nature portfolio

https://doi.org/10.1038/s41586-026-10720-3

Supplementary information

Backreaction of stimulated Hawking radiation in an optical analogue

In the format provided by the authors and unedited

Supplement: Backreaction of stimulated Hawking radiation in an optical analogue

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May 12, 2026

Methods

1 Propagation equation

A light pulse propagating in a nonlinear medium, such as an optical fibre, produces a wide range of phenomena that can be described by a formalism based on the electric field $E(z,t)$. These phenomena include the well-known optical Kerr effect [1], resonant radiation (RR) (also called dispersive wave or Cherenkov radiation) [2], Raman-induced frequency shift (RIFS) [3], and supercontinuum generation [4].

We describe light in optical fibres by using Maxwell's equations without free charges or currents. Considering linearly polarised light in a given transverse mode and only its forward propagating component, we arrive at the unidirectional pulse propagation equation (UPPE) for the electric field E [5]:

$$i\partial_z E_\omega(z) + \beta(\omega)E_\omega(z) + \frac{\omega}{2cn(\omega)}P_\omega^{(\mathrm{NL})}(z) = 0, \tag{1}$$

where the subscript ω denotes a Fourier transform, $\beta(\omega)$ is the full dispersion relation, and $P^{(\mathrm{NL})}(z) = \epsilon_0\chi^{(3)}E^3(z,t)$ is the nonlinear polarisation term.

One of the most critical parts of any propagation model is its dispersion description. We consider the full propagation constant $\beta(\omega) = \omega n(\omega)/c$ to describe ultra-short pulses (~ 8 fs). For photonic-crystal fibres, the refractive index $n(\omega)$ includes material and geometrical contributions [1, 6]. The nonlinear term $P^{(\mathrm{NL})}(z,t)$ enables frequency mixing during propagation.

We can further reduce the UPPE to the paradigmatic nonlinear Schrödinger equation (NLSE) after some approximations, in particular, a second-order expansion in the dispersion $\beta(\omega)$ and restricting the nonlinear interaction to $P^{(\mathrm{NL})}(z) = \chi^{(3)}|E(z,t)|^2E(z,t)$. However, the dynamics of the optical analogue of Hawking radiation require ultra-short pulses—comprising only a few optical cycles—such that the NLSE is insufficient. For this reason we implement the UPPE. In fact, we must generalise it further to include the so-called negative-frequency signals, as we will see next.

Let us consider the analytic signal $a(z,t)$, defined by

$$a(z,t) = \frac{1}{\pi}\int_0^\infty E_\omega(z)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega, \tag{2}$$

as the propagation variable [7, 8]. From this definition it follows that the analytic signal is complex and includes only positive frequencies. The resulting propagation equation for the analytic signal

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in the co-moving frame $\tau = t - z/u$ and $\zeta = z/u$ (u is the velocity of the moving frame) including negative frequencies is

$$i\partial_\zeta a_\omega(z) + u\beta(\omega)a_\omega(z) + u\gamma(a^3 + 3|a|^2 a + 3|a|^2 a^*)_{\omega,+} = 0, \tag{3}$$

where $\gamma = \omega\epsilon_0\chi^{(3)}/(16cn(\omega))$. The subscript $+$ denotes a positive-frequency filter implemented via a Heaviside step function $\Theta(\omega)(\cdots)_\omega = (\cdots)_{\omega,+}$, used to retain only positive-frequency components in the propagation of the analytic signal field [9], as the nonlinear product with a^* introduces back negative frequencies. The three non-linear terms are, respectively, the third-harmonic generation (THG), the self-phase modulation (SPM) and the conjugated self-phase modulation (SPM*).

