
Supplementary information

Observation of critical topological phase transition

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Supplementary Information for “Observation of critical topological phase transition”

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This Supplementary Information comprises 8 Supplementary Notes, 19 Supplementary Figures, and 10 Supplementary Tables. [Supplementary Note 1](#) presents theoretical discussions of our model, including the calculation of topological invariants, topological edge states, bulk-edge correspondence, and entanglement entropy. In [Supplementary Note 2](#), we describe the design process of the acoustic waveguides used to realize samples discussed in the main text. In [Supplementary Note 3](#), we provide the eigenstates corresponding to these samples. In [Supplementary Note 4](#), we detail the procedure for retrieving Floquet operators from both simulation and experimental data. In [Supplementary Note 5](#), we present the phononic band structures of the designed acoustic lattice and show that conventional phononic bands are insufficient to characterize the topological properties of the Floquet system. In [Supplementary Note 6](#), we construct the interference intensity, which serves as a reliable indicator for the presence of 0 and π edge states. In [Supplementary Note 7](#), we introduce structure defects and verify the robustness of the topological edge states through both numerical simulations and experimental measurements. Finally, in [Supplementary Note 8](#), we detail the reasons why the entanglement coefficient obtained from simulations differs from its tight-binding counterpart.

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Supplementary Note 1. THEORY

In this section, we provide the theoretical description of the one-dimensional (1D) Floquet lattice model that we realized in experiments. We first define our model and characterize its bulk properties, including the spectrum, phase boundaries, symmetries, topological invariants, and different topological phases under the periodic boundary condition (PBC). Next, we take the open boundary condition (OBC) for our model and obtain exact solutions of its Floquet edge states at 0 and π quasienergies. The principle of bulk-edge correspondence for our model is further established in both its gapped phases and gapless phase boundaries. Finally, we analyze the scaling law of entanglement entropy for our system.

A. Model, spectrum and phase boundaries

In momentum space, the Bloch Hamiltonian of our periodically driven system takes the form

$$H(k, t) = \begin{cases} H_2(k), & t \in [\ell T, \ell T + T_1), \\ H_1(k), & t \in [\ell T + T_1, \ell T + T_1 + T_2), \end{cases} \quad (\text{S1-1})$$

where t denotes time and $\ell \in \mathbb{Z}$. $k \in [-\pi, \pi)$ denotes quasimomentum. The driving period $T \equiv T_1 + T_2$, and

$$H_\alpha(k) = J_\alpha h_\alpha(k) = J_\alpha [\cos(\alpha k) \sigma_x + \sin(\alpha k) \sigma_y], \quad \alpha = 1, 2, \quad (\text{S1-2})$$

where σ_x and σ_y are Pauli matrices in their usual representations. The Floquet operator is defined as the evolution operator of the system over a complete driving period, i.e., $F(k) = \hat{T} e^{-\frac{i}{\hbar} \int_t^{t+T} H(k, t') dt'}$, where $\hat{T}$ performs the time ordering. Consider the evolution period from $t = 0$ to T , we find

$$F(k) = e^{-\frac{i}{\hbar} T_2 H_1(k)} e^{-\frac{i}{\hbar} T_1 H_2(k)}. \quad (\text{S1-3})$$

To simplify the notations, we introduce dimensionless parameters $t_1 \equiv J_1 T_2 / \hbar$ and $t_2 \equiv J_2 T_1 / \hbar$. The Floquet operator can then be rewritten as

$$F(k) = e^{-it_1 h_1(k)} e^{-it_2 h_2(k)}. \quad (\text{S1-4})$$

The Floquet spectrum of $F(k)$ can be obtained by solving the eigenvalue equation $F(k) |\psi_\pm(k)\rangle = e^{-i\varepsilon_\pm(k)} |\psi_\pm(k)\rangle$, yielding the two Floquet bands

$$\varepsilon_\pm(k) = \pm \varepsilon(k) = \pm \arccos(\cos t_1 \cos t_2 - \sin t_1 \sin t_2 \cos k). \quad (\text{S1-5})$$

The two bands can meet with each other at the quasienergies 0, π , or both, yielding gapless critical points for the Floquet system under the conditions

$$\cos t_1 \cos t_2 - \sin t_1 \sin t_2 \cos k = \pm 1. \quad (\text{S1-6})$$

As $\cos k \in [-1, 1]$, the left-hand-side takes the extreme value when $\cos k = 1$ (at $k = 0$) and $\cos k = -1$ (at $k = \pm\pi$), yielding critical lines in the t_1 - t_2 parameter space as

$$t_1 \pm t_2 = \nu\pi, \quad \nu \in \mathbb{Z}. \quad (\text{S1-7})$$

Here, ν takes even (odd) integers when the gap between the two Floquet bands $\varepsilon_\pm(k)$ closes at the quasienergy 0 (π).

