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Supplementary information

Dynamical topological phase realized in a trapped-ion quantum simulator

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Supplementary information
"Dynamical topological phase realized in a trapped-ion quantum simulator"

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A. Relation of models to AKLT chain

In this section, we relate the dynamical topology of the (quasi)-periodically driven models in the main text to the more familiar equilibrium topology of the AKLT chain.[16,17] While the models defined in the text can be defined for either even- or odd- length chains, the AKLT model is only defined for integer-spin systems. Hence, for the purposes of this discussion, we will assume an even number of spins and mentally group the spins into neighbouring pairs, such that each pair only has integer spin (1 or 0) configurations. Again, we stress that this grouping is not essential for the driven models which do not rely on symmetries that can distinguish integer or half-integer spins, but is helpful to enable an analogy to the AKLT model.

A definition for the AKLT chain Hamiltonian is:

$$H_{\mathrm{AKLT}} = K \sum_{i=1}^{L/2-1} \vec{\sigma}_{2i} \cdot \vec{\sigma}_{2i+1}. \tag{A1}$$

By inspection, H_{AKLT} commutes with the end-spin operators $\vec{\sigma}_{1,L}$, i.e. has two topological spin-1/2 edge states. While H_{AKLT} enjoys a full spin-rotation symmetry, we will focus only on a $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup generated by global π rotations:

$$R_{x,z}(\pi) = \prod_i e^{-i\pi/2\sigma_i^{x,z}} = (-i)^L \prod_i \sigma_i^{x,z}, \tag{A2}$$

which are sufficient to protect the edge states — this phenomenon is called symmetry protected topological (SPT) order. For example, the edge spins can be pinned by a local magnetic field $H_B = -\vec{B} \cdot \vec{\sigma}_{1,L}$, but such terms would break the protecting symmetry. More generally the fact that R_x anticommutes with R_z when acting only on the edge spin-1/2 states, whereas all bulk excitations are bond-triplets for which R_x and R_z commute, ensures that any symmetry-preserving perturbation to the system that does not close the gap to excitations will preserve the topological edge states in the ground-space of H_{AKLT}.

When the spin-degeneracy is reduced to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup, and the couplings K are sufficiently strongly disordered to ensure many-body localization (MBL), the SPT order survives not only in the ground-state, but also in highly excited eigenstates, and in dynamics from an any initial short-range entangled state (not necessarily an eigenstate). The possibility of this excited state topological order has been theoretically known for some time, but, to our knowledge, this work presents the first-ever experimental realization of topology in the excited states of an interacting many-body system.

Next, we relate these properties of the AKLT chain to the models implemented experimentally and discussed in the main text.

Quasiperiodic model – Consider the model defined in Fig. 1a. First consider, the limit $J, \vec{B} = 0$, i.e. with only the K-couplings. At long times (large n) the time evolution converges to time-independent evolution with respect to an AKLT-type Hamiltonian $U_n|_{\text{K-only}} \to e^{-i\varphi^n H'_{\mathrm{AKLT}}}$ with:

$$H'_{\mathrm{AKLT}} = \sum_{i=1}^{L/2-1} \left(K_x^{[i]} \sigma_{2i}^x \sigma_{2i+1}^x + \varphi^{-1} K_z^{[i]} \sigma_{2i}^z \sigma_{2i+1}^z \right), \tag{A3}$$

where L is the length of the chain (assumed to be even), and the factor of the Golden-ratio, $\varphi = \lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1}{2}\left(1+\sqrt{5}\right)$ accounts for the fact that the X-pulses occur relatively more often then Z-pulses in this sequence.

We note that, while H'_{AKLT} has reduced symmetry compared to the more commonly used spin-rotation invariant H_{AKLT} model defined above. Namely, in H'_{AKLT}, we have intentionally broken the SU(2) symmetry down to a $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup by retaining only x- and z- spin exchange terms with different couplings (since non-Abelian symmetries are incompatible with MBL[S1]), and introduced random bond-disorder to the couplings (also to favour MBL, which requires disorder). Importantly, this anisotropy does not affect the topological aspects of the model. By inspection, H'_{AKLT} commutes with $\vec{\sigma}_{1,L}$, i.e. has two topological edge states.

As described in the text, when we turn on the J-terms with J sufficiently close to π, these act to dynamically enforce the protecting $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry generated by

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$R_{x,z}(\pi)$, by repeated flip-flopping the qubits around the x and z axis in an interleaved quasiperiodic sequence. Loosely speaking, we can understand that these $R_{x,z}(\pi)$ flips cancel out any symmetry breaking terms, such as random fields like the $\vec{B}$-layer, "on average". More precisely, one can find a quasiperiodically rotating frame[6,9] (see also Section C) in which these pulses cancel the symmetry breaking couplings to a very high accuracy. As discussed in Section C, for the non-smooth (recursive/pulsed) drive implemented experimentally, we can prove that this accuracy goes as a large polynomial of the deviation from π-pulses, but numerical evidence suggests this is a loose bound and that the accuracy is actually super-polynomial, likely exponential).

Floquet model – We can also understand the Floquet model (Fig. 2a) via the lens of the AKLT chain by dividing the J pulses into odd- and even- bond terms (where we define bond i to be the bond connecting qubits i and $i+1$, and label the bond by whether i is even/odd). For $J=\pi$, the even bond terms take the form of the Z-only version of H'_{AKLT} above (i.e. with $K_x=0$), which again has topological edge states $\vec{\sigma}_{1,L}$. To see this, note that: $XY_{J=\pi}=e^{-i\pi/2(\sigma^x\otimes\sigma^x+\sigma^y\otimes\sigma^y)}=\sigma^z\otimes\sigma^z=ie^{-i\pi/2(\sigma^z\otimes\sigma^z)}$. For $J=\pi$, the odd bond terms implement a global π-rotation around z, and, similar to the Z-type J-terms in the quasiperiodic model, enforce this z-rotation symmetry when the motion is averaged over an even number of periods. Again, more precisely, one can define a periodically rotating frame in which this cancellation of $R_z(\pi)$ symmetry-breaking fields becomes exact so long as one is in the MBL regime.[19–22] Since only the R_z symmetry is dynamically enforced, the edge states rely on a microscopic $R_x(\pi)$ symmetry.

