single-qubit depolarizing error channel $\Phi^{(1)}$ and two-qubit depolarizing error channel $\Phi^{(2)}$, i.e.,

$$\Phi^{(n)}(\rho) = \left(1 - p_e^{(n)}\right) \rho + \frac{p_e^{(n)}}{2^n} \mathbb{1}_{2^n}, \quad (5)$$

acting on the single-qubit rotation $\hat{R}_\alpha(\theta)$ (see Eq. (3) of the Methods) and CZ gate in the unitary, respectively (see Eq. (2) of the Methods) where ρ is the density matrix for a single qubit, $\mathbb{1}$ is the maximally mixed state up to a normalization factor, and $p_e^{(1)}$ ($p_e^{(2)}$) is the single-qubit (two-qubit) depolarizing error probability. Using a Choi–Jamiołkowski isomorphism, the density matrix $\rho = \sum_{i,j} \rho_{ij} |i\rangle\langle j|$ can be encoded as a ket state in the double Hilbert space as $|\rho\rangle\rangle = \sum_{ij} \rho_{ij} |i\rangle |j\rangle$, where the quantum channel $\Phi^{(n)}$ is then described as

$$\hat{\Phi}^{(n)}|\rho\rangle\rangle \equiv |\Phi^{(n)}(\rho)\rangle\rangle = \left[\left(1 - p_e^{(n)}\right) + p_e^{(n)} |\mathbb{1}\rangle\rangle\langle\mathbb{1}| \right] |\rho\rangle\rangle, \quad (6)$$



Supplementary Figure 1 | Experimental device. Device layout showing the chosen qubits and their specifications for all the experimental realizations on `ibm_fez` corresponding to the data shown in Fig. 2 of the main text. Unused and uncharacterized qubits and couplings have been grayed out. The mean T_1 and T_2 times, shown as dotted vertical lines in the plot above, were $1.4 \times 10^2 \mu s$ and $1.0 \times 10^2 \mu s$, respectively. Single and two-qubit durations were set at 24 ns and 84 ns and the readout duration was 1.6×10^3 ns. On average, single-qubit, two-qubit and readout errors were respectively 3.0×10^{-4} , 1.8×10^{-2} , 1.3×10^{-2} .

where $|1\rangle\rangle = 2^{-n} \sum_{i=0}^{2^n-1} |i\rangle\rangle$ is the maximally entangled Einstein–Podolsky–Rosen (EPR) pair between the bra and ket space, and $n = 1, 2$ for the single and two-qubit channels, respectively.

Therefore, the two-qubit density matrix, on which the corresponding Bernoulli map acts, averaged over Haar randomness is

$$\mathbb{E}_{\text{Haar}} [\hat{U}|\rho\rangle\rangle] = \mathbb{E}_{\text{Haar}} [\hat{U}] |\rho\rangle\rangle = \mathbf{J}_4 \oplus \mathbf{0}_{12} |\rho\rangle\rangle \quad (7)$$

where $\hat{U}$ is the Haar random unitary U in the double Hilbert space. Here, $\mathbf{J}_4$ is an all-one matrix of the size 4 by 4 acting on the basis in $|\rho\rangle\rangle$ that corresponds to

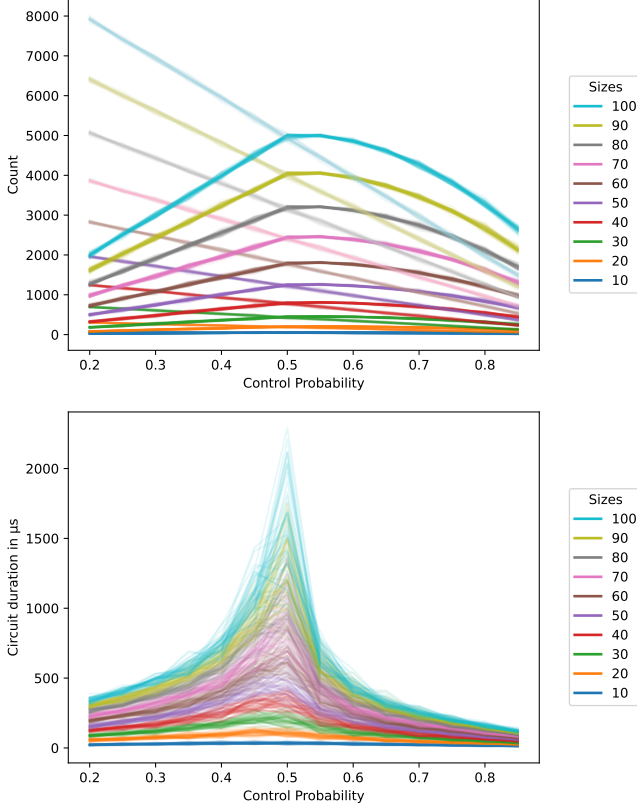
the diagonal elements of ρ , and $\mathbf{0}_{12}$ is a zero matrix of the size 12 by 12 acting on those basis states $|\rho\rangle\rangle$ associated with the off-diagonal elements of ρ because the off-diagonal element of ρ will be averaged to zero within Haar randomness.

Similarly, the single-qubit density matrix, on which the control map acts, is

$$\hat{C}|\rho\rangle\rangle = (|00\rangle\rangle\langle\langle 00| + |00\rangle\rangle\langle\langle 11|)|\rho\rangle\rangle, \quad (8)$$

where $\hat{C}$ is the control map C in the double Hilbert space, such that it flips $|1\rangle\langle 1|$ to $|0\rangle\langle 0|$ while leaving $|0\rangle\langle 0|$ intact.