B. Symmetries and topological invariants

To characterize the topological properties of our driven system, we need to first identify the relevant symmetries of its Floquet operator. This is achieved by transforming our $F(k)$ into symmetric time frames [1] upon unitary transformations, yielding two complementary Floquet operators

$$F_1(k) = e^{-\frac{i}{2} t_2 h_2(k)} e^{-it_1 h_1(k)} e^{-\frac{i}{2} t_2 h_2(k)}, \quad (\text{S1-8})$$

$$F_2(k) = e^{-\frac{i}{2} t_1 h_1(k)} e^{-it_2 h_2(k)} e^{-\frac{i}{2} t_1 h_1(k)}. \quad (\text{S1-9})$$

Unitary equivalence implies that $F_{1,2}(k)$ share the same Floquet spectrum $E_{\pm}(k)$ with the $F(k)$. Meanwhile, both $F_1(k)$ and $F_2(k)$ have the chiral (sublattice) symmetry $\mathcal{S} = \sigma_z$, in the sense that

$$\mathcal{S}F_s(k)\mathcal{S} = F_s^\dagger(k), \quad s = 1, 2, \quad (\text{S1-10})$$

where $\sigma_z \equiv \text{diag}(1, -1)$ is the Pauli matrix. In addition, $F_s(k)$ ($s = 1, 2$) has the inversion symmetry $\mathcal{I} = \sigma_x$ with $\mathcal{I}F_s(k)\mathcal{I}^\dagger = F_s(-k)$, the time-reversal symmetry $\mathcal{T} = \sigma_0 \equiv \text{diag}(1, 1)$ with $\mathcal{T}F_s^*(k)\mathcal{T} = F_s^\dagger(-k)$, and the particle-hole symmetry $\mathcal{C} = \sigma_z$ with $\mathcal{C}F_s^*(k)\mathcal{C}^\dagger = F_s(-k)$. Therefore, our 1D Floquet system belongs to the BDI symmetry class, whose topological phases are characterized by a pair of integer quantized topological invariants [2].

To find the topological invariants of our Floquet system, we follow the general theoretical framework developed in Refs. [3, 4]. Let us consider a more generic Floquet operator

$$F'(k) \equiv e^{-\frac{i}{2}t_\alpha h_\alpha(k)} e^{-it_\beta h_\beta(k)} e^{-\frac{i}{2}t_\alpha h_\alpha(k)}, \quad (\text{S1-11})$$

where $\alpha, \beta \in \mathbb{N}$ and $\alpha \neq \beta$. Working out the Taylor expansion of $F'(k)$, we obtain

$$F'(k) = \cos[\varepsilon'(k)] \sigma_0 - i[d_x(k)\sigma_x + d_y(k)\sigma_y], \quad (\text{S1-12})$$

where

$$\begin{aligned} d_x(k) &= \frac{i}{2} \text{Tr}[\sigma_x F'(k)] \\ &= \sin t_\alpha \cos t_\beta \cos(\alpha k) + \sin t_\beta \cos^2 \frac{t_\alpha}{2} \cos(\beta k) - \sin t_\beta \sin^2 \frac{t_\alpha}{2} \cos[(2\alpha - \beta)k], \end{aligned} \quad (\text{S1-13})$$

$$\begin{aligned} d_y(k) &= \frac{i}{2} \text{Tr}[\sigma_y F'(k)] \\ &= \sin t_\alpha \cos t_\beta \sin(\alpha k) + \sin t_\beta \cos^2 \frac{t_\alpha}{2} \sin(\beta k) - \sin t_\beta \sin^2 \frac{t_\alpha}{2} \sin[(2\alpha - \beta)k]. \end{aligned} \quad (\text{S1-14})$$

Next, we define the Floquet effective Hamiltonian $H'(k)$ of $F'(k)$ as

$$H'(k) \equiv \frac{1}{2i} [F'^\dagger(k) - F'(k)] = \begin{bmatrix} 0 & f'^*(k) \\ f'(k) & 0 \end{bmatrix}, \quad (\text{S1-15})$$

where

$$\begin{aligned} f'(k) &\equiv d_x(k) + id_y(k) \\ &= \sin t_\alpha \cos t_\beta e^{ik\alpha} + \sin t_\beta \cos^2 \frac{t_\alpha}{2} e^{ik\beta} - \sin t_\beta \sin^2 \frac{t_\alpha}{2} e^{ik(2\alpha - \beta)}. \end{aligned} \quad (\text{S1-16})$$

Applying analytic continuation $e^{ik} \rightarrow z$, we arrive at the complex continued function

$$f'(z) = \sin t_\alpha \cos t_\beta z^\alpha + \sin t_\beta \cos^2 \frac{t_\alpha}{2} z^\beta - \sin t_\beta \sin^2 \frac{t_\alpha}{2} z^{2\alpha - \beta}, \quad (\text{S1-17})$$

where $z \in \mathbb{C}$. For our symmetric frame Floquet operators $F_{1,2}(k)$, these complex continued functions are respectively given by

$$f_1(z) = \sin t_2 \cos t_1 z^2 + \sin t_1 \cos^2 \frac{t_2}{2} z - \sin t_1 \sin^2 \frac{t_2}{2} z^3, \quad (\text{S1-18})$$

$$f_2(z) = \sin t_1 \cos t_2 z + \sin t_2 \cos^2 \frac{t_1}{2} z^2 - \sin t_2 \sin^2 \frac{t_1}{2}. \quad (\text{S1-19})$$

