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Supplementary information

Spin-cooling of the motion of a trapped diamond

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Spin-Cooling of the Motion of a Trapped Diamond
Supplementary Material : Spin-cooling and phonon-lasing theory

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CONTENTS

A. The mechanical oscillator

The considered mechanical oscillator is the librational mode of a microdiamond levitating in a Paul trap. It was shown in [3, 8] that the Paul trap induces a confinement of the angle due both the trap and diamond particle asymmetry. In the manuscript, we directly observe the three librational modes under vacuum using the speckle detection detailed in the Method section. We fit the power spectral density of the librational modes in Fig. 1-b) of the main text using the equations that are derived from the following linear response theory.

In the following we consider the simple case of a single excited angle, assuming that the other angles are not coupled to it. The librational motion of the diamond in the Paul trap is ruled by the equation of motion for the angle between the main axis of the diamond and the main axis of the trap [8]. Noting this angle ϕ, we get

$$I\ddot{\phi} = -I\omega_\phi^2\phi - I\gamma\dot{\phi} + \Gamma_T(t) \tag{1}$$

where I is the moment of inertia, ω_ϕ is the angular frequency, γ is the damping rate due to collisions with the background gas and $\Gamma_T(t)$ is the associated Langevin torque.

Fourier Transforming this equation yields

$$\phi(\omega) = \chi(\omega)\Gamma_T(\omega)$$

where

$$\chi(\omega) = \frac{1}{I(\omega_\phi^2 - \omega^2 + i\omega\gamma)}.$$

The Langevin torque obeys the relation

$$\langle \Gamma_T(\omega)\Gamma_T(\omega')\rangle = 2\pi\delta(\omega+\omega')S_T(\omega)$$

where

$$S_T(\omega) = -\frac{2kT}{\omega}\mathrm{Im}\Big[\frac{1}{\chi(\omega)}\Big]$$

when the mean number of phonon excitations is larger than 1. One finds

$$S_T(\omega) = 2kT\gamma I.$$

The librational spectrum is then found to be

$$\begin{aligned} S_\phi(\omega) &= |\chi(\omega)|^2 S_T(\omega) \\ &= \frac{2\gamma kT}{I((\omega_\phi^2-\omega^2)^2+\gamma^2\omega^2)}. \end{aligned} \tag{2}$$

This formula describes very well the observed librational motion shown in Fig. 1 b) of the main text. Integrating this expression over ω, we obtain

$$\frac{1}{2}I\omega_\phi^2\langle\phi^2\rangle = \frac{1}{2}kT$$

where

$$\langle\phi^2\rangle = \int S_\phi(\omega)d\omega,$$

in agreement with the equipartition theorem. In principle, the area below the Lorentzian curves observed in Fig. 1 b) gives us direct access to the temperature (see Methods, for a discussion on temperature calibration).

B. Rotating a levitating diamond using the spin of nitrogen-vacancy centers

We now discuss the theory behind the spin-induced torque from the nitrogen-vacancy centers. In the secular approximation of the Paul trap potential, the Hamiltonian describing the angular motion takes the form

$$H_{\rm meca} = \frac{1}{2}I\omega_\phi^2\phi^2 + \frac{L^2}{2I}. \tag{3}$$

In our experiment, the angular position of the particle in the trap is shifted due to a torque induced by spins of NV centers in the presence of a magnetic field, which will add a magnetic potential energy.

The NV centers is a spin triplet system due to the presence of two paired electrons. The singlet state lies at much higher energies and only contributes to the triplet state polarization via the green laser. The dipolar interaction between the two electron spins lifts the degeneracy between the $|m_s' = 0\rangle$ and $|m_s' = \pm 1\rangle$ states, which sets a preferential quantization axis that is not given by the external magnetic field as it would normally be for free electrons [9]. The degeneracy is already lifted by $D = 2.87$ GHz at room temperature in the absence of magnetic field. This property is at the core of the proposals put forward in [8, 10] to let the NV centers act on the angle of a levitating diamond.