For $J=\pi$, we could alternatively have achieved the same dynamics by periodically repeating the U_z layer of the quasiperiodic model, with only B^x-fields. However, this model maps to a non-interacting fermion system under the standard Jordan-Wigner transformation, and hence would remain integrable and non-thermalizing for all values of J. To avoid this non-generic, fine-tuned integrability and give a fair chance for MBL to fail, we instead chose to use XY_J gates, which yield non-integrable interactions for $J\neq\pi$, and can be compiled into the same number of native two-qubit gates as the XX_θ gates.

B. Detection of FSPT order through many-body interferometry

In this section we theoretically introduce a non-local bulk "order parameter" for the FSPT phase, and exploit the flexible qubit connectivity of the QCCD architecture to measure it through a many-body interferometry. As with the local qubit correlations presented in the main text, these demonstrations are limited to transient short time regimes due to the accumulation of coherent errors.

The premise of the non-local order parameter, which we dub the "Loschmidt flux echo" is to consider a ring with closed, periodic boundary conditions (PBC), such that there are no edge states and any topology is evident only in global features. Next, we effectively "gauge" the protecting symmetry, i.e. promote the global $\mathbb{Z}_2$ symmetry generated by $g=\prod_i\sigma_i^x$ to a local gauge-redundancy, and explore the affect of a dynamically inserting a classical (non-fluctuating) background gauge-flux by measuring the trace-overlap of the time-evolution unitaries with- and without- a flux inserted:

$$\mathcal{Z}(t)=\frac{\text{tr}\,\mathcal{U}_F^\dagger(t)\mathcal{U}(t)}{\text{tr}\mathbb{1}} \tag{B1}$$

where $\mathcal{U}_F(t)$ denotes the unitary for time-evolution with a symmetry-flux inserted (defined in detail below), and t are assumed to be integers (i.e. multiples of the Floquet period).

Based on general properties of MBL and SPT systems, we argue that the long-time behaviour of $\mathcal{Z}(t)$ can sharply distinguish trivial MBL, FSPT, symmetry-breaking, and thermal behaviours. In particular, we will argue below that:

$$\lim_{t\to\infty}\lim_{L\to\infty}\overline{\mathcal{Z}(t)}=\begin{cases}1 & \text{trivial-MBL}\\ (-1)^t & \mathbb{Z}_2\text{-FSPT}\\ 0 & \text{symmetry-breaking MBL}\\ 0 & \text{thermal}\end{cases} \tag{B2}$$

where $\overline{(\dots)}$ denotes disorder averaging. Finite size corrections to these formulas are $\mathcal{O}(e^{-L/\xi})$ where ξ is the localization length. We also note that thermal and symmetry-breaking MBL can be distinguished by the latter having non-vanishing $\overline{|\mathcal{Z}(t)|}$. This predicted behaviour is confirmed by numerically exact diagonalization (ED) simulations of the FSPT model (see Extended Data Fig. 5).

Intuitive picture – Intuitively, we can understand the FSPT result as follows: the $\mathbb{Z}_2$-FSPT is characterized by a quantized pumping of symmetry parity (henceforth called "charge").[13,21,22] Namely, during each period, for a system with closed, periodic boundary conditions an odd number of symmetry charges encircle the system. In the presence of a symmetry flux, the $\mathbb{Z}_2$ analogue of the Aharonov-Bohm effect dictates that a unit charge encircling a π-flux acquires phase (-1). Otherwise, the local LIOM dynamics is insensitive to the global flux sector (to see this note that for any spatially-well localized operator, one can always choose a gauge such that the local action of flux insertion is equivalent to a physically-inconsequential gauge transformation). Hence, in $t=n$ periods, the evolution with- and without- the flux will then differ by $(-1)^n$.

Locally-implementable approximation – We then construct a strictly-local approximate flux-insertion operator, F, which has the property that, in MBL phases, $F|\psi\rangle$ has finite overlap with the exact flux-inserted state:

$|\psi_F\rangle$. Denoting by $\tilde{\mathcal{Z}}(t)$ the local-approximation obtained by evaluating $\mathcal{Z}(t)$ with the exact flux insertion operator replaced by its local approximation: $\mathcal{U}_F \to F^\dagger \mathcal{U} F$, we argue that $\overline{\tilde{\mathcal{Z}}(t)} \sim c \cdot \overline{\mathcal{Z}(t)}$ where $0 < c < 1$ is model-dependent constant of order $\log c \sim -\xi$, i.e. that the local approximation to the Loschmidt flux echo tracks the true value with finite fidelity. We further design and implement a circuit to measure $\tilde{\mathcal{Z}}$ for the FSPT model utilizing an ancillary qubit. Of course, the experimental implementation suffers from the same limitations due to coherent errors described in the main text, and yields good agreement with simulations only up to ~ 5 Floquet periods.