Following the Cauchy's argument principle, $f_1(z)$ and $f_2(z)$ have the topological winding numbers $\omega_1 \equiv N_{1z} - N_{1p}$ and $\omega_2 \equiv N_{2z} - N_{2p}$, where N_{sz} and N_{sp} denote the numbers of zeros and poles of $f_s(z)$ inside the unit circle $|z| = 1$ for $s = 1, 2$. The gapped topological phases of our Floquet system can then be characterized by the topological invariants [3, 4]

$$\omega_0 \equiv \frac{\omega_1 + \omega_2}{2}, \quad \omega_\pi \equiv \frac{\omega_1 - \omega_2}{2}, \quad (\text{S1-20})$$

which are also integer quantized. The demand for two such invariants to characterize a single Floquet phase comes from the fact that the two bands of our driven system could touch at both the quasienergies 0 and π , yielding two distinct paths of topological transitions and two separate gaps that could each become topologically nontrivial.

C. Topology of gapped Floquet phases

To proceed, we reorganize the complex functions $f_1(z)$ and $f_2(z)$ in Eqs. (S1-18) and (S1-19) as

$$f_1(z) = -\sin t_1 \sin^2 \frac{t_2}{2} z \left(z + \tan \frac{t_1}{2} \cot \frac{t_2}{2} \right) \left(z - \cot \frac{t_1}{2} \cot \frac{t_2}{2} \right), \quad (\text{S1-21})$$

$$f_2(z) = \sin t_2 \cos^2 \frac{t_1}{2} \left(z + \tan \frac{t_1}{2} \cot \frac{t_2}{2} \right) \left(z - \tan \frac{t_1}{2} \tan \frac{t_2}{2} \right). \quad (\text{S1-22})$$

Their zeros and poles are found by solving $f_1(z) = 0$ and $f_2(z) = 0$. Away from the critical lines (i.e., $t_1 \pm t_2 \neq \nu\pi$), there are four possible Floquet topological phases, whose topological invariants (ω_0, ω_π) , number of edge states and different phase regions in the t_1 - t_2 parameter space are summarized in [Supplementary Table 1](#). The number of edge states is solved explicitly in [Supplementary Note 1 E](#).

Supplementary Table 1 | Gapped Floquet phases, their topological invariants (ω_0, ω_π) and numbers of 0 and π edge states (N_0, N_π) . The phase indices II, IV, VI, and VIII are consistent with those indexed in the phase diagram Fig. 2a of the main text.

Phase	Conditions	Topological invariants	Edge states
II	$ \tan(t_1/2)/\tan(t_2/2) < 1$ and $ \tan(t_1/2)\tan(t_2/2) > 1$	$(\omega_1, \omega_2) = (3, 1)$, $(\omega_0, \omega_\pi) = (2, 1)$	$(N_0, N_\pi) = (4, 2)$
IV	$ \tan(t_1/2)/\tan(t_2/2) < 1$ and $ \tan(t_1/2)\tan(t_2/2) < 1$	$(\omega_1, \omega_2) = (2, 2)$, $(\omega_0, \omega_\pi) = (2, 0)$	$(N_0, N_\pi) = (4, 0)$
VI	$ \tan(t_1/2)/\tan(t_2/2) > 1$ and $ \tan(t_1/2)\tan(t_2/2) > 1$	$(\omega_1, \omega_2) = (2, 0)$, $(\omega_0, \omega_\pi) = (1, 1)$	$(N_0, N_\pi) = (2, 2)$
VIII	$ \tan(t_1/2)/\tan(t_2/2) > 1$ and $ \tan(t_1/2)\tan(t_2/2) < 1$	$(\omega_1, \omega_2) = (1, 1)$, $(\omega_0, \omega_\pi) = (1, 0)$	$(N_0, N_\pi) = (2, 0)$

To include all these phases into a single phase diagram, it is enough to consider the two-dimensional parameter domain $(t_1, t_2) \in (0, \pi) \times (0, \pi)$. There are two lines of phase boundaries in this domain, i.e., $t_2 = t_1$ with the gap closing at quasienergy $\varepsilon = 0$, and $t_2 = \pi - t_1$ with the gap closing at quasienergy $\varepsilon = \pi$. They are crossed at a multicritical point $t_1 = t_2 = \pi/2$, dividing the domain $(t_1, t_2) \in (0, \pi) \times (0, \pi)$ into four distinct regions (see Fig. 2a in the main text). Referring to [Supplementary Table 1](#), we summarize the gapped phases, their topological invariants, and the number of edge states within this parameter domain in [Supplementary Table 2](#), which are in perfect consistency with the gapped phases shown in Fig. 2a in the main text. The number of edge states is solved explicitly in [Supplementary Note 1 E](#).