We only track the diagonal elements of the density matrix because the off-diagonal elements will not mix with



Supplementary Figure 2 | Top: the total number of entangling two-qubit operations (lighter curves) and measurement-induced conditional resets (darker curves). Bottom: Estimated circuit duration as a function of control probability for all 50 circuit realizations.

the diagonal ones. Although the mapping is not strict in the classical model, it suffices to study the change of the order parameter in the control-induced phase transition. We thus evolve the bit-string probabilistically following the transition matrix in Eq. (7) and (8) with $1 - p$ and p , in analogy to a Markov process.

We apply a single-qubit depolarizing noise channel in Eq. (6) after each measurement with $p_e^{(1)}$ on the qubit where it was acted on, and two-qubit noise channel in Eq. (7) after the Haar random gate with $p_e^{(2)}$. Finally, we compute the magnetization averaged over different trajectories as shown in Supplementary Figs. 3 a and b with the insets showing the data collapse. We find that although the ferromagnetic fixed point state deep inside the controlled phase is suppressed, it still demonstrates a sharp phase transition near $p = 0.5$, leading to a similar critical exponent, which is consistent with the experimental findings in Fig. 2a of the main text. In Supplementary Figs. 3 c and d, we show fitted p_c and ν as a function of the depolarizing error with a fixed relation $p_e^{(1)} = 10p_e^{(2)}$. We find that the depolarizing error does not affect the critical exponents within the error bar. Experimentally, the error rates are reported as 10^{-4} to 10^{-2} for the two-qubit gate, and 10^{-3} to 10^{-2} for the measurement error,

which implies our noise analysis does not underestimate the impact of errors.

V. CLASSICAL MODELS

In this work, we have considered circuits where the sequence of scrambling unitaries sample the Bernoulli map $B_{i,i+1}$ in each application of the chaotic map that is chosen randomly from the distribution $\mathcal{D}$. Under the assumption that this distribution over unitaries $\mathcal{D}$ is invariant under conjugation by single-qubit unitaries from either a one-design or a two-design, we can compute up to first-moment or second-moment quantities, respectively, of observables linear in ρ . Specifically, the k 'th moment of an observable is

$$\mathbb{E}_{C \sim \mathcal{B}(p)} [\text{tr}[(\rho_C \mathcal{O})^{\otimes k}]], \quad (9)$$

where ρ_C is the output of the circuit when the sampled unitaries are $C = (U_1, \dots, U_m)$. The state ρ_U is in general mixed because the circuit has nonunitary channels corresponding to the control map in addition to the unitary channels coming from $U_1, \dots, U_m$. By exchanging the trace and the expectation value, we can rewrite the k 'th moment as

$$\text{tr} [\mathbb{E}_{C \sim \mathcal{B}(p)} [\rho_C^{\otimes k}] \mathcal{O}^{\otimes k}]. \quad (10)$$

In this section, we deal with classical techniques for directly accessing the average k -copy state $\mathbb{E}_{C \sim \mathcal{B}(p)} [\rho_C^{\otimes k}]$ for $k = 1$ and 2, which lets us compute the first and second moments of the observable $\mathcal{O}$. We denote the average k -copy state by $\bar{\rho}$. Also denote the partial trace of $\bar{\rho}$ at a subset A by $\bar{\rho}_A := \text{tr}_{A^c} \bar{\rho}$. It should be noted that this technique only gives us access to averages over unitaries from some suitable ensemble, and not the specific k 'th moment for any particular sequence of unitaries C .

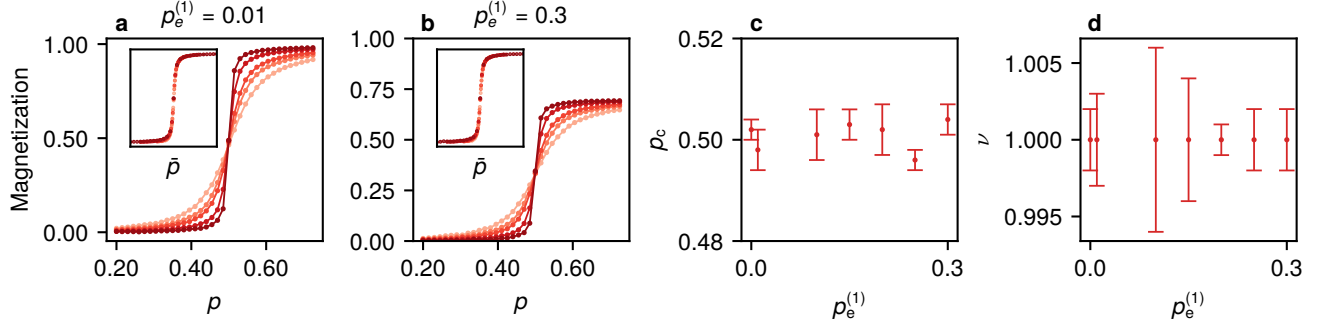
A. First moment

When $k = 1$, it suffices to consider the average state after the chaotic and control maps. Contained in the average $\mathbb{E}_{C \sim \mathcal{B}(p)}$ over the Bernoulli circuits lies an average $\mathbb{E}_{U_i \sim \mathcal{D}}$ over random circuits, which we first tackle. After a chaotic map at sites j, j' , the 1-copy average state evolves as:

$$\bar{\rho}_{jj'} \rightarrow \mathbb{E}_{U_i \sim \mathcal{D}} [U_i \bar{\rho}_{jj'} U_i^\dagger] \quad (11)$$

$$= \frac{\mathbb{1}_{jj'}}{4}. \quad (12)$$

This map is operationalized by replacing the classical representation of the state at these sites by a uniformly random bit. On the other hand, after a control map on site j , the state $\bar{\rho}_j$ gets reset to $|0\rangle\langle 0|_j$, which is also easy to track using a classical bit. These two simple rules suffice