1. Formalism

a. Gauging the symmetry

Following a standard "gauging" procedure, we define a formal symmetry-gauging procedure in two steps. First, one expands the Hilbert space to include gauge-link variables $\boldsymbol{\tau}_{i,i+1}$ on each nearest neighbour bond where τ^z represents a $\mathbb{Z}_2$-gauge connection (the discrete analogue of the Wilson line segment $e^{i\int \boldsymbol{A}\cdot d\boldsymbol{r}}$ of electromagnetism), and τ^x represents the $\mathbb{Z}_2$ gauge-electric field. Second, one projects into the subspace of this enlarged Hilbert space that obeys the Gauss' law constraint: $G_i = 1\ \forall i$ with:

$$G_i = \tau^x_{i-1,i}\sigma^x_i\tau^x_{i,i+1}, \tag{B3}$$

(the discrete and lattice analogue of Gauss' law $\nabla\cdot\boldsymbol{E}(\boldsymbol{r}) = \rho(\boldsymbol{r})$ for electromagnetism). Crucially, we consider τ-variables to be completely non-dynamical "background" gauge variables, i.e. which have Hamiltonian $= 0$.

Any $\mathbb{Z}_2$-symmetric local Hamiltonian can be similarly gauged by adding a connected string of τ^z's connecting between every pair of symmetry-charged single site operators ($\sigma^{y,z}$) in each term of the Hamiltonian [1].

For example, the $\mathbb{Z}_2$ FSPT model in the main text can be generated by a local time-dependent Hamiltonian which changes as follows under the gauging procedure:

$$H = \sum_i \left[J_i(t)\left(X_iX_{i+1} + Y_iY_{i+1}\right) + h_i(t)X_i\right]$$

$$\Big\downarrow \text{gauge}$$

$$H_G = \sum_i \left[J_i(t)\left(X_iX_{i+1} + Y_i\tau^z_{i,i+1}Y_{i+1}\right) + h_i(t)X_i\right], \tag{B4}$$

where $J_i(t), h_i(t)$ the (piecewise constant) time-dependent coefficients that reproduce the unitary circuit dynamics shown in Fig. 2a.

b. Formal Flux Insertion

After projection, the system retains 2^L (gauge-invariant) degrees of freedom consisting of: 2^{L-1} spin configurations that satisfy the charge-neutrality constraint implied by Gauss' law: $\prod_i \sigma^x_i = \prod_i \tau^x_{i-1,i}\tau^x_{i,i+1} = 1$, and a global $\mathbb{Z}_2$ "magnetic" flux: $\Phi = \prod_i \tau^z_{i,i+1} = \pm 1$. This foreshadows that we will be able to effectively simulate the gauged system without introducing extra qubits to represent the τ variables.

We can formally define the action of a flux-insertion operator, F, on any local gauge-invariant operator $\mathcal{O}_i$ with bounded support, centred at site i as follows: Choose a reference bond, (i_a, i_a+1) at which to insert the flux, and divide the system (with periodic boundary conditions) into an interval A of length $L/2$ centred at i_a and its complement, B centred at the antipodal point $i_b = (L/2 - i_a) \bmod L$. Then define the "flux-inserted" version of $\mathcal{O}$ as:

$$\mathcal{O}(x) \to \mathcal{O}_F(x) = \begin{cases} f_+^\dagger \mathcal{O}_{F,i} f_+ & x\in A \\ \mathcal{O}(x) & x \in B\end{cases},$$
$$f_+ = \left(\prod_{x_b<x<x_a}\sigma^x_i\right)\tau^x_{i_a,i_a+1}. \tag{B5}$$

For any operator in A away from site i_a, f_+ acts like a pure gauge transformation. Furthermore, this definition is actually independent of the reference points $i_{a,b}$ (up to a gauge transformation). While the above definition holds sharply for strictly local operators (with finitely-bounded support), one can extend it to exponentially-well localized operators with the minor caveat that the definitions depend on choice of $i_{a,b}$ with sensitivity $\sim \mathcal{O}(e^{-L/2\xi})$.

As an application, we can use this definition to identify pairs of symmetry-preserving gauged-MBL eigenstates in different flux sectors, and show that each element of the pair has nearly identical energy to accuracy $O(e^{-L/\xi})$. Following standard practice, we define a Floquet system as being MBL if its Floquet operator (time evolution for a single time-step/circuit layer) can be written as:

$$\mathcal{U} = W^\dagger \Xi e^{-iE[\sigma^x_i]}W \tag{B6}$$

where $E[\sigma^x_i]$ is an exponentially-well localized function of σ^x_i's, and W is a (symmetric) finite-depth local unitary that implements the weak-local dressing of single-site qubits σ^x_i into conserved local integrals of motion (LIOMs):

$$\ell_i = W^\dagger\sigma^x_i W. \tag{B7}$$

[1] Note that there are guaranteed to be an even number of these factors in any term of a symmetric Hamiltonian. In $1d$, imposing strictly locality in all terms removes any ambiguity for different choices of pairings.

In the gauged system (with non-dynamical background gauge fields), the gauged analogue of W (defined by applying the gauging procedure to the local generator of W: $-i\log W$, and which, in a slight notational abuse, we will denote by the same symbol as the ungauged version) depends on the gauge-variables only through τ^z , i.e. W commutes with the gauge-flux operators. Here, Ξ is a symmetric operator satisfying $[\Xi, E] = 0$. For all the phases we consider here, $\Xi^2 = 1$. For example, in the gauged $\mathbb{Z}_2$ FSPT with periodic boundary conditions:

$$\Xi_{\text{FSPT}} = \Phi = \prod_i \tau^z_{i,i+1}, \tag{B8}$$

is the $\mathbb{Z}_2$ gauge flux operator (with open boundary conditions, Ξ would capture the quantized spin echo dynamics of the topological edge states).