Supplementary Table 2 | Gapped Floquet phases, their topological invariants (ω_0, ω_π) and numbers of 0 and π edge states (N_0, N_π) in the parameter domain $(t_1, t_2) \in (0, \pi) \times (0, \pi)$. The phase indices II, IV, VI, and VIII are those indexed in the phase diagram Fig. 2a of the main text.

Phase	Conditions	Topological invariants	Edge states
II	$t_2 > \pi - t_1$, $t_1 \in (0, \pi/2)$ and $t_2 > t_1$, $t_1 \in (\pi/2, \pi)$	$(\omega_1, \omega_2) = (3, 1)$, $(\omega_0, \omega_\pi) = (2, 1)$	$(N_0, N_\pi) = (4, 2)$
IV	$t_1 < t_2 < \pi - t_1$ and $t_1 \in (0, \pi/2)$	$(\omega_1, \omega_2) = (2, 2)$, $(\omega_0, \omega_\pi) = (2, 0)$	$(N_0, N_\pi) = (4, 0)$
VI	$\pi - t_1 < t_2 < t_1$ and $t_1 \in (\pi/2, \pi)$	$(\omega_1, \omega_2) = (2, 0)$, $(\omega_0, \omega_\pi) = (1, 1)$	$(N_0, N_\pi) = (2, 2)$
VIII	$t_2 < t_1$, $t_1 \in (0, \pi/2)$ and $t_2 < \pi - t_1$, $t_1 \in (\pi/2, \pi)$	$(\omega_1, \omega_2) = (1, 1)$, $(\omega_0, \omega_\pi) = (1, 0)$	$(N_0, N_\pi) = (2, 0)$

D. Topology of gapless Floquet phase boundaries

Next, we consider the topological characterization of critical points, which correspond to phase boundaries I, III, V, VII, and IX in Fig. 2a of the main text. Along the critical line $t_2 = t_1$, the bulk Floquet spectrum is gapless at quasienergy 0. Referring to [Supplementary Table 2](#) and the phase diagram Fig. 2a in the main text, we notice that the gapped phases VI and VIII are smoothly connected to the gapless critical segments III and VII. Therefore, we conclude that the phase boundary $t_2 = t_1$ with

$t_1 \in (0, \pi/2)$ [$t_1 \in (\pi/2, \pi)$] is characterized by the topological invariants $(\omega_0, \omega_\pi) = (1, 0)$ [$(\omega_0, \omega_\pi) = (1, 1)$]. These two segments of phase boundaries are thus topologically distinct and separated by a critical topological phase transition (CTPT) at $t_1 = t_2 = \pi/2$ (multicritical point V). It is accompanied by the closing and reopening of the bulk Floquet gap at quasienergy π and the changing of ω_π between zero and one. Meanwhile, the spectrum remains gapless at quasienergy 0, and the ω_0 retains its value $\omega_0 = 1$ during this transition.

The phase boundary $t_2 = \pi - t_1$ consisting of the critical segments I, V, and IX in Fig. 2a of the main text should be treated more carefully. If we insert $t_2 = \pi - t_1$ into the expressions of $f_1(z)$ and $f_2(z)$, we would obtain

$$f_1(z) = -\sin t_1 \cos^2 \frac{t_1}{2} z \left(z + \tan^2 \frac{t_1}{2} \right) (z - 1), \quad (\text{S1-23})$$

$$f_2(z) = \sin t_1 \cos^2 \frac{t_1}{2} \left(z + \tan^2 \frac{t_1}{2} \right) (z - 1). \quad (\text{S1-24})$$

If we directly perform the zero-pole counting of these functions, we would obtain $(\omega_1, \omega_2) = (1, 0)$ or $(2, 1)$, generating half-integer quantized (ω_0, ω_π) in both cases. To fix this issue, we refine the rule of zero-pole counting for cases with gapless Floquet spectrum at quasienergy π , i.e., we define $\omega_s \equiv N_{sz} - \frac{N_{sp}}{2}$ ($s = 1, 2$) along the phase boundaries $t_1 \pm t_2 = (2\nu + 1)\pi$ for all $\nu \in \mathbb{Z}$ (see Refs. [3, 4] for more theoretical details). Following this rule, we start with the complex continued functions

$$f_1(z) = -\sin t_1 \cos^2 \frac{t_1}{2} z \left(z + \tan^2 \frac{t_1}{2} \right) (z - 1), \quad (\text{S1-25})$$

$$f_2(z) = \sin t_1 \cos^2 \frac{t_1}{2} \frac{\left(z + \tan^2 \frac{t_1}{2} \right) (z - 1) z^2}{z^2}. \quad (\text{S1-26})$$

It is then straightforward to see that along $t_2 = \pi - t_1$ with $0 < t_1 < \pi/2$, we have $(\omega_1, \omega_2) = (2, 2)$ and $(\omega_0, \omega_\pi) = (2, 0)$. Along $t_2 = \pi - t_1$ with $\pi/2 \leq t_1 < \pi$, we instead have $(\omega_1, \omega_2) = (1, 1)$ and $(\omega_0, \omega_\pi) = (1, 0)$. These two critical segments are thus also topologically distinct, separated by a CTPT at $t_1 = t_2 = \pi/2$ (multicritical point V). It is accompanied by the closing and reopening of the bulk Floquet gap at quasienergy 0 and the changing of ω_0 between two and one. Meanwhile, the spectrum remains gapless at quasienergy π and the ω_π retains its value $\omega_\pi = 0$ during this transition. To sum up, we present the phase boundaries of our system, their topological characterizations in terms of (ω_0, ω_π) , and the number of edge states in [Supplementary Table 3](#). The number of edge states is solved explicitly in [Supplementary Note 1 E](#).