Here, we consider a large ensemble of NV centers (about 10^9 NV centers), which are all along one of the four $\langle 111\rangle$ crystalline axes. Under an external magnetic field, the orientation dependent Zeeman effect lifts the degeneracy between these NV spin states. Therefore, when the microwave frequency is close to the resonance corresponding to only one axis, only those spins that are along this axis are in a magnetic state.

To estimate the maximal torque exerted by N magnetized NV centers onto the diamond, one must search for the steepest variation of the Zeeman splitting with the NV angle with respect to the B field. Let us define the average spin operators as

$$\hat{S}_\alpha = \frac{1}{N}\sum_{i=1}^{N}\sigma_\alpha^i,$$

where α denotes the three spin directions x, y and z and σ_α^i are the Pauli matrices of the NV number i corresponding to the direction α.

We assume for simplicity that the ensemble of spins are aligned along one crystalline axis. The hamiltonian for such NV ensemble reads

$$\hat{H}_{\rm NV} = \hbar ND\hat{S}_z^2 + \hbar\gamma_e N\mathbf{B}\cdot\hat{\mathbf{S}} \tag{4}$$

where γ_e is the electron gyromagnetic factor, $\hbar$ the reduced Planck constant, $\hat{\mathbf{S}} = (\hat{S}_x, \hat{S}_y, \hat{S}_z)$ and $\mathbf{B}$ the external magnetic field. The first term describes the aforementioned spin-spin interaction which is the dominant energy contribution and the second term, the Zeeman energy.

The greatest angular variation of the three eigenenergies with angle can be found between the ground and lowest excited state at approximately $\phi_{\rm opt} = \pi/4$ under moderate magnetic fields (less than 100 G)[10]. The derivative of the energy as a function of the angle can also be estimated close to this angle and can be found to be around $\gamma_e B$.

We can then write the Hamiltonian in the new eigenstate basis, and consider the NV electronic spins as being a simple two-level system in the $\{|m_s' = 0\rangle, |m_s' = -1\rangle\}$ basis. Assuming that the spins are promoted to the excited $|m_s' = -1\rangle$ spin state by a resonant microwave, and linearizing the energy dependance with respect to ϕ at $\pi/4$, the total energy is changed to

$$E = \frac{1}{2}I\omega_\phi^2\phi^2 + \frac{L^2}{2I} - \hbar N\gamma_e B\phi + \text{cte}$$

The center of the angular potential is thus shifted to

$$\phi_0 = \frac{\hbar N\gamma_e B}{I\omega_\phi^2}.$$

The number of NV centers that are magnetized during a typical microwave scan is at most one fourth of the total number of NV centers due to the 4 involved NV orientations. Including the detrimental influence of the transverse component of the magnetic field on the spin polarization, only 50% of the population is transferred to the $|0\rangle$ state by the laser before being excited by the microwave, so in total, only about 10^8 NV centers are in a magnetized state. Taking a 15μm diameter diamond, using typical librational mode frequency of $\omega_\phi/2\pi = 500$ Hz and an external magnetic field of 100 G, we obtain an angle shift of around $\phi_0 = 5$ mrad.

Let us note that the spin-induced angular displacement will significantly increase when decreasing the particle size. Indeed, the moment of inertia scales like the fifth power of the particle diameter. Magnetizing a single spin inside a 500 nm diameter diamond with similar trapping conditions than the presented experiments would lead to an angular displacement of 0.5 mrad. Our sensitive detection technique (via the speckle) imposes the minimum size of the employed particles to be the illumination wavelength, but detecting such angular displacement within the spin lifetime does not seems out of reach for future experiments.