The key property of symmetric MBL is that the (quasi)-energy eigenstates can be uniquely specified by listing the eigenvalues of the LIOM operators. In a gauged system, for each state in the even flux sector: $|\{s_i\}, \Phi = +1\rangle$ defined by $\ell_i|\{s_i\}, \Phi = +1\rangle = s_i|\{s_i\}, \Phi = +1\rangle$, there is a partner state in the odd gauge-flux sector: $|\{s_i, \Phi = -1\rangle = F|\{s_i\}, \Phi = +1\rangle$ defined by $\ell_{F,i}|\{s_i, \Phi = -1\rangle = s_i|\{s_i, \Phi = -1\rangle$ (i.e. labelled by the same LIOM eigenvalues but with flux-inserted LIOM operators $\ell_{F,i}$). Crucially, since the action of F is locally equivalent to a gauge transformation everywhere, the partner states have the same (quasi)-energy, E_F, (up to $\mathcal{O}(e^{-L/\xi})$ corrections).

From these definitions and observations, one can readily verify Eq. B2. In particular, for the FSPT:

$$\begin{aligned}\mathcal{Z}(t = n) &= \frac{\text{tr}\ \mathcal{U}_F^\dagger(n)\mathcal{U}(n)}{\text{tr}\mathbb{1}} &\text{(B9)}\\ &= \frac{\text{tr}\ W^\dagger F^\dagger \Phi^n F e^{+inE_F} W W^\dagger \Phi e^{-inE} W}{\text{tr}\mathbb{1}} \\ &= \frac{1}{\text{tr}\mathbb{1}} \underbrace{\left(F^\dagger \Phi^n F \Phi^n\right)}_{(-1)^n} \underbrace{\left(W^\dagger e^{+i(E_F - E)} W\right)}_{1+\mathcal{O}\left(e^{-L/\xi}\right)} \\ &= (-1)^n + \mathcal{O}\left(e^{-L/\xi}\right) &\text{(B10)}\end{aligned}$$

where we have used that $[W, \Phi] = 0$ since W is diagonal in the τ^z basis.

Behaviour of flux-echo in other phases – Similar arguments show that a trivial MBL phase ($\Xi = \mathbb{1}$) would have $\mathcal{Z}(t)_{\text{trivial-MBL}} = 1$. By contrast, in a thermal phase the localization length diverges, and the system becomes sensitive to the global flux (e.g. the ratio of the Thouless energy to level spacing diverges[S2]), and to exhibit chaotic response, such that the evolutions with and without flux insertions deviate exponentially with time, resulting in exponential decay of $\mathcal{Z}$.

Finally, we note that in MBL systems with spontaneous symmetry breaking (SSB), the above arguments fail. For example, there is not quite a complete set of symmetry-preserving LIOMs. For example, if we consider a ferromagnetic spin glass, a natural set of LIOMs would be a locally dressed version of $\{\sigma^z_i \sigma^z_{i+1}\}$, which count the parity of domain walls in the magnetic order, but require one additional integral of motion for completeness (e.g. either a local asymmetric operator like σ^z_i or a symmetric, but non-local operator such as $\prod_i \sigma^x_i$). In the gauged system, the total parity of domain walls is locked to the gauge flux in the system, so that there is a local energy cost to inserting flux, and the quasi-energies with and without flux differ by an $O(1)$ constant. As a result, for SSB-MBL, $\mathcal{Z}(t) \sim e^{-i\varepsilon t}$ where ε depends on the location of flux insertion and disorder configuration. In particular, the disorder average $\overline{\mathcal{Z}(t)} = 0$ at long times, but the average modulus remains $\overline{|\mathcal{Z}(t)|} = 1$. The latter property distinguishes the SSB-MBL behaviour from that of thermal systems for which $\overline{|\mathcal{Z}(t)|} = 0$.

Together, these behaviours show that the flux-echo provides a complete non-local "order parameter" diagnosing all possible thermal and MBL phases with symmetric drives. This property makes the flux-echo a potentially useful diagnostic. For example, a previously introduced non-local string-order parameter of the FSPT phase[13] could not distinguish between FSPT and SSB-MBL requiring measurement of multiple order parameters to uniquely diagnose the phase.

Local approximation to flux-echo – While the above arguments demonstrate the existence of a formally exact flux-insertion procedure, in practice, this procedure requires explicit knowledge of LIOMs. However, we now argue that it is sufficient to locally approximate the action of flux insertion without knowledge of the LIOMs, which enables a practical measurement scheme for the flux-echo.

Since the LIOMs, ℓ_i of a symmetric MBL system are related by local dressing, W to single-site qubit operators σ^x_i, one can write the exact flux insertion operator $F = W^\dagger \tau^x_{i_a} W$ (where, recall that i_a is an arbitrarily chosen location of flux-insertion). While the exact flux insertion operator F cannot be implemented without knowledge of W, one can approximate the action of F by simply acting with $\tau^x_{i_a}$ which differs only by quasi-local, and symmetric dressing from F, and its correlation function:

$$\tilde{\mathcal{Z}}(t) = \frac{\text{tr}\ \tau^x_{i_a} \mathcal{U}^\dagger(t) \tau^x_{i_a} \mathcal{U}(t)}{\text{tr}\ \mathbb{1}} \tag{B11}$$

would hence have finite overlap with the exact flux Loschmidt echo, $\mathcal{Z}(t)$. Approximating the action of W as randomly scrambling operators within a localization length ξ, generically, we expect the disorder averaged magnitude of this quantity $\overline{|\tilde{\mathcal{Z}}(t)|}$ to be smaller than the $\overline{|\mathcal{Z}(t)|}$ by a finite constant factor of order $0 < c \sim e^{-\xi} < 1$.

These arguments are supported by numerical simulations of the FSPT model (see Extended Data Fig. 5).