Supplementary Table 3 | Gapless Floquet phase boundaries, their topological invariants (ω_0, ω_π) and numbers of 0 and π edge states (N_0, N_π) in the parameter domain $(t_1, t_2) \in (0, \pi) \times (0, \pi)$. The phase boundary indices I, III, V, VII, and IX correspond to those indexed in the phase diagram Fig. 2a of the main text.

Phase boundary	Condition	Topological invariants	Edge states
I	$t_2 = \pi - t_1, t_1 \in (0, \pi/2)$	$(\omega_0, \omega_\pi) = (2, 0)$	$(N_0, N_\pi) = (4, 0)$
III	$t_2 = t_1, t_1 \in (\pi/2, \pi)$	$(\omega_0, \omega_\pi) = (1, 1)$	$(N_0, N_\pi) = (2, 2)$
V	$t_1 = t_2 = \pi/2$	$(\omega_0, \omega_\pi) = (1, 0)$	$(N_0, N_\pi) = (2, 0)$
VII	$t_2 = t_1, t_1 \in (0, \pi/2)$	$(\omega_0, \omega_\pi) = (1, 0)$	$(N_0, N_\pi) = (2, 0)$
IX	$t_2 = \pi - t_1, t_1 \in (\pi/2, \pi)$	$(\omega_0, \omega_\pi) = (1, 0)$	$(N_0, N_\pi) = (2, 0)$

Supplementary Table 4 | Topological invariants (ω_0, ω_π) and numbers of 0 and π edge states (N_0, N_π) of all gapped Floquet phases and their intervening gapless phase boundaries in the parameter domain $(t_1, t_2) \in (0, \pi) \times (0, \pi)$. The indices I–XI correspond to the same indices shown in the phase diagram Fig. 2a of the main text. Indices of gapped phases are highlighted in blue.

Phases and phase boundaries	Conditions	Topological invariants	Edge states
II	$t_2 > \pi - t_1, t_1 \in (0, \pi/2)$ and $t_2 > t_1, t_1 \in (\pi/2, \pi)$	$(\omega_0, \omega_\pi) = (2, 1)$	$(N_0, N_\pi) = (4, 2)$
IV + I	$t_1 < t_2 \leq \pi - t_1$ and $t_1 \in (0, \pi/2)$	$(\omega_0, \omega_\pi) = (2, 0)$	$(N_0, N_\pi) = (4, 0)$
VI + III	$\pi - t_1 < t_2 \leq t_1$ and $t_1 \in (\pi/2, \pi)$	$(\omega_0, \omega_\pi) = (1, 1)$	$(N_0, N_\pi) = (2, 2)$
VIII + V + VII + IX	$t_2 \leq t_1, t_1 \in (0, \pi/2]$ and $t_2 \leq \pi - t_1, t_1 \in (\pi/2, \pi)$	$(\omega_0, \omega_\pi) = (1, 0)$	$(N_0, N_\pi) = (2, 0)$

Up to now, we have completed the characterization of all gapped phases and gapless phase boundaries of our Floquet model via the bulk topological invariants (ω_0, ω_π). For convenience of reference, we collect all these informations in a single [Supplementary Table 4](#), which is graphically demonstrated by the phase diagram Fig. 2a in the main text. The number of edge states in [Supplementary Table 4](#) and their relationship with bulk topological invariants are analyzed in the next two subsections.

We emphasize that the critical topology discussed in CTPT explicitly excludes topological semimetals, which are often referred to as gapless topological phases in the literature [5]. The fundamental distinctions between critical topologies and topological semimetals manifest in two key aspects [6]. First, the Fermi arcs—regarded as boundary states of topological semimetals—exist only on *gapped* momentum slices of the bulk Brillouin zone, whereas the topological edge states associated with critical topologies are located precisely at the *gapless* phase boundaries. Second, the stability of topological semimetals relies crucially on translational symmetry and is therefore vulnerable to strong disorder. By contrast, the topological edge states emerging at critical topologies remain robust even in the presence of symmetry-preserving strong disorder [3].

E. Topological edge states

The Floquet 0 and π topological edge states of our system can be found analytically by considering half-infinite chains with the OBC taken at one end.