Supplementary Figure 3 | Control transition in the presence of noise. Results are shown for statistical mechanics model calculations (red circles; $L = 16 - 128$). **a** The magnetization density averaged over initial states, measurement outcomes and circuits (denoted $\langle M_z \rangle$) as a function of the control probability p for various system sizes L , and single-qubit depolarizing error $p_e^{(1)} = 0.01$ and two-qubit depolarizing error $p_e^{(2)} = 0.001$. (Inset) Data collapse as a function of $\tilde{p} = L^{1/\nu}(p - p_c)$ with $p_c = 0.498(1)$ and $\nu = 1.000(2)$. **b** The magnetization density for single-qubit depolarizing error $p_e^{(1)} = 0.3$ and two-qubit depolarizing error $p_e^{(2)} = 0.03$. (Inset) Data collapse as a function of $\tilde{p} = L^{1/\nu}(p - p_c)$ with $p_c = 0.504(3)$ and $\nu = 1.000(2)$. **c** The fitted p_c as a function of the single-qubit depolarizing error rate $p_e^{(1)}$, where $p_e^{(2)} = p_e^{(1)}/10$ is fixed. **d** The fitted ν as a function of the single-qubit depolarizing error rate $p_e^{(1)}$. In **c** and **d** the error bar is the Levenberg–Marquardt standard error of the fitted parameter (Sec. III), with $n = 180$.

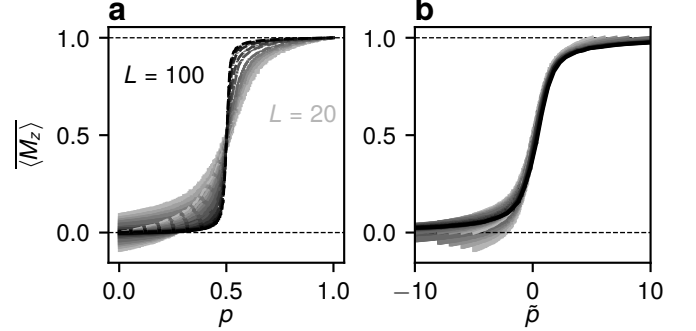
to yield a classical random walk that reproduces the order parameter $\mathbb{E}_{C \sim \mathcal{B}(p)}[\text{tr } \rho_C \mathcal{O}]$ as well as the features of the shot noise seen in Figs. 2 and 3 of the main text.

We reproduce the magnetization density using the first-moment stat-mech model as shown in Supplementary Fig. 4a similar to Fig. 2a of the main text. In Supplementary Fig. 4b, we perform the finite-size scaling analysis, which manifests the same critical behavior as in the inset of Fig. 2a of the main text.

In Supplementary Fig. 5(a-c), we reproduce the distribution of the quantum fluctuations (shot-to-shot fluctuation) for the chaotic regime (Supplementary Fig. 5a), the critical point (Supplementary Fig. 5b), and the controlled regime (Supplementary Fig. 5c), under various system sizes from $L = 20$ to 100. We find that, although they do not quantitatively reproduce the distribution of the shot-to-shot fluctuation (as we will show in Sec. VI that the dephasing model is the closest classical model to the quantum case, yet the dephasing model still fails to capture all the features deep inside the chaotic regime and the controlled regime)—because it is by construction only accurate up to the first-moment while the shot-to-shot fluctuation is a second-moment quantity, it does capture the distribution near the critical points, and reproduces the same scaling dimension $\beta = 1$, as shown in Supplementary Fig. 5(d,e).

B. Second moment

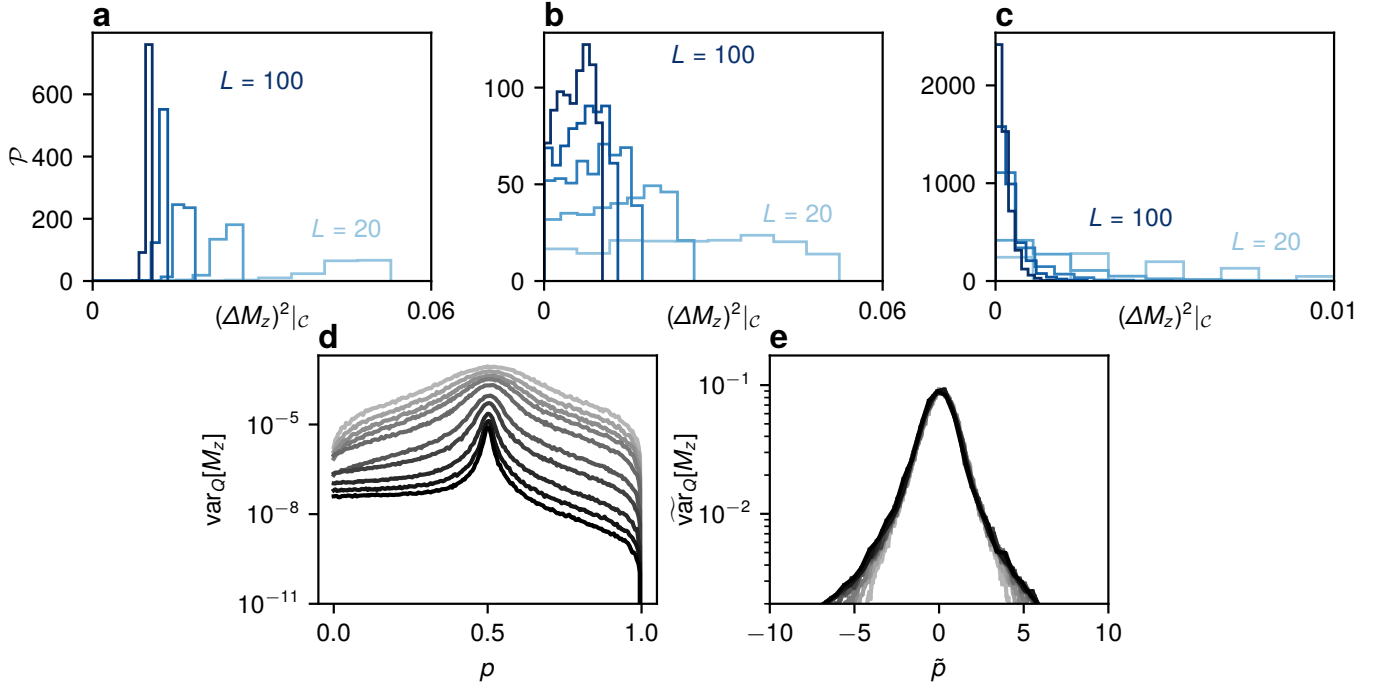
We can also generalize the previous calculation to the second moment. For both unitary [4–7] and nonunitary random circuits [8–10], there is prior work computing several second-moment quantities such as expected purity, fidelity with a target state, the cross-entropy benchmark, and the variance of local observables. In these settings,