2. Quantum Circuit Implementation

The flux-echo witness $\tilde{\mathcal{Z}}$ can be measured interferometrically (see Extended Data Fig. 6) using an ancilla qubit initialized in an equal superposition of $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, to control the time-evolution, such that the system evolves under $H(t)$ or $\tau^x_{i_a} H(t) \tau^x_{i_a}$ when the ancilla is in the $|0\rangle$ or $|1\rangle$ state respectively. For the FSPT model, this amounts to simply allowing the ancilla qubit to flip the sign of a single $e^{-iJ\theta\sigma^y_i\sigma^y_{i+1}} \to e^{+iJ\theta\sigma^y_i\sigma^y_{i+1}}$, which can be accomplished with only two extra CNOT gates per Floquet period as shown in Extended Data Fig. 6b. Then, measuring $\langle \sigma^{x,y} \rangle$ for the ancilla reveals the real or imaginary parts of $\tilde{\mathcal{Z}}$ respectively.

We have implemented this protocol both in classical simulations and experimentally, with results shown in Extended Data Fig. 6. The experimentally measured flux-echo in the FSPT phase initially shows oscillations that survive up to ≈ 5 Floquet periods, but then are dephased due to coherent errors (as indicated by the oscillations with large amplitude, but wrong phase compared to numerical simulations). We note that this dephasing occurs $\approx$ 2-3$\times$ more rapidly than that for the OBC circuits, suggesting that the more complicated ion transport paths required to implement periodic (ring) boundary conditions exacerbate the effects of errors, consistent with the expected results of spatial inhomogeneities in the magnetic-field as discussed above. In contrast, in the thermal phase the flux-echo immediately dies, and in the trivial MBL phase the data is consistent with a saturation to a constant value up to an overall slow decay due to weakly-open MBL effects.

C. Long-time (meta)stability of recursively-generated quasiperiodic drives

This section addresses the long-time fate of the EDSPT model, in an idealized perfectly isolated system. We will see that the EDSPT behaviour is not a rigidly stable phase, but rather a very-long-lived but ultimately metastable behaviour. However, the lifetime for the edge modes can be made exponentially long in a certain parameter regime, and hence can easily be made much longer than any experimental lifetime.

Topologically-trivial paramagnetic and time-quasicrystalline behaviour arising in such recursive Fibonacci drives were previously studied theoretically and numerically in Ref. 12, leveraging the exponential growth of t_n with number of recursions n to efficiently reach exponential long times in simulations[2]. There, it was observed that, for strong-disorder, MBL only occurs as a meta-stable phenomena in these recursively driven systems: rather than saturating to finite constant, qubit-autocorrelators decayed logarithmically slowly, with MBL dynamics eventually melting away into infinite temperature incoherent at ultra-long ‘heating time-scales $t_h \sim e^{1/\delta}$ where δ represents the drive strength (more precisely, deviation from an idealized perfectly-commuting drive). Numerical simulations (Extended Data Fig. 7) indicate that this long-lived MBL behaviour also arises in the EDSPT model.

Additionally, in this section we extend a recursive adaptation of high-frequency (Magnus-type) expansion to show that the dynamics are governed by an effective time-independent Hamiltonian with emergent dynamical symmetries, for times up to $t_* \sim \delta^{-p}$ with $p = 3, 5$ for topologically non-trivial (see below) and trivial[12] phases respectively. The long-time dynamics between t_* and t_h appears to be beyond the purview of *any* effective Hamiltonian description, and a controlled theoretical description of this regime remains elusive. However, the recursive nature of the drive permits efficient numerical access to long-time dynamics, and our simulations indicate that the topological edge qubits and emergent dynamical symmetries persist well beyond t_* up to t_h.

1. Recursive Magnus expansion

In Ref. 12, we previously developed a recursive high-frequency (“Magnus”) expansion technique to reduce the evolution under a weak quasiperiodic drive generated by a Fibonacci sequence of two circuit layers to an effective time-independent Hamiltonian evolution. This technique was accurate when the unitary of the generating circuit layers was close to the identity by an amount $\sim \delta$. The resulting effective Hamiltonian accurately captured the evolution up to time $t_* \sim \delta^{-5}$. Comparison to numerical simulations showed that, beyond t_*, the driven system exhibited logarithmically slow heating causing complete thermalization in timescale $t_h \sim e^{1/\delta}$.

Here, we generalize this technique to the case relevant for the idealized EDSPT model, i.e. when generating unitaries $\mathcal{U}_{x,z}$ are near-perfect π-pulses about the x, z axes respectively. I.e. $\mathcal{U}^2_{x,z} \approx \mathbb{1} + \mathcal{O}(\delta)$ but $\mathcal{U}_{x,z}$. We will show that, in this parameter regime, one can again obtain an approximate time-independent Hamiltonian description up to times $t_* \sim \delta^{-3}$. Furthermore, the resulting effective Hamiltonian has an emergent pair of $\mathbb{Z}_2$ (Ising) symmetries whose generates g^α with $\alpha \in \{x, z\}$ are related to $\prod_i \sigma^\alpha_{2i}\sigma^\alpha_{2i+1}$ by a finite-depth local unitary transformation whose precise form depends on the details of the drive. Our theoretical picture of the EDSPT phase, is that these emergent dynamical symmetries protect the SPT edge modes of an effective AKLT chain, “encrypted” in a quasiperiodically rotating frame of the drive.

While this technique provides our best known analytic handle on recursively generated quasiperiodic drives, numerical simulations and experimental results suggest that the recursive Magnus expansion dramatically underesti-

[2] Asymptotically $t_n \sim \varphi^n$ where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

mates the stability and survival time scales for the emergent dynamical symmetries. First, we note that the model implemented in the main text, which has maximal disorder strength 4π for the K-couplings is actually not in the small δ-regime, yet it still exhibits long-lived signatures of topological edge states. Moreover, based on the numerical simulations that access long times on modest system sizes observe that these oscillations decay exponentially-slowly with respect to $1/\delta$, suggesting that the emergent dynamical symmetries survive up to this much longer t_h timescale.