To obtain the wave functions of edge states, we first transform our Hamiltonian $H(k, t)$ from the momentum to lattice representations, yielding

$$H_\alpha = J_\alpha \sum_n (b_n^\dagger a_{n+\alpha} + \text{H.c.}), \quad (\text{S1-27})$$

where $\alpha = 2$ and $\alpha = 1$ in the time spans $t \in [\ell T, \ell T + T_1)$ and $t \in [\ell T + T_1, \ell T + T_1 + T_2)$ of each driving period, respectively. $a_n^\dagger$ (a_n) and $b_n^\dagger$ (b_n) denote the creation (annihilation) operator of a spinless fermion on the sublattices A and B of the corresponding Su-Schrieffer-Heeger (SSH) α -chain, respectively. In the position representation, the Floquet operator of our system takes the form

$$F = e^{-it_1 h_1} e^{-it_2 h_2} = e^{-it_1 \sum_{n=1}^{\infty} (b_n^\dagger a_{n+1} + \text{H.c.})} e^{-it_2 \sum_{n=1}^{\infty} (b_n^\dagger a_{n+2} + \text{H.c.})}, \quad (\text{S1-28})$$

where the dimensionless parameters $t_1 \equiv J_1 T_2 / \hbar$ and $t_2 \equiv J_2 T_1 / \hbar$. We have also taken the OBC at the left end of the lattice, leading to the summation of the unit cell index n from 1 to $+\infty$.

We may now view t_1 and t_2 as two time variables and consider the dynamics of $a_n^\dagger$ and $b_n^\dagger$ following the Schrödinger equation of h_α ($\alpha = 1, 2$), i.e., $a_n^\dagger(t) = e^{ith_\alpha} a_n^\dagger e^{-ith_\alpha}$ and $b_n^\dagger(t) = e^{ith_\alpha} b_n^\dagger e^{-ith_\alpha}$. Solving the Heisenberg equations of motion for $a_n^\dagger(t)$ and $b_n^\dagger(t)$ yield

$$\frac{d}{dt} a_n^\dagger(t) = -i [a_n^\dagger, h_\alpha](t) = -i \sum_{n'} [a_n^\dagger, b_{n'}^\dagger a_{n'+\alpha} + \text{H.c.}](t) = \begin{cases} 0, & n = 1, \dots, \alpha, \\ ib_{n-\alpha}^\dagger(t), & n > \alpha \end{cases}, \quad (\text{S1-29})$$

$$\frac{d}{dt} b_n^\dagger(t) = -i [b_n^\dagger, h_\alpha](t) = -i \sum_{n'} [b_n^\dagger, b_{n'}^\dagger a_{n'+\alpha} + \text{H.c.}](t) = ia_{n+\alpha}^\dagger(t), \quad n = 1, \dots, \infty. \quad (\text{S1-30})$$

Taking the second-order derivatives, we further obtain

$$\frac{d^2}{dt^2} a_n^\dagger(t) = \begin{cases} 0, & n = 1, \dots, \alpha, \\ -a_n^\dagger(t), & n > \alpha \end{cases}, \quad (\text{S1-31})$$

$$\frac{d^2}{dt^2} b_n^\dagger(t) = -b_n^\dagger(t), \quad n = 1, \dots, \infty. \quad (\text{S1-32})$$

These equations are both in harmonic forms, yielding the general solutions

$$a_n^\dagger(t) = \begin{cases} a_n^\dagger, & n = 1, \dots, \alpha, \\ c_1 \cos t + c_2 \sin t, & n > \alpha \end{cases}, \quad (\text{S1-33})$$

$$b_n^\dagger(t) = c_3 \cos t + c_4 \sin t, \quad n = 1, \dots, \infty. \quad (\text{S1-34})$$

Taking into account the initial conditions, the coefficients c_1 – c_4 are found to be

$$c_1 = a_n^\dagger, \quad c_2 = ib_{n-\alpha}^\dagger, \quad c_3 = b_n^\dagger, \quad c_4 = ia_{n+\alpha}^\dagger. \quad (\text{S1-35})$$

Therefore, the evolution of $a_n^\dagger(t)$ and $b_n^\dagger(t)$ are governed by

$$a_n^\dagger(t) = e^{ith_\alpha} a_n^\dagger e^{-ith_\alpha} = \begin{cases} a_n^\dagger, & n = 1, \dots, \alpha, \\ a_n^\dagger \cos t + ib_{n-\alpha}^\dagger \sin t, & n > \alpha \end{cases}, \quad (\text{S1-36})$$

$$b_n^\dagger(t) = e^{ith_\alpha} b_n^\dagger e^{-ith_\alpha} = b_n^\dagger \cos t + ia_{n+\alpha}^\dagger \sin t, \quad n = 1, \dots, \infty. \quad (\text{S1-37})$$

Due to the chiral symmetry, degenerate edge states can only appear at the quasienergies 0 and π of our Floquet lattice. The creation operators $\psi_0^\dagger$ and $\psi_\pi^\dagger$ of 0 and π eigenstates then must satisfy

$$F^\dagger \psi_0^\dagger F = \psi_0^\dagger, \quad F^\dagger \psi_\pi^\dagger F = -\psi_\pi^\dagger. \quad (\text{S1-38})$$

On the left end of the Floquet chain, we find that

$$F^\dagger a_1^\dagger F = e^{it_2 h_2} e^{it_1 h_1} a_1^\dagger e^{-it_1 h_1} e^{-it_2 h_2} = a_1^\dagger. \quad (\text{S1-39})$$