Supplementary Figure 4 | Magnetization in the first-moment stat-mech model. **a** Magnetization density as a function of p for different system sizes. ($L = 20 - 100$ from light to dark curves.) **b** Finite-size scaling of **a**. The fitted critical exponents are $p_c = 0.4970(1)$ and $\nu = 1.0000(1)$ with a fitting range of $p \in [0.4, 0.6]$.

the calculation is greatly simplified by virtue of some assumptions on the ensemble of random circuits, which do not hold in our case. We show how to overcome this hurdle by relaxing two fundamental assumptions underlying these calculations.

First, one major assumption in several of the above works is that each operation in the random circuit is chosen independently of the others. In our setting, this assumption is invalid because after the application of every map, the choice of the next qubit(s) to apply the next map to depends on whether we applied the chaotic map or the control map. In other words, the random circuit has important spatial correlations. In fact, without this feature, our class of circuits would not feature a phase transition for a local observable. We have already accounted for this correlation in the previous subsection when we calculate the first moment. The tradeoff is that we do not analytically compute expectation values



Supplementary Figure 5 | Shot-to-shot fluctuations in the first-moment stat-mech model. Behavior of shot-to-shot fluctuations of the order parameter from the classical stat-mech model for the first moment. The fitted critical exponent is $p_c = 0.5010(1)$, $\nu = 1.00(2)$, and $\beta = 1.011(4)$ with a fitting range of $p \in [0.4, 0.6]$.

of first- or second-moment quantities but instead supply an *efficient algorithm* to compute them.

Second, and more importantly, another assumption underlying all the above works is that the ensemble of random circuits is *locally scrambling* [11], i.e., is invariant under the insertion of random single-qubit gates from a 2-design after every operation. In our case, this assumption is manifestly invalid, since after the control map there is no such single-qubit operation. Indeed, if our ensemble of circuits were locally scrambling, there would be no signal to discern in a local observable and we would have $\text{tr}[\mathbb{E}_C[\rho_C]\mathcal{O}] = 0$. We can nevertheless define a custom and novel stat-mech model for this situation, which can again be classically efficiently simulated. We illustrate the technique by calculating the collision probability.

First, observe that it is enough to understand the physics of the two maps and the effect they have on the two-copy average state $\mathbb{E}_{C \sim \mathcal{B}(p)}[\rho_C^{\otimes 2}] =: \bar{\rho}$. Since we are not considering any postselection and hence probe quantities linear in ρ , the effect of these maps may be computed by considering an input reduced density matrix supported at the location of these maps. For example, for a control map acting on qubit j , it is enough to consider its action on $\bar{\rho}_j$, which we recall is defined as $\bar{\rho}_j = \text{tr}_{j^c} \bar{\rho}$. The collision probability is given by

$$\begin{aligned} \sum_{\mathbf{x}} p_{\mathbf{x}}^2 &= \sum_{\mathbf{x}} \text{tr} \rho^{\otimes 2} |\mathbf{x}^{\otimes 2}\rangle\langle \mathbf{x}^{\otimes 2}| \\ \Rightarrow \mathbb{E}_C \sum_{\mathbf{x}} p_{\mathbf{x}}^2 &= \sum_{\mathbf{x}} \text{tr} \bar{\rho} |\mathbf{x}^{\otimes 2}\rangle\langle \mathbf{x}^{\otimes 2}|. \end{aligned} \quad (13)$$

A certain subset A of the $[n]$ qubits have a chaotic map

last applied to them, and their complement $A^c = [n] \setminus A$ have a control map last applied to them. Since there is a control map on this subset A^c , these qubits are in the state $\bar{\rho}_{A^c} = (|00 \dots 0\rangle\langle 00 \dots 0|_{A^c})^{\otimes 2}$. The two-copy state on the other qubits in A lies in $\text{span}\{I, S\}^{|A|}$, where I and S are the identity and SWAP operators. Denote the entire two-copy state

$$\bar{\rho} = \left(\sum_{\mathbf{x} \in \{0,1\}^{|A|}} c_{\mathbf{x}} S^{\mathbf{x}} \right) \otimes (|00 \dots 0\rangle\langle 00 \dots 0|_{A^c})^{\otimes 2}. \quad (14)$$

The expected collision probability is

$$\sum_{\mathbf{x}: \mathbf{x}_{A^c} = 00 \dots 0, \mathbf{y}} \text{tr} c_{\mathbf{y}} S^{\mathbf{y}} |\mathbf{x}^{\otimes 2}\rangle\langle \mathbf{x}^{\otimes 2}| = 2^{|A|} \sum_{\mathbf{y}} c_{\mathbf{y}}. \quad (15)$$

We have seen that it suffices to keep track of the quantity $\sum_{\mathbf{y}} c_{\mathbf{y}}$. We compute the action of the two maps and state some useful facts about the dynamics of the two-copy state through the following claims.