The above analysis however neglects an important effect that is at the core of the spin-mechanical coupling, that is the back-action of the oscillator onto the spin state. The root of this effect is that the magnetic state population also depends on ϕ. The reason for this is that when the diamond rotates, the B field projection onto the NV axis changes. The microwave will then be brought in or out of resonance, which will in turn change the total magnetization. This mechanism is truly analogous to the radiation pressure back-action force onto moveable mirrors in a high-finesse optical cavity. There, when the light enters the cavity, it can push the mirror and displace it enough so that the cavity resonance condition is changed. This change in the resonance condition can in turn reduce or enhance the intra-cavity light intensity [11], which can increase again, or reduce, the radiation pressure force. The analogy with the present system is very strong since the interpretation of our experiment can be carried out by essentially replacing the NV spin degree of freedom with the cavity field and the mirror position by the particle angle.

Let us analyze the mechanisms at play including such, potentially delayed, interaction between the spin and librational degree of freedom.

C. The linear regime : Spin-spring effect and spin-cooling

The above mentioned spin-mechanical coupling mechanism was discussed in [8, 12].

The analysis of [8] was carried out for diamonds in a Paul trap in the sideband resolved regime, where the motional frequency is larger than the spin decoherence rate. Reducing the mean phonon number in the librational mode can be done in a very similar way than with single ions/atoms by tuning the microwave frequency close to the motional red sideband transition. Fast cooling can be realized by using a green laser that polarizes the magnetized spins back to the ground state. In a regime where the spin-excitation rate on the red-sideband is faster than any heating mechanism of the angular degree of freedom (typically collisions with the background gas), such a combination of microwave and laser excitation can lead to ground state cooling of the librational mode.

In our diamonds, paramagnetic impurities anf NV-NV dipolar interactions broaden the NV spin resonance to about 7 MHz, making it difficult to reach the sideband resolved regime. Another difficulty is that the particles that we trap have a large moment of inertia (in the 10^{-22}kg/m^2 range). Technical limitations on the trap parameters (size and voltage) finally impose an upper limit to the librational frequency.

Using a large particle means that we can use many NV centers so the spin-dependent torque can in fact be very large. Further, although we are far from the sideband resolved regime, since $T_2^* = 50$ns and the librational mode period is in the millisecond range, significant cooling of the oscillation can be obtained because of a significant delay between the magnetization of the NV centers (measurements of the magnetization rate are shown in the extended data) and the librational motion. Such spin-cooling mechanism is sketched in Fig. 2-b). We now quantify the cooling efficiency using a numerical model.

1. Numerical model

To fit the experimental data Fig.2-c), we study the dynamics of the coupled angle and spin degrees of freedom using Bloch equations. The green laser induces decay from the $|m_s = \pm 1\rangle$ levels to the ground state $|m_s = 0\rangle$ at a rate $\gamma_{\rm las}$ of a few kHz. We note $\Omega/2\pi$, the Rabi frequency of the microwave signal and $\Delta/2\pi$ the frequency detuning with respect to the $|m_s = 0\rangle$ to the $|m_s = -1\rangle$ transition at equilibrium.

The dynamics of the paramagnetic spin bath (μs) is faster than the timescales of the motion and spin-spring shifts so we can treat the paramagnetic impurities as a general Markovian reservoir that limits the spin coherence time T_2^* to 50 ns [1], much below the T_1 time of the spin populations ($\approx ms$ in our experiment). We write S the mean expectation value of the Pauli operator $\hat{\sigma}_i^-$ of the individual spins on the subspace $\{|0\rangle, |-1\rangle\}$. We also write $\hat{S}_z^\beta = |\beta\rangle\langle\beta|$, the population in the three spin states $|\beta\rangle$ and S_z^β their expectation values. The evolution of these expectation values is ruled by the following set of equations

$$\begin{aligned}
\frac{\partial S}{\partial t} &= \big[-\frac{1}{T_2^*} + i(\Delta + \gamma_e B\phi)\big]S + i\frac{\Omega}{2}(S_z^{-1} - S_z^0) \\
\frac{\partial S_z^{+1}}{\partial t} &= -\frac{1}{T_1}(S_z^{+1} - S_z^0) - \gamma_{\rm las} S_z^{+1} \\
\frac{\partial S_z^{-1}}{\partial t} &= -\frac{1}{T_1}(S_z^{-1} - S_z^0) - \gamma_{\rm las} S_z^{-1} + i\frac{\Omega}{2}(S - S^*) \\
\frac{\partial S_z^0}{\partial t} &= -\frac{\partial S_z^{-1}}{\partial t} - \frac{\partial S_z^{+1}}{\partial t}.
\end{aligned}$$