As a preview, the main result of this section is that, up to third order in a small parameter, δ that quantifies the deviation from some exactly-solvable ideal drive, we can approximately reduce the unitary evolution to the form:

$$\mathcal{U}_{6n} = V^\dagger \mathrm{e}^{-i\varphi^{6n}(D+\dots)} V \tag{C1}$$

where, D is an effective time-independent Hamiltonian that obeys an emergent $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry generated by $g^{x,z} = V^\dagger \prod_i \sigma_i^{x,z} V$ respectively. Here, V is a finite depth local unitary that we explicitly construct, and $(\dots)$ includes i) terms with sub-leading (and generally oscillatory) n-dependence (e.g. with coefficients decreasing as $\sim \varphi^{-6n}$ or faster), and ii) $\mathcal{O}(\delta^3)$ terms beyond the validity of the expansion. We note that, for reasons that will become apparently shortly, it is convenient to express the evolution at Fibonacci-indices that are multiples of 6, and note that similar expressions can be obtained for $\mathcal{U}_{6n+k}$ with $k \in \{1, \dots 5\}$.

As with the closely-related expansion for topologically trivial Fibonacci drives previously derived in,[12] this expansion has the peculiar property that it breaks down at finite order independent of expansion parameter δ. Namely, at $O(\delta^3)$, the expansion results in terms in D that grow faster that φ^{6n} regardless of δ and finite system size, whereas on general grounds for sufficiently small δ one must always be able to reduce $\mathcal{U}_{6n} = e^{-i\varphi^{6n} D_n'}$ for some bounded, Hermitian D_n' (though not necessarily one that is time- i.e. n-independent) since the circuit contains only $\sim \varphi^{6n}$ layers. Hence, the generation of terms with coefficients growing faster than φ^{6n} signal a breakdown in the recursive Magnus expansion, and hint at a possible obstruction to describing the dynamics beyond t_* by *any* effective time-independent Hamiltonian.

2. Inflation rule

To derive the recursive Magnus expansion, we exploit a self-similar fractal structure of Fibonacci sequences under "inflating" the generating unitaries $\mathcal{U}_{x,z}$. Let us consider a Fibonacci drive generated by two pulses ($\mathcal{U}_x = X\mathrm{e}^A, \mathcal{U}_z = Z\mathrm{e}^B$) where $X = \prod_i \sigma_{2i}^x \sigma_{2i+1}^z$, $Z = \prod_i \sigma_{2i}^z \sigma_{2i+1}^z$ commute and square to one, and A, B are anti-Hermitian operators with small norm $\delta \sim \|A, B\| \ll 1$ that represent perturbations to the ideal drive. For convenience, we also define $Y = XZ$ (note that in our notation X, Y, Z are *not* single qubit Pauli operators, but rather strings of even numbers of $\sigma^{x,y,z}$ products).

The Fibonacci drive is generated by the inflation rule

$$\left(Xe^A, Ze^B\right) \to \left(Ze^B, Xe^A Ze^B\right) = \left(Ze^B, Ye^{A_x} e^B\right), \tag{C2}$$

where we have defined $A_x = XAX$, and similarly define $A_{y,z}, B_{x,y,z}$. It is convenient to recurse three times with the inflation rule, and move all X, Z operators to the left (conjugating A,B operators along the way) until the X, Z prefactors return to their original form:

$$\begin{aligned}\left(Xe^A, Ze^B\right) &\overset{(1)}{\to} \left(Ze^B, Xe^A Ze^B\right) = \Big(Ze^B, XZe^{A^z} e^B\Big) \\ &\overset{(2)}{\to} \left(XZe^{A_z} e^B, Ze^B XZe^{A_z} e^B\right) = \left(XZe^{A_z} e^B, Xe^{B_y} e^{A_z} e^B\right) \\ &\overset{(3)}{\to} \left(Xe^{B_y} e^{A_z} e^B,\ Ze^{A_y} e^{B_x} e^{B_y} e^{A_z} e^B\right).\end{aligned} \tag{C3}$$

This "3x-inflated" inflation rule yields a recursion relation for A, B alone:

$$(\mathrm{e}^A, \mathrm{e}^B) \xrightarrow{(3x)} (\mathrm{e}^{B_y} \mathrm{e}^{A_z} \mathrm{e}^B, \mathrm{e}^{A_y} \mathrm{e}^{B_x} \mathrm{e}^{B_y} \mathrm{e}^{A_z} \mathrm{e}^B), \tag{C4}$$

where the arrow now represents this "3x" recursion.

At this point, it turns out to be easier to think of the unitaries: $\mathcal{U}_n$ as being functions of an "alphabet" of 8 "letters": $(A, B, A_x, B_x, A_y, B_y, A_z, B_z)$. To avoid a proliferation of subscripts, in what follows we change our notation to label Fibonacci times with indices that are multiples of 3, by defining:

$$U_n \equiv \mathcal{U}_{3n}, \tag{C5}$$

where U_n corresponds to $t_{3n} = F_{3n} \sim \varphi^{3n}$ circuit layers. The recursion rules for A_x, A_y, A_z and B_x, B_y, B_z follow from those of A, B simply by conjugating by X, Y, Z. We have the following recursion relation for the unitary operator:

$$U_{n+1}(\mathrm{e}^A, \mathrm{e}^B, \mathrm{e}^{A_x}, \mathrm{e}^{B_x}, \mathrm{e}^{A_y}, \mathrm{e}^{B_y}, \mathrm{e}^{A_z}, \mathrm{e}^{B_z}) = U_n(\mathrm{e}^{B_y} \mathrm{e}^{A_z} \mathrm{e}^B, \mathrm{e}^{A_y} \mathrm{e}^{B_x} \mathrm{e}^{B_y} \mathrm{e}^{A_z} \mathrm{e}^B, \mathrm{e}^{B_z} \mathrm{e}^{A_y} \mathrm{e}^{B_x}, \dots), \tag{C6}$$

where we emphasize that n now labels the number of "3x" recursions, and $U_1 = \mathrm{e}^A$.