Claim 1 (Two-copy state after chaotic map). *For any arbitrary subset $A \subseteq [n]$, the reduced two-copy state in the circuit dynamics after the application of a chaotic map lies in the space $\text{span}\{I, S\}^{\otimes |A|}$, where $|A|$ is the number of qubits in A and the operators I and S are the identity and SWAP operators, respectively, acting on a qubit and its copy.*

This follows from the property that the chaotic map is locally scrambling. For a locally scrambling distribution over two-qubit unitaries, the distribution is invariant under insertion of random single-qubit gates from a

2-design. Let us denote these V . Averaging over these random single-qubit gates, we project on to the symmetric subspace, which is spanned by $\{I, S\}^n$. We can explicitly compute this projection using linearity of trace:

$$\begin{aligned} & \mathbb{E}_V[(V \otimes V)\bar{\rho}(V \otimes V)^\dagger] \\ &= \frac{\text{tr } \bar{\rho} - \frac{1}{2} \text{tr } \bar{\rho} S}{3} I + \frac{\text{tr } \bar{\rho} S - \frac{\text{tr } \bar{\rho}}{2}}{3} S. \end{aligned} \quad (16)$$

Claim 2 (Two-copy state after control map). *The reduced two-copy state in the circuit dynamics after the application of a control map on qubit j is $|0\rangle\langle 0|^{\otimes 2}$.*

This fact follows from the fact that the control map is a reset operation, producing the state

$$|00\rangle\langle 00| = \frac{1}{4}(II + IZ + ZI + ZZ). \quad (17)$$

These two claims, together with the observation that at every point in time, the reduced two-copy state at a particular site is either $|0\rangle\langle 0|^{\otimes 2}$ or in $\{I, S\}$, suffice for the classical simulation of the collision probability.

When the chaotic map is a Haar-random two-qubit unitary, we have the following rules for the classical random walk on $\{I, S\}^2$:

$$\begin{aligned} II &\rightarrow II; \quad SS \rightarrow SS, \\ IS, SI &\rightarrow \frac{4}{5} \left(\frac{II + SS}{2} \right). \end{aligned} \quad (18)$$

This yields the following update rules for the collision probability. We examine three cases. i) Both qubits are in a pure state, ii) One qubit is pure and the other already lies in $\{I, S\}$, iii) Both qubits are in $\{I, S\}$. For convenience, we note that if the initial state is pure, for example in the $|0\rangle$ state, upon applying a single-qubit Haar average we get:

$$|0^{\otimes 2}\rangle\langle 0^{\otimes 2}| \rightarrow \frac{I + S}{6}. \quad (19)$$

i) In this case, the input two-copy average state at sites j, j' where the chaotic map is applied is $|0\rangle\langle 0|_j^{\otimes 2} (aI + bS)_{j'}$. In the input state, the contribution of these qubits to the collision probability is $1 \times 2(a+b)$. Here the factor 1 comes from the qubit j being in a pure state, while the second factor $2(a+b)$ comes from the qubit j' being in $(aI + bS)_{j'}$ (see Equation (15)). After application of the chaotic map, the output two-copy state is given by

$$II \left(\frac{7a}{30} + \frac{2b}{30} \right) + SS \left(\frac{2a}{30} + \frac{7b}{30} \right). \quad (20)$$

This yields a contribution to the collision probability of

$$2^2 \times \left(\frac{9a}{30} + \frac{9b}{30} \right) = \frac{6}{5}(a+b). \quad (21)$$

Therefore, the contribution to the collision probability has decreased from $2 \rightarrow \frac{6}{5}$, a factor of $\frac{3}{5}$.

ii) The input two-copy average state is $|00\rangle\langle 00|_{jj'}^{\otimes 2}$, since both sites last had a control map applied to them. The initial contribution to the collision probability is 1. After the chaotic map, the output two-copy state is now

$$\frac{II + SS}{20}. \quad (22)$$

This has a contribution of $2^2 \times \frac{2}{20} = \frac{2}{5}$ to the collision probability. In this case, the reduction in collision probability is by a factor of $\frac{2}{5}$.

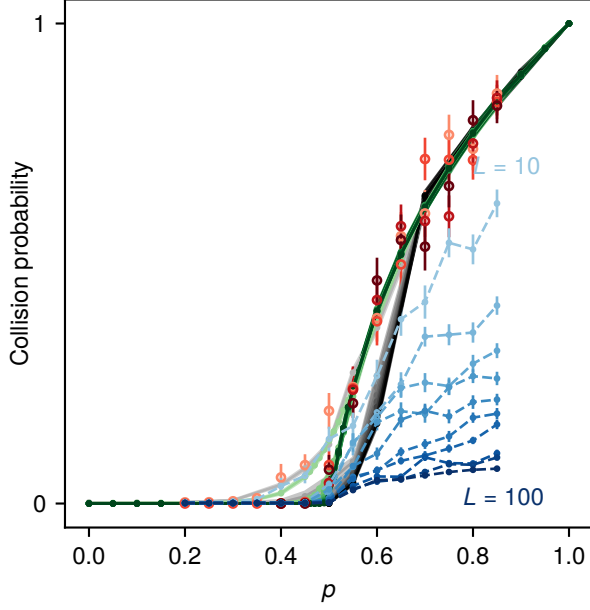
iii) If the reduced two-copy state on qubits j, j' is in $\text{span}\{I, S\}^{\otimes 2}$, we apply the rules in Equation (18). The classical simulation of the stat-mech model samples “words” in $\{I, S\}^{\otimes 2}$. If the sampled words are II (respectively, SS), the update rules keep them in II (resp. SS). The collision probability is left unchanged. If the word is either IS or SI , we pick with probability $1/2$ each whether to update it to either II or SS . The factor $4/5$ from Equation (18) is accounted for by multiplying the collision probability by $4/5$ in this case.