The SPM* term is often discarded in standard SVEA regimes or for being non-physical [10] (called the conjugated Kerr term in the reference). It acts as a source term, generating signals from mixing positive and negative frequencies, which have been predicted and observed numerically and experimentally in the positive part of the spectrum [9]. This term produces two signals: the negative-frequency resonant radiation (NRR) [11,12] and the negative-frequency Hawking radiation (NHR) [12]. It is worth noting that the so-called negative-frequency signals are only significant in extreme nonlinear optics (XNLO) conditions and, as such, do not commonly appear in standard fibre optical experiments. These conditions include the use of ultrashort pulses (~ 10 fs) and highly-nonlinear fibres (as the photonic-crystal fibres) with nonlinear coefficient $\gamma \sim 100\ \mathrm{W}^{-1}\mathrm{km}^{-1}$.

2 Numerical methods

We obtain numerical solutions corresponding for the propagation equation (3) by implementing the Dormand-Prince algorithm. This algorithm is part of the Runge-Kutta family of solvers for ordinary differential equations where fourth- and fifth-order solutions are obtained, with their difference providing an error estimate. The fast Fourier transform (FFT) accounts for most of the computation cost, as it is required to switch between temporal and frequency domains multiple times per step.

To match the experiment, we propagate each run to $z = 7$ mm with a fixed step size of 0.5 μm. The frequency and time vectors in the FFT have a size of 2^{16}. The frequency window ranges from -16.2 rad/fs to 16.2 rad/fs, resulting in a temporal window of 12.9 ps. The initial pump field is a soliton with central frequency $\lambda_1 = 800$ nm, FWHM$_1 = 8$ fs, and average power $P_1 = 45$ mW (with repetition rate of 80 MHz). The initial probe field is another soliton with $\lambda_2 = 1400$ nm, FWHM$_2 = 30$ fs, and $P_2 = 4$ mW. We then vary several of these parameters to obtain different experimental conditions.

Our implementation follows two approaches: firstly, using Wolfram Mathematica for our code; and secondly, employing the publicly accessible pulse-propagation packages developed in Python by Melchert and Demircan [13], to validate our results. We adapted this second code to incorporate negative frequencies, following the method outlined in Ref. [9], and confirmed consistency with our own results, including the error assessment. The highest relative error achieved in any solution is approximately 10^{-5}.

We can also separate the pump and probe without any additional approximation and solve them as a system of coupled equations. This simplifies the identification of the NRR, NHR, and backreaction signals that are spectrally very close to one another. In Supplement Figure 1, we present the results of varying the probe wavelength, pump power, and pump chirp. The results match our experimental measurements in terms of the spectral features. The signal wavelengths differ the most for probe wavelengths closer to the horizon in (a) and for increasing pump power in (b). Moreover, the maximum efficiency of all signals is achieved for zero chirp in (c). However, our results do not match in terms of intensity, as our numerics predict much weaker signals than those measured in the laboratory. These signals are exponentially dependent on the precise pulse shapes and so minute imperfections in modelling may have significant consequences. Our model excludes the Raman effect and the optical shock term [1] that are difficult to implement correctly for near–single–cycle pulses. We hope that future work will resolve this issue.

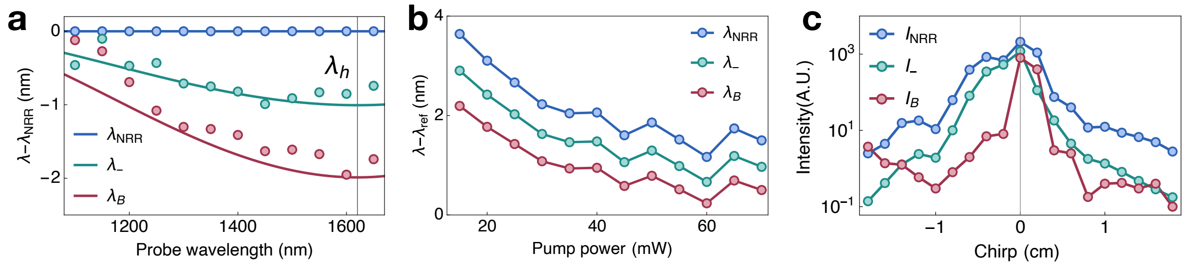

Figure 1: Simulations. Wavelength of the three signals with probe wavelength λ_2 **a** and pump power P_1 **b**. Counts of the three signals varying the pump chirp **c**.

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