The last equation ensures that the spin population in this manifold is preserved. One important ingredient is the angular dependent detuning that appears in the first equation, which couples the equations of motion of the spins to the equation for angular motion. The former reads

$$\frac{\partial^2\phi}{\partial t^2} = -\omega_\phi^2\phi - \gamma\frac{\partial\phi}{\partial t} + \Gamma_{-1}S_z^{-1} - \Gamma_{+1}S_z^{+1} + \Gamma_T(t), \tag{5}$$

where $\Gamma_{-1} \approx \hbar N\gamma_e B/I$ and $\Gamma_{+1} \approx \hbar N\gamma_e B/I/\sqrt{2}$ under the condition $\gamma_e B \ll D$. This equation describes the angle evolution under the NV spin torque. $\Gamma_T(t)$ is the delta-correlated Langevin noise term.

In the experiment, the polarization dynamics and librational motion evolve on ms timescales, much longer than the T_2^* time (ns). We can thus adiabatically eliminate the evolution of S and use rate equations that describe the spin populations only. They read

$$\frac{\partial S_z^{+1}}{\partial t} = -\frac{1}{T_1}(S_z^{+1} - S_z^0) - \gamma_{\rm las} S_z^{+1} \tag{6}$$

$$\begin{aligned}
\frac{\partial S_z^{-1}}{\partial t} &= (-\frac{1}{T_1} + \Omega^2 P(\Delta,\phi))(S_z^{-1} - S_z^0) - \gamma_{\rm las} S_z^{-1} \\
\frac{\partial S_z^0}{\partial t} &= -\frac{\partial S_z^{-1}}{\partial t} - \frac{\partial S_z^{+1}}{\partial t}
\end{aligned} \tag{7}$$

where $P(\Delta,\phi)$ is given by

$$P(\Delta,\phi) = \frac{1}{2\sigma}\frac{1}{1 + (\frac{\Delta+\gamma_e B\phi}{\sigma})^2} \tag{8}$$

where $\sigma = 1/T_2^*$. $P(\Delta,\phi)$ is a Lorentzian function quantifying the change in the polarization rate of the NV centers in the magnetic state as the microwave detuning or particle angle change. Using this model, the spin resonance lineshape would not accurately model our experiment because the coupling of the NV centers to a paramagnetic spin bath gives rise to a Gaussian lineshape. Here, since the spin bath dynamics is fast compared the measurement time, it does not play any other role than changing the actual lineshape, we can thus instead write P as a Gaussian function

$$P(\Delta,\phi) = \frac{1}{2\sigma}\exp\Big(-\frac{(\Delta+\gamma_e B\phi)^2}{\sigma^2}\Big) \tag{9}$$

with a width given by $\sigma\sqrt{\ln(2)}/2\pi$.

The solutions for ϕ obtained from this numerical model can be used to estimate the effective damping and spin-spring effects and hence the libration temperature.

2. *Relation between the spin population fluctuations and the final temperature*

To understand how to relate the results from our numerical simulations to the effective temperature, we can again write the equation of motion for ϕ in the Fourier domain

$$\phi(\omega) = \chi(\omega)(\Gamma_T(\omega) + \Gamma_{\rm mag}(\omega)) \tag{10}$$