Lastly, we examine what happens when we apply a control map at site j . If the reduced two-copy state at site j is already in $|0\rangle\langle 0|^{\otimes 2}$, the control map leaves it unchanged. If we have $\bar{\rho}_j = \frac{a}{4}I + \frac{b}{4}S$ (with $\text{tr } \bar{\rho}_j = a+b=1$), we effectively sample I with probability a and S with probability b . When the sampled word is I , the contribution to the collision probability prior to the control map is $2 \times \frac{a}{4} = \frac{a}{2}$, and the final contribution is a . Similarly, when the sampled word is S , the prior contribution to the collision probability is $2 \times \frac{b}{4} = \frac{b}{2}$, and the final contribution is b . Therefore, upon application of the control map, if the input sampled word is I , we pick up a factor of 2 in the collision probability. If the sampled word is S , we leave the collision probability unchanged.

Thus, we can see that it is the control map that results in an increase in collision probability. This is natural, because the control map tends to reset qubits to $|0\rangle$, and in the extreme case the output distribution is peaked on bit-strings with many $|0\rangle$ s.

These rules, taken together, allow for a classical simulation of the expected collision probability, as shown in Supplementary Fig. 6. At $p = 0$, we numerically find that the predicted collision probability is close to the ideal value of $2/(2^n + 1) \approx 2^{-(n-1)}$, and at $p = 1$, it approaches the value 1, which is expected for the fixed point state $|00\dots 0\rangle$, which has a peaked distribution. We also roughly see that in the control phase $p > 0.5$, the collision probability is nonzero even for large L . This illustrates that the classical bit-string distribution in this phase has a large peak, even though it may be formally supported on many other bit-strings. When $p < 0.5$, the collision probability is exponentially small and the stat-mech model needs a large number of samples to precisely predict the ideal expected collision probability. This behavior is also expected, since in this phase the classical distribution is supported on exponentially many bit-strings.

Finally, we note that we can also extract similar



Supplementary Figure 6 | Collision probability. Blue dashed curves for the experimental data. Red curves for MPS simulation using the same 49 circuits. Green curves for the first-moment stat-mech model, and the gray curves for the second-moment stat-mech model. The error bar is $\sigma/\sqrt{n}$ with $n = 49$ circuit realizations for the experimental and MPS data; the shaded band for the two stat-mech models is a 95% bootstrap percentile interval over 100 resamples.

physics from the first-moment stat-mech model, although we do not expect it to accurately model second-moment quantities. Formally, we predict a collision probability from the first-moment stat-mech model by pretending that the true state is given by $|00\dots 0\rangle_A \otimes \prod_{j \notin A} (\frac{1}{2})_j$, where A is again the set of qubits that last encountered a reset operation. This predicts a collision probability of $2^{-(n-|A|)}$. For $p = 0$, since $|A| = 0$, this predicts a collision probability of 2^{-n} , which differs from the ideal collision probability of $\approx 2^{-1-n}$. This difference is not detectable from the experiment because at large system sizes ($L \gtrsim 40$), the number of outcomes is much larger than the number of shots taken.

VI. DEPHASING MODEL

In this section, we introduce the details of the noise model in the dephasing channel, which we dub the “dephasing model” for brevity. The motivation of the dephasing model is to try to find a noise channel that is as close as possible to classically “spoofing” the observer when trying to measure the quantum fluctuations defined in Eq. (5) of the Methods. This is distinct from the statistical mechanics model we have investigated, which is more akin to the depolarizing noise channel that averages over Haar unitaries to define the classical process (see Sec. V). Instead, we can retain more classical fluctu-

ations by focusing on a “classical” circuit (i.e., without quantum coherence) while keeping the notation of “circuit realization” sharp (i.e., without first taking the Haar random average of the density matrix).

Therefore, the dephasing model only focuses on the evolution of the *amplitude* of the wave function, which is an L_1 -normed probability mass function $|\psi\rangle$ supported on the computational basis states $\mathbf{b} = b_1 b_2 \dots b_L \equiv \sum_{i=1}^L b_i 2^{-i}$. For example, for a Haar random unitary, $\hat{U} \sim U(4)$, the transition matrix $\hat{U}_t$ can be constructed as

$$[\hat{U}_t]_{ij} = |\hat{U}_{ij}|^2, \quad (23)$$

such that $(\mathbf{b}|\hat{U}_t|\psi) \equiv |\langle \mathbf{b}|\hat{U}|\psi\rangle|^2$.

For the control map, we simply reset the corresponding qubit i . Namely,

$$\hat{C} = \bigotimes_{j=1}^L \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (1 - \delta_{i,j}) + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \delta_{i,j} \right], \quad (24)$$

where the order of the basis on each site is 0 and 1.

In the end, we sample the same circuit by fixing the realization of $\hat{U}_T$ and its spacetime position. The “quantum fluctuations” are in analogy defined as

$$(\Delta M_z)^2|_C = \mathbb{E}_\psi [M_z^2] - (\mathbb{E}_\psi [M_z])^2 \quad (25)$$

where $\mathbb{E}_\psi [\mathcal{O}] = \sum_{\mathbf{b}} p_{\mathbf{b}}(\psi) \mathcal{O}_{\mathbf{b}}$, and $p_{\mathbf{b}}(\psi) = (\mathbf{b}|\psi\rangle$.

In numerics, since Eq. (23) and Eq. (24) are both essentially Markov processes, we can simulate the time evolution of a single bit-string $\mathbf{b}$ under the stochastic dynamics of transition matrix $\hat{U}_t$ and $\hat{C}$ in replacement of the full probability mass function simulation of $|\psi\rangle$.