Therefore, the isolated 0 edge state $|\psi_L^{(0,1)}\rangle \equiv a_1^\dagger |\emptyset\rangle$ is a left edge state of the Floquet chain with quasienergy 0, where $|\emptyset\rangle$ denotes the vacuum state. Similarly, if we consider a half-infinite chain with the OBC taken at its right end of cell index N , we will find another edge 0 state $|\psi_R^{(0,1)}\rangle \equiv b_N^\dagger |\emptyset\rangle$ at the right end of the chain. The two eigenstates $|\psi_{L,R}^{(0,1)}\rangle$ form a pair of degenerate Floquet edge states at quasienergy 0. They persist throughout the t_1 - t_2 parameter space, making all the gapped and gapless Floquet phases of our system topologically nontrivial.

Despite the two 0 states $|\psi_{L,R}^{(0,1)}\rangle$, there could be other Floquet edge states at 0 and π quasienergies, whose presence or absence depends on the system parameters. To find them, we express their ansatz creation operators as

$$\psi_0^\dagger = \sum_{n=1}^{\infty} (A_n a_n^\dagger + B_n b_n^\dagger), \quad \psi_\pi^\dagger = \sum_{n=1}^{\infty} (C_n a_n^\dagger + D_n b_n^\dagger). \quad (\text{S1-40})$$

As the 0 states satisfy $\psi_0^\dagger = F^\dagger \psi_0^\dagger F$, we obtain

$$\sum_{n=1}^{\infty} (A_n a_n^\dagger + B_n b_n^\dagger) = \sum_{n=1}^{\infty} (A_n F^\dagger a_n^\dagger F + B_n F^\dagger b_n^\dagger F). \quad (\text{S1-41})$$

Direct calculations yield

$$\begin{aligned} F^\dagger A_n a_n^\dagger F &= e^{it_2 h_2} e^{it_1 h_1} A_n a_n^\dagger e^{-it_1 h_1} e^{-it_2 h_2} \\ &= \begin{cases} A_1 a_1^\dagger, & n = 1 \\ i \sin t_1 \cos t_2 A_2 b_1^\dagger + \cos t_1 A_2 a_2^\dagger - \sin t_1 \sin t_2 A_2 a_3^\dagger, & n = 2, \\ i \cos t_1 \sin t_2 A_n b_{n-2}^\dagger + i \sin t_1 \cos t_2 A_n b_{n-1}^\dagger + \cos t_1 \cos t_2 A_n a_n^\dagger - \sin t_1 \sin t_2 A_n a_{n+1}^\dagger, & n > 2 \end{cases} \end{aligned} \quad (\text{S1-42})$$

$$\begin{aligned} F^\dagger B_n b_n^\dagger F &= e^{it_2 h_2} e^{it_1 h_1} B_n b_n^\dagger e^{-it_1 h_1} e^{-it_2 h_2} \\ &= \begin{cases} \cos t_1 \cos t_2 B_1 b_1^\dagger + i \sin t_1 B_1 a_2^\dagger + i \cos t_1 \sin t_2 B_1 a_3^\dagger, & n = 1 \\ -\sin t_1 \sin t_2 B_n b_{n-1}^\dagger + \cos t_1 \cos t_2 B_n b_n^\dagger + i \sin t_1 \cos t_2 B_n a_{n+1}^\dagger + i \cos t_1 \sin t_2 B_n a_{n+2}^\dagger, & n > 1 \end{cases} \end{aligned} \quad (\text{S1-43})$$

Letting the coefficients in front of $a_n^\dagger$ and $b_n^\dagger$ be equal on both sides, we obtain up to a normalization constant that

$$\psi_0^\dagger = \sum_{n=1}^{\infty} \left(-\tan \frac{t_1}{2} \cot \frac{t_2}{2} \right)^{n-1} \left(\cos \frac{t_1}{2} a_{n+1}^\dagger - i \sin \frac{t_1}{2} b_n^\dagger \right). \quad (\text{S1-44})$$

If we consider a half-infinite chain with the OBC taken at its right end of cell index N , we will find another 0 state $|\psi_R^{(0,2)}\rangle$. These two eigenstates form a pair of degenerate Floquet states at quasienergy 0, which are edge states if and only if

$$\left| \tan \frac{t_1}{2} \right| < \left| \tan \frac{t_2}{2} \right|. \quad (\text{S1-45})$$

As the π states satisfy $\psi_\pi^\dagger = -F^\dagger \psi_\pi^\dagger F$, we obtain

$$\sum_{n=1}^{\infty} (C_n a_n^\dagger + D_n b_n^\dagger) = - \sum_{n=1}^{\infty} (C_n F^\dagger a_n^\dagger F + D_n F^\dagger b_n^\dagger F). \quad (\text{S1-46})$$

In parallel with the calculation of 0 states, we find the solution

$$\psi_\pi^\dagger = \sum_{n=1}^{\infty} \left(\cot \frac{t_1}{2} \cot \frac{t_2}{2} \right)^{n-1} \left(\sin \frac{t_1}{2} a_{n+1}^\dagger + i \cos \frac{t_1}{2} b_n^\dagger \right). \quad (\text{S1-47})$$