As was done in [13], in the linear regime, one can also introduce an effective susceptibility $\chi_{\rm eff}(\omega)$ such that

$$\phi(\omega) = \chi_{\rm eff}(\omega)\Gamma_T(\omega).$$

In order to find $\chi_{\rm eff}$, one needs a relationship between the angle and the spin population. To do this, we decompose the spin and angles as the sum of a mean value and a fluctuating component. $S_z = \overline{S_z} + S_z(\omega)$ for instance. Linearizing the Bloch equations will then allow to get a relationship between δS_z and ϕ

$$S_z(\omega) = \xi(\omega)\phi(\omega).$$

where ξ is a complex number that depends on all the parameters of the spin-mechanical interaction, including the mean values. Injecting this relation back into Eq. (10), we get

$$\phi(\omega) = \chi(\omega)\Gamma_T + \chi(\omega)\hbar\gamma_e BN\xi(\omega)\phi(\omega).$$

The power spectral density of the librational motion ϕ can finally be found to be

$$\begin{aligned} S_\phi(\omega) &= \Big|\frac{\chi(\omega)}{1-\hbar\gamma_e BN\xi\chi(\omega)}\Big|^2 S_T(\omega) \\ &= |\chi_{\rm eff}|^2 S_T(\omega). \end{aligned} \tag{11}$$

It was shown in [13], that the real part of ξ gives either an extra binding or anti-binding confinement (depending on its sign), also know as a spring effect. The imaginary part of ξ, is related to delayed action of the spin onto the angle and provides the cooling/heating mechanism. Provided that the dynamics is still that of a damped harmonic oscillator, effective damping and spring effects can be found.

We checked this by solving the full set of equations numerically. The set of experimental curves shown in Fig. 2-d) where performed by exciting parametrically the librational motion using the NV spins inside the diamond. The ring down of the librational mode was then observed and was very well approximated by exponentially decaying curves (see Methods).

For the most part of the main text, we model the experiment numerically using Monte-Carlo simulations and the XMDS package [14]. For the spin-cooling and spin-spring effects shown in Fig. 2.c), we used the measured librational frequency $\omega_\phi = 2\pi\times$ 480 Hz, the measured damping rate $\gamma/2\pi = 16$ Hz and the magnetic field splitting $\gamma_e B = 260$ MHz as parameters. The number of NV centers, polarization rate, Rabi frequency and the longitudinal spin lifetime T_1 in the excited state are not known with a high precision and are left as free parameters. The number of NV centers in particular can be deduced from the detected count rate, but this can only be an order of magnitude estimate since we do not know precisely the collection efficiency of the apparatus, how the laser light is coupled to the NV centers in the diamond and how the PL is refracted by the diamond. Doing this we found around 10^9 NV centers in total, but use it as a free parameter when determining the best value in the experiment. Similarly, the moment of inertia is not known precisely due to the imprecision in the quoted diamond diameter d and shape, which translates into a large error in the moment of inertia (which scales like d^5). We thus leave the parameter Γ as a free parameter.

Numerically, we turn on a microwave at a time t=20 ms, and record the librational mode ring down. When the microwave is quasi-resonant, it induces a spin torque which displaces the angle from the equilibrium position. To find the optimum values for the resulting cooling and spring-constants, and to obtain a fit to the experimental data, we analyze S_z, θ, $\gamma_{\rm eff}$ and $\omega_{\rm eff}$ as a function of the microwave detuning from the spin transition. The numerical results of the Monte-Carlo simulation are adjusted to match the experimental parameters corresponding to the Fig.2-c) of the paper. Doing a Monte-Carlo simulation was important to describe damping of the oscillations on the blue-side. Without the Brownian motion of the angle included in the numerical simulations, phonon-lasing (described below) takes place in the regime where the damping on the red matches the experimental data. The Brownian motion induces phase diffusion of the phonon-laser, which effectively damps out our averaged signal.

Using these numerical data, we can then fit a cosine function multiplied by an exponentially decaying curve to deduce both the spin spring and damping rates. We observe an increase/decrease in the trapping frequency and a decrease/increase of the damping rate when the microwave is tuned to the blue/red. The exponentially decay is in very good agreement with the ring down numerical data with $1/T_1 = 0.6$, $\gamma_{\rm las} = 2$ and $\Omega/2\pi = 30$ in kHz units, as free parameters.