VII. SYSTEM SIZE DEPENDENCE OF THE KULLBACK-LEIBLER DIVERGENCE

The Kullback-Leibler (KL) divergence is a measure of how one probability distribution differs from a second, reference probability distribution. In our case, we can use the KL divergence to quantify the difference among the probability density function of the variance of the magnetization obtained from the MPS simulation, the dephasing model, and the experimental data. All results in this section are shown *without* readout error (pristine simulations), in contrast to the main-text Fig. 4 where all MPS and dephasing simulations include readout error (see Sec. VIII). The pristine $\mathcal{D}_{\text{KL}}(\text{MPS}, \text{Deph})$ as a function of p for various system sizes is shown in Supplementary Fig. 7a; the corresponding comparison with readout error included is shown in Fig. 4e of the main text. In Supplementary Fig. 7a, we show $\mathcal{D}_{\text{KL}}(\text{MPS}, \text{Deph})$ versus p for various system sizes without readout error, establishing that the quantum-classical distinction grows with system size in the chaotic phase and vanishes in the controlled phase. In Supplementary Fig. 7b and c, we compare the “experimental explainability” of the quantum

“MPS” simulation versus that of the classical dephasing model, and we find that the latter KL divergence is always larger than the former inside the chaotic phase for all system sizes. In the controlled phase, both KL divergences increase to a large value because of the intrinsic noise in the experiments that will always prevent the full suppression of quantum fluctuations. Supplementary Fig. 7d shows $\mathcal{D}_{\text{KL}}(\text{Exp, MPS})$, $\mathcal{D}_{\text{KL}}(\text{Exp, Deph})$, and $\mathcal{D}_{\text{KL}}(\text{Exp, SM})$ as a function of p at $L = 40$ without readout error; comparing with Fig. 4a of the main text (which includes readout error) shows that including readout error brings both simulations closer to the experimental data.

VIII. READOUT ERROR

To study the effect of readout error on quantum fluctuations, we model the readout error as an independent per-qubit classical channel that stochastically flips the measurement. For each qubit i , the confusion matrix is

$$\mathbf{C}_i = \begin{pmatrix} 1 - \epsilon_i^{(0)} & \epsilon_i^{(1)} \\ \epsilon_i^{(0)} & 1 - \epsilon_i^{(1)} \end{pmatrix}, \quad (26)$$

where $\epsilon_i^{(b)} = P(1 - b|b)$ is the bit-flip readout error rate given the true bit value being $b = \{0, 1\}$ during the readout.

Since our experiments are performed on different arrangements of qubits for different circuit realizations $\mathcal{C}$ and different shots $\mathbf{m}_C$, we assume a simplified model with a uniform readout error by averaging over all qubits from the actual devices, i.e., $\epsilon_i^{(0)} \equiv \epsilon^{(0)} = 0.007$ and $\epsilon_i^{(1)} \equiv \epsilon^{(1)} = 0.0166$.

The quantum fluctuation $(\Delta M_z)^2|_C = \mathbb{E}_{\mathbf{m}_C} [\langle \psi_{\mathbf{m}_C} | M_z^2 | \psi_{\mathbf{m}_C} \rangle] - (\mathbb{E}_{\mathbf{m}_C} [\langle \psi_{\mathbf{m}_C} | M_z | \psi_{\mathbf{m}_C} \rangle])^2$ averaged over readout error $\mathbb{E}_{\text{RO}} [(\Delta M_z)^2|_C]$ can be simply derived from the pristine simulation data by noting that the readout error is a classical stochastic process that acts on the bit-string outcomes of the measurement, and does not affect the underlying quantum state $|\psi_{\mathbf{m}_C}\rangle$.

Here, the first moment of M_z averaged over different trajectories $\mathbf{m}_C$ and readout error is

$$\begin{aligned} & \mathbb{E}_{\text{RO}} [\mathbb{E}_{\mathbf{m}_C} [\langle \psi_{\mathbf{m}_C} | M_z | \psi_{\mathbf{m}_C} \rangle]] \\ &= \mathbb{E}_{\mathbf{m}_C} \left[\frac{1}{L} \sum_{i=1}^L \mathbb{E}_{\text{RO}} [\langle \psi_{\mathbf{m}_C} | \hat{Z}_i | \psi_{\mathbf{m}_C} \rangle] \right] \\ &= \mathbb{E}_{\mathbf{m}_C} \left[\frac{1}{L} \sum_{i=1}^L \alpha \langle \psi_{\mathbf{m}_C} | \hat{Z}_i | \psi_{\mathbf{m}_C} \rangle + \beta \right] \\ &= \alpha \mathbb{E}_{\mathbf{m}_C} [\langle \psi_{\mathbf{m}_C} | M_z | \psi_{\mathbf{m}_C} \rangle] + \beta, \end{aligned} \quad (27)$$

where the first equal sign follows the definition of $M_z =$

$\frac{1}{L} \sum_i \hat{Z}_i$, the second equal sign is because

$$\begin{aligned} & \mathbb{E}_{\text{RO}} [\langle \psi_{\mathbf{m}_C} | \hat{Z}_i | \psi_{\mathbf{m}_C} \rangle] \\ &= \sum_{b_i, b'_i \in \{0, 1\}} (-1)^{b'_i} P(b'_i | b_i) P(b_i) \\ &= \sum_{b_i} (-1)^{b_i} (1 - 2\epsilon^{(b_i)}) P(b_i) \\ &= \sum_{b_i} (-1)^{b_i} (1 - 2\epsilon^{(b_i)}) \frac{1 + (-1)^{b_i} \langle \psi_{\mathbf{m}_C} | \hat{Z}_i | \psi_{\mathbf{m}_C} \rangle}{2} \\ &= \alpha \langle \psi_{\mathbf{m}_C} | \hat{Z}_i | \psi_{\mathbf{m}_C} \rangle + \beta, \end{aligned} \quad (28)$$

where $\alpha = 1 - \epsilon^{(0)} - \epsilon^{(1)}$, $\beta = \epsilon^{(1)} - \epsilon^{(0)}$.