3. Effective Hamiltonian and generalized Magnus expansion

Our goal is to check the emergence of an effective Hamiltonian for the dynamics by computing order by order $H_n = \log U_n$ (defined here to be anti-Hermitian), which obeys

$$H_{n+1}(A, B, A_x, B_x, A_y, B_y, A_z, B_z) = H_n(B_y \star A_z \star B, A_y \star B_x \star B_y \star A_z \star B, B_z \star A_y \star B_x, \dots), \tag{C7}$$

where $A \star B \star C \star \cdots = \log(\mathrm{e}^A \mathrm{e}^B \mathrm{e}^C \dots)$.

To proceed, we relabel those letters $(K_1, K_2, \dots, K_8) = (A, B, A_x, B_x, A_y, B_y, A_z, B_z)$, and expand

$$H_n = \sum_{\alpha=1}^{8} c_\alpha(n) K_\alpha + \sum_{\alpha\beta} C_{\alpha\beta}(n) \left[K_\alpha, K_\beta\right] + \dots . \tag{C8}$$

where $(\dots)$ indicate nested commutators with larger number of $K's$ that contribute at $\mathcal{O}(\delta^3)$.

Substituting this expression into the recursion relation (C7), we find that the leading order coefficients are given by

$$\vec{c}_\alpha(n+1) = \hat{M}^T \vec{c}_\alpha(n), \tag{C9}$$

with the recursion matrix

$$\hat{M} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} . \tag{C10}$$

This recursion relation can be solved straightforwardly by diagonalizing $\hat{M}$. The largest eigenvalue is φ^3, with an eigenvector $(1/\varphi, 1, 1/\varphi, 1, 1/\varphi, 1, 1/\varphi, 1)$. This implies an *emergent* symmetry between A, A_x, A_y, A_z (and same for B, B_x, B_y, B_z) at long times. Namely, from this dominant eigenvector coefficients we can see that $c_1 \sim c_3 \sim c_5 \sim c_7$ at large n, and $c_2 \sim c_4 \sim c_6 \sim c_8$. In other words, at this order H_n commutes with the pules X and Z at large n (long times).

Let us now consider the next order terms $C_{\alpha,\beta}$. There are two different types of contributions: one coming from applying the substitution rule $\tilde{K}_\alpha \to \sum_\beta M_{\alpha\beta} K_\beta + \dots$ to leading order to the term $\sum_{\alpha\beta} C_{\alpha,\beta}(n) \left[\tilde{K}_\alpha, \tilde{K}_\beta\right]$ on the right-hand side, and the other one coming from the Baker, Campbell, Hausdorff (BCH) formula from $\sum_\alpha c_\alpha(n) \tilde{K}_\alpha$. The first type is readily taken into account, and we find:

$$\hat{C}(n+1) = \hat{M}^T \hat{C}(n) \hat{M} + \hat{r}_n, \tag{C11}$$

where $\hat{r}_n$ is a skew-symmetric matrix that originates from the second type of terms mentioned above.

Keeping track of the contributions to $\hat{r}_n$ is straightforward albeit cumbersome. For example, we have a term $c_1(n)\tilde{K}_1 = c_1 \tilde{A}$ with $\tilde{A} \to B_y \star A_z \star B$. This will generate terms $\frac{c_1(n)}{2}[B_y, A_z] + \frac{c_1(n)}{2}[B_y, B] + \frac{c_1(n)}{2}[A_z, B]$ that should be included in $\hat{r}_n$. The contributions from $c_3(n), c_5(n), c_7(n)$ can be obtained by conjugating by X, Y, Z, and the other contributions can be dealt with similarly (except there are now five exponentials to expand using the BCH formula). With this expression for $\hat{r}_n$, Eq. (C11) can be solved by going to the eigenbasis of $\hat{r}_n$ and then rotating back. The explicit expression for the matrix $\hat{C}(n)$ is not particularly illuminating, but we have checked that it grows with n as φ^{3n}, corresponding to time, which means that one can write $H_n = -i\varphi^{3n}(D + \dots)$, where D can be interpreted as an effective Hamiltonian for the dynamics. We will show that H_n has an *emergent* $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry at large n (long times), which protects the topological edge modes.

This expansion breaks down at the next order: including nest commutators in the expression of H_n, we find that the coefficients of such terms grow with n exponentially faster than φ^{3n}. Not only does the Hamiltonian interpretation therefore break down at this order, but higher orders become more important earlier in time: this appears to be a general property of recursive drives as noted in Ref. 12. This means that this high-frequency, Magnus-type expansion is only strictly valid for times up to $t_\star \sim \delta^{-3}$, with $\delta \sim \|A, B\| \ll 1$.

However, as previously commented, we find numerically that even away from this high-frequency regime, strong disorder and MBL protect this dynamical phase up to much longer, exponential time scales (see Extended Data Fig. 7).