We can then deduce effective damping and spring coefficients when the microwave is tuned to the red side of the spin resonance, and write the net susceptibility of the levitating diamond angle to the spin-torque as

$$\chi_{\rm eff}(\omega) = \frac{1}{I(\omega_{\rm eff}^2 - \omega^2 + i\omega\gamma_{\rm eff})}.$$

from which we can deduce the final temperature using the equipartition theorem.

The imaginary part of $\xi(\omega)$ determines the damping term in the mechanical susceptibility, so the delay between the librational motion and the depolarization is the cause of cooling/heating. This model reproduces qualitatively the experimental results. To optimize the cooling, the laser induced re-polarization rate should thus be on the order of the trapping frequency. Again, this simplified analysis neglects the T_1 longitudinal relaxation, which tends to demagnetize the whole spin ensemble on millisecond times scales. The librational frequency must thus be on the order of the T_1 time, which is the main limitation to the cooling efficiency in the experiment. Increasing the microwave and laser powers makes the librational mode evolution become bistable when the microwave is tuned to the red, and motional lasing occurs on the blue as we describe next.

D. The non-linear regime : Bistability and phonon-lasing

The above calculations were performed in the regime where the spins were weakly polarized by the laser and microwaves. Here, we use the simplified two-level model to express the effective spin-mechanical potential energy in the non-linear regime. We will explain Fig. 3 of the main text, where bistability and phonon-lasing of the librational modes are observed. Let us first discuss the regime where the microwave is tuned to the red.

1. *Bistable regime*

The phenomenon of bistability has been analyzed and experimentally realized by many research groups and can be observed in a vast range of experimental settings. Bistability is associated with a wealth of interesting phenomena [15], one prominent example being the possibility to observe driven-dissipative phase transitions [16]. One situation that has been extensively studied is when atoms are placed in an optical cavity, which was realized in the early days of cavity Quantum electrodynamics [17]. The present system bears strong analogy with these experiments, since the NV spin degree of freedom plays the role of the cavity field in the experiments.

In order to describe this observed bistability, we can start from a simple two-level model, with spontaneous decay rate $\gamma_{\rm las}$ and decoherence rate $1/T_2^*$. In the steady state limit, the Bloch equations give the following equation for S_z

$$S_z = \frac{2\Omega^2}{\gamma_{\rm las}T_2^*}\Big[\frac{1}{(1/T_2^*)^2+(\Delta+\gamma_e B\phi)^2+4\Omega^2/(\gamma_{\rm las}T_2^*)}\Big].$$

Using the equation of motion for the angle in the steady state limit gives

$$\phi = \mathcal{A}\Big[\frac{1}{\kappa+(\Delta+\gamma_e B\phi)^2}\Big],$$

where $\kappa=(1/T_2^*)^2+4\Omega^2/(\gamma_{\rm las}T_2^*)$ and $\mathcal{A}=\frac{\Gamma}{\omega_\phi^2}\frac{2\Omega^2}{\gamma_{\rm las}T_2^*}$. This relation is a third order polynomial equation for ϕ. We solve it numerically using the parameters of our experiment and found that when $\Delta<0$ it can have two real and one imaginary solutions. Physically, this effectively corresponds to a stable and unstable (saddle point) for the angle. The result of the numerical simulations is shown in Fig. 3-a) as a function of the detuning, where a characteristic S-shape curve is observed. When the microwave is tuned to the blue of the spin resonance transition, two stable solution for the angle (at positions A and B) are seen to co-exist.

The probability of being in either one of these two points depends on the history of the angle motion, which often implies a hysteretic behavior. In the experiment, we expect that if the microwave is scanned from the blue to the red, the particle will follow the upper angular trajectory (where the saddle point is B) up to the end, and finally drop to the small angle solution. On the other hand, if the microwave is scanned from the red to the blue, the particle will stay in the saddle point A, longer until the spin torque pulls it towards large angles. If the scan is performed on times scales of the librational period then hysteresis behavior is expected. This hysteresis behavior was observed in the experiment (Fig.3-b)) and numerical simulations of the coupled Bloch and torque equation very well describe the data.