Similarly, the second moment of M_z averaged over readout error is

$$\begin{aligned} & \mathbb{E}_{\text{RO}} [\mathbb{E}_{\mathbf{m}_C} [\langle \psi_{\mathbf{m}_C} | M_z^2 | \psi_{\mathbf{m}_C} \rangle]] \\ &= \mathbb{E}_{\mathbf{m}_C} \left[\frac{1}{L^2} \left[\sum_i 1 + \sum_{i \neq j} \mathbb{E}_{\text{RO}} [\langle \psi_{\mathbf{m}_C} | \hat{Z}_i \hat{Z}_j | \psi_{\mathbf{m}_C} \rangle] \right] \right] \\ &= \alpha^2 \mathbb{E}_{\mathbf{m}_C} [\langle \psi_{\mathbf{m}_C} | M_z^2 | \psi_{\mathbf{m}_C} \rangle] + \frac{1 - \alpha^2}{L} \\ & \quad + 2\alpha\beta \left(1 - \frac{1}{L}\right) \mathbb{E}_{\mathbf{m}_C} [\langle \psi_{\mathbf{m}_C} | M_z | \psi_{\mathbf{m}_C} \rangle] \\ & \quad + \beta^2 \left(1 - \frac{1}{L}\right), \end{aligned} \quad (29)$$

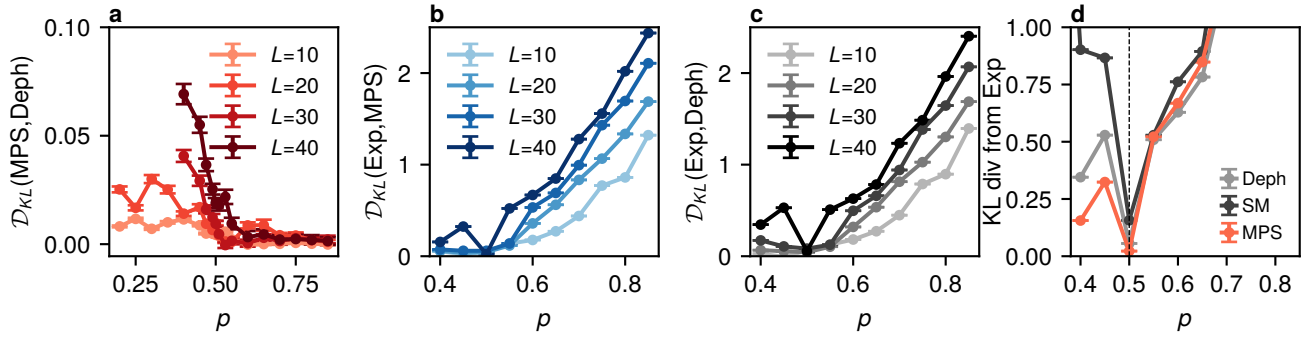
where the first equal sign separates $M_z^2 = \frac{1}{L^2} \sum_{i,j} \hat{Z}_i \hat{Z}_j$ into diagonal ($i = j$, with $\hat{Z}_i^2 = 1$) and off-diagonal ($i \neq j$) contributions, the second equal sign is because the off-diagonal readout expectation factorizes due to the independence of readout errors across qubits,

$$\begin{aligned} & \mathbb{E}_{\text{RO}} [\langle \psi_{\mathbf{m}_C} | \hat{Z}_i \hat{Z}_j | \psi_{\mathbf{m}_C} \rangle] \\ &= \sum_{b_i, b_j} (\alpha(-1)^{b_i} + \beta)(\alpha(-1)^{b_j} + \beta) \\ & \quad \times P(b_i, b_j | \psi_{\mathbf{m}_C}) \\ &= \alpha^2 \langle \psi_{\mathbf{m}_C} | \hat{Z}_i \hat{Z}_j | \psi_{\mathbf{m}_C} \rangle + \beta^2 \\ & \quad + \alpha\beta \left(\langle \psi_{\mathbf{m}_C} | \hat{Z}_i | \psi_{\mathbf{m}_C} \rangle + \langle \psi_{\mathbf{m}_C} | \hat{Z}_j | \psi_{\mathbf{m}_C} \rangle \right), \end{aligned} \quad (30)$$

where we summed over readout outcomes b'_i, b'_j using the single-qubit marginal from above and expanded the product. The last equal sign in Equation (29) follows from substituting the above and using $\sum_{i \neq j} \langle \psi | \hat{Z}_i \hat{Z}_j | \psi \rangle = L^2 \langle \psi | M_z^2 | \psi \rangle - L$.

Combining the above, the quantum fluctuation $(\Delta M_z)^2|_C$ transforms as

$$\begin{aligned} & \mathbb{E}_{\text{RO}} [(\Delta M_z)^2|_C] = \alpha^2 (\Delta M_z)^2|_C \\ & \quad + \frac{1 - \alpha^2 - \beta^2}{L} - \frac{2\alpha\beta}{L} \mathbb{E}_{\mathbf{m}_C} [\langle \psi_{\mathbf{m}_C} | M_z | \psi_{\mathbf{m}_C} \rangle]. \end{aligned} \quad (31)$$



Supplementary Figure 7 | The system size dependence of the Kullback-Leibler divergence (without readout error). All results in this figure are shown *without* readout error, in contrast to the main-text Fig. 4 where all simulations include readout error (see Sec. VIII). **a** The KL divergence $\mathcal{D}_{KL}(\text{MPS}, \text{Deph})$ between the MPS simulation and the dephasing model as a function of p for various system sizes $L = 10, 20, 30, 40$ (lightest to darkest red). **b** The KL divergence between the experimental (Exp) data and matrix product state (MPS) simulation and for various system sizes. **c** The KL divergence between the experimental (Exp) data and dephasing model (Deph) for various system sizes. **d** The KL divergence $\mathcal{D}_{KL}(\text{Exp}, \text{model})$ from the experimental data as a function of p at $L = 40$, *without* readout error. Comparing with Fig. 4a of the main text (which includes readout error) shows that including readout error brings both simulations closer to the experimental data. The error bar is defined as in Fig. 4 of the main text.

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