4. Emergent symmetry

Let us now analyze the symmetries of H_n including second order terms. Let

$$\hat{g}_x = \begin{pmatrix} 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0 \end{pmatrix}, \quad \hat{g}_z = \begin{pmatrix} 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0 \end{pmatrix}. \tag{C12}$$

This is a representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on the 8-dimensional "alphabet", whose action corresponds to conjugating letters by X and Z, and $\hat{g}_x^2 = \hat{g}_z^2 = 1$ and $[\hat{g}_z, \hat{g}_z] = 0$. Note that the recursion matrix is compatible with this symmetry:

$$[\hat{M}, \hat{g}_z] = [\hat{M}, \hat{g}_x] = 0. \tag{C13}$$

The leading order coefficients $c_\alpha(n)$ are symmetric only at long times, as the initial condition $c_\alpha(n=0) = \delta_{\alpha,1}$ breaks the symmetry. Because of this, the matrix $\hat{r}_n$ is also "almost" symmetric up to $\mathcal{O}(1)$ errors: $[\hat{r}_n, \hat{g}_{x,z}] = \mathcal{O}(1)$. However, these non-symmetric terms get amplified in the recursion relation (C11), so $\hat{C}(n)$ contains some exponentially large non-symmetric terms of order $\mathcal{O}(\varphi^{3n})$. We will write

$$\hat{C}(n) = \hat{C}_s(n) + \delta\hat{C}(n), \tag{C14}$$

with the symmetrized matrix $\hat{C}_s = (\hat{C} + \hat{g}_x\hat{C}\hat{g}_x + \hat{g}_z\hat{C}\hat{g}_z + \hat{g}_x\hat{g}_z\hat{C}\hat{g}_x\hat{g}_z)/4$, and $||\delta\hat{C}(n)|| \sim \mathcal{O}(\varphi^{3n})$.

Let us now try to cancel out those symmetric term by a (finite-depth unitary) change of frame:

$$V = \mathrm{e}^{\frac{1}{2}\sum_\gamma v_\gamma K_\gamma}, \tag{C15}$$

with $U'_n = \hat{V} U_n \hat{V}^\dagger$.

We find

$$H'_{2n} = \log U'_{2n} = \frac{1}{2}\sum_{\gamma\alpha} c_\alpha(2n) v_\gamma \left[K_\gamma, K_\alpha\right] + \sum_\alpha c_\alpha(2n) K_\alpha + \sum_{\alpha\beta} C_{\alpha\beta}(2n)\left[K_\alpha, K_\beta\right] + \dots, \tag{C16}$$

where we restricted ourselves to even times to avoid odd/even effects. Our goal is to choose v_γ to cancel out the non-symmetric terms $\delta\hat{C}(n)$, in (C14), or more precisely, cancel out the leading non-symmetric terms $\delta\hat{C}(2n) = \varphi^{6n}\left(\delta\hat{C}_0 + \dots\right)$, where the dots represent exponentially decaying terms. We find that this can be achieved by choosing $v_\alpha = \left(-\frac{2+\sqrt{5}}{4} - \frac{1-\sqrt{5}}{2}c, c, -\frac{\sqrt{5}}{4} - \frac{1-\sqrt{5}}{2}c, c, \frac{1}{4} - \frac{1-\sqrt{5}}{2}c, c, \frac{3}{4} - \frac{1-\sqrt{5}}{2}c, c-1\right)$ with c a constant. Note that this is only possible since the leading order terms $\sum_{\alpha=1}^{8} c_\alpha(n) K_\alpha$ in H_n are symmetric at large n.

We conclude that for this choice of dressing unitary, we can write

$$U_{2n} = V^\dagger \mathrm{e}^{-i\varphi^{6n}(D+\dots)} V, \tag{C17}$$

with an effective Hamiltonian D that has a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry generated by the pulses, and where the dots denote exponentially small in n non-symmetric terms. In other words, the unitary evolution operator U_{2n} commutes with the "dressed" symmetry generators

$$g_x = V^\dagger X V, \;\; g_z = V^\dagger Z V, \tag{C18}$$

at long times. This symmetry is manifestly *emergent*, as V depends on A and B. The effective Hamiltonian has the following expression

$$-iD = \frac{1}{\sqrt{5}}\left(B_s + \varphi^{-1}A_s\right) + \frac{3\sqrt{5}-5}{40}\left[A,B\right]_s + + \frac{\sqrt{5}-5}{40}\left[A,B_x\right]_s + \frac{1}{4\sqrt{5}}\left[A,B_z\right]_s. \tag{C19}$$

with $A_s = (A + A_x + A_y + A_z)/4$, and similarly for all symmetrized quantities, *e.g.* $\left[A,B\right]_s = (\left[A,B\right] + \left[A_x,B_x\right] + \left[A_y,B_y\right] + \left[A_z,B_z\right])/4$.

To investigate the long-time dynamics beyond the regime of validity of the recursive Magnus expansion, we have performed numerically exact simulations (full Hilbert space evolution) of the EDSPT model for various

J. The recursive structure enables access to exponentially long times $t_n \sim F_n$ with n recursions. Extended Data Fig. 7 shows the resulting edge-qubit correlators, averaged over disorder, for various J values. For J close to π, we observe that the quasiperiodic oscillations of the edge qubits survive up to very long times. We estimate the heating time t_h as the time at which the edge-oscillation drops below an arbitrary small threshold: $\epsilon = 0.02$. As shown in the inset of Extended Data Fig. 7, t_h shows super-polynomial growth $1/\delta = \pi/|J-\pi|$ for small δ, indicating that the edge qubit oscillations survive *far beyond the timescale at which recursive Magnus expansion fails.* This behaviour is similar to that numerically observed in topologically-trivial recursive Fibonacci drives,[12] and suggests that ultra-long-lived but ultimately metastable MBL-like dynamics are generic features for recursive drives with strong disorder. However, a detailed theoretical picture of the logarithmically slow entanglement growth and heating in these models remains elusive at this time.

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