If now the microwave is parked to the red of the spin resonance, there may be sudden stochastic jumps between the two metastable angular positions. External noise sources may indeed push the angle from one point to the other so that sharp jumps from the two stable positions A and B may occur on times scales of the motional period, and remain at the stable points for times scales dictated by the amplitude of the external driving noise. The effect was observed in the experiment, and is shown in Fig. 3-c) of the main text. The main origin of these jumps is likely to collisions with the gas particles, but any other sources of noise, such as the microwave signal generator amplitude noise or laser noise, could also explain our observations.

We also measured the relative time spent in each of these stable dynamical potential wells in these condition (long measurement times) as a function of the microwave detuning. The evolution of the histograms showing the time spent in each of those wells is shown in the extended data.

These results point towards a naive interpretation that the two local potential minima are effectively separated by a potential barrier whose height depends upon the microwave detuning. Going further in this interpretation, one could estimate that the jump rate between two local minima from Kramer's theory [18, 19]. In the overdamped regime, the transition rate from the well A to B is :

$$R_{\mathrm{AB}} = \frac{1}{2\pi} \prod_{i=x,y,z} \frac{\omega_i^A}{|\omega_i^C|} \left[\sqrt{|\omega_i^C|^2 + \gamma^2/4} - \gamma/2\right] \exp\left(\frac{-U_A(\Delta)}{k_B T}\right) \tag{12}$$

The reverse rate is obtained by swapping indices A and B. $\omega_i^{A,B,C}$ are the angular confinement frequencies of the mode i at the point A,B or C.

Such a dependence of the jump rate with temperature of the bath is also confirmed by the Monte-Carlo simulations and offers a rich playground to studies of thermalization in out-of-equilibrium systems.

2. *Librational lasing*

Phonon-lasing takes place when stimulated emission overcomes absorption of phonons from the mechanical mode. This effect has recently been observed using the mechanical modes of micro-toroids in [20]. Here the phonon laser is replenished via the dual pumping mechanism provided by the microwave (why magnetizes the NV states) and the green laser that polarises the NV in the ground state. In addition, delay also plays a crucial role in the phonon-lasing process. Indeed, the critical point where phonon-lasing occurs is to be found when the damping γ_{eff} becomes negative, that is when the delay between the spin and Paul trap torques makes the system unstable. Fig. 6-a) shows the result of numerical simulations in this regime. Numerical simulations show the damping and spin-spring shifts with torque coefficients $\Gamma = 0.7, 0.9, 1.2, 1.5$ kHz from traces i) to iv) and resonant microwave Rabi frequency of 100 kHz. On the blue side, the damping coefficient becomes negative and the spring effect stabilizes to a value that is slightly greater than the librational mode frequency defined solely by the Paul trap (here 480 Hz). This is observed experimentally in the Fig. 3.b)-i) of the main text. In this regime, the oscillator runs unstable and self sustained oscillations then take place. In the transient regime, the librational mode undergoes amplification, until it settles to a constant coherent oscillation. This is shown Fig. 6-b), where a microwave signal detuned by 7 MHz from the spin resonance spin transition is applied at a time t=20 ms. Here we set Γ to 1.5 kHz. At this detuning, the signal is amplified for 120 ms, up to a point where a regime of self-sustained oscillation sets-in with an angular amplitude that settles at 0.05 rad.

Phonon-lasing occurs at a threshold value that depends on the ratio between losses and gain. The losses are related to residual gas damping and the longitudinal spin relaxation time T_1 limits the population inversion. Numerical simulations of the librational mode velocity are plotted as a function of microwave Rabi frequency in Fig. 6-c) for three different values of the spin-lattice relaxation rate $1/T_1 = \{1, 2, 3\}$kHz for trace i)-ii) and iii) respectively. A threshold behavior is observed at microwave Rabi frequencies of $24, 38$ and 59 kHz for the traces i),ii) and iii). Numerical simulations are in good agreement with the experimental observations shown in Fig. 3.b)-ii) in the main text.

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