If we consider a half-infinite chain with the OBC taken at its right end of cell index N , we will find another π state $|\psi_R^{(\pi)}\rangle$. These two eigenstates form a pair of degenerate Floquet states at quasienergy π , which are edge states if and only if

$$\left| \tan \frac{t_1}{2} \tan \frac{t_2}{2} \right| > 1. \quad (\text{S1-48})$$

To sum up, we find the following exact solutions of Floquet 0 eigenstates ($|\psi_L^{(0,1)}\rangle$ and $|\psi_L^{(0,2)}\rangle$) and π eigenstates ($|\psi_L^{(\pi)}\rangle$) for a half-infinite chain with the OBC taken at its left end ($n = 1$), i.e.,

$$|\psi_L^{(0,1)}\rangle = a_1^\dagger |\emptyset\rangle, \quad (\text{S1-49})$$

$$|\psi_L^{(0,2)}\rangle = \sum_{n=1}^{\infty} \left(-\tan \frac{t_1}{2} \cot \frac{t_2}{2} \right)^{n-1} \left(\cos \frac{t_1}{2} a_{n+1}^\dagger - i \sin \frac{t_1}{2} b_n^\dagger \right) |\emptyset\rangle, \quad (\text{S1-50})$$

$$|\psi_L^{(\pi)}\rangle = \sum_{n=1}^{\infty} \left(\tan \frac{t_1}{2} \tan \frac{t_2}{2} \right)^{1-n} \left(\sin \frac{t_1}{2} a_{n+1}^\dagger + i \cos \frac{t_1}{2} b_n^\dagger \right) |\emptyset\rangle. \quad (\text{S1-51})$$

In the parameter domain $(t_1, t_2) \in (0, \pi) \times (0, \pi)$, we notice that the degenerate 0 states $|\psi_{L,R}^{(0,2)}\rangle$ are edge localized if and only if $t_2 > t_1$. Therefore, they exist in the gapped phases II, IV, and survive along the critical line I in the phase diagram Fig. 2a of the main text. The degenerate π states $|\psi_{L,R}^{(0,2)}\rangle$ are edge localized if and only if $t_2 > \pi - t_1$. Therefore, they exist in the gapped phases II, VI, and survive along the critical line III in the phase diagram Fig. 2a of the main text. The 0 states $|\psi_{L,R}^{(0,1)}\rangle$ persist throughout the parameter space of the system, including all its gapped phases and gapless critical lines in the phase diagram Fig. 2a of the main text. The numbers (N_0, N_π) of these 0 and π Floquet edge states, counted by our exact solutions above, are summarized in the last columns of [Supplementary Tables 1–5](#).

F. Bulk-edge correspondence

Finally, we arrive at the principle of bulk-edge correspondence between the topological invariants (ω_0, ω_π) and the numbers of Floquet 0 and π edge states (N_0, N_π) , i.e.,

$$N_0 = 2|\omega_0|, \quad N_\pi = 2|\omega_\pi|. \quad (\text{S1-52})$$

Referring to the last two columns of the [Supplementary Tables 1–4](#), we find that this relation is in full agreement with the exact results of our Floquet model. Note in passing that, although we have only demonstrated this principle for our experimentally realized Floquet system, it holds generally for any 1D, two-band Floquet lattice model with chiral symmetry, regardless of whether the Floquet spectrum is gapped or gapless [3]. In this paper, we focus on demonstrating the applicability of this principle at the topologically nontrivial, gapless phase boundaries between different gapped Floquet phases. For convenience of reference, we summarize the theoretical results regarding the phase diagram, topological invariants, and bulk-edge correspondence of our Floquet model in [Supplementary Table 5](#), with the conditions listed applicable to the whole parameter space $(t_1, t_2) \in \mathbb{R} \times \mathbb{R}$.

Supplementary Table 5 | Summary of the theoretical results. Indices of gapped phases are highlighted in blue.

Phases and phase boundaries	Conditions	Topological invariants	Edge states $(N_0, N_\pi) = 2(\omega_0 , \omega_\pi)$
II	$ \tan(t_1/2)/\tan(t_2/2) < 1$, $ \tan(t_1/2)\tan(t_2/2) > 1$	$(\omega_0, \omega_\pi) = (2, 1)$	$(N_0, N_\pi) = (4, 2)$
IV + I	$ \tan(t_1/2)/\tan(t_2/2) < 1$, $ \tan(t_1/2)\tan(t_2/2) \leq 1$	$(\omega_0, \omega_\pi) = (2, 0)$	$(N_0, N_\pi) = (4, 0)$
VI + III	$ \tan(t_1/2)/\tan(t_2/2) \geq 1$, $ \tan(t_1/2)\tan(t_2/2) > 1$	$(\omega_0, \omega_\pi) = (1, 1)$	$(N_0, N_\pi) = (2, 2)$
VIII + V + VII + IX	$ \tan(t_1/2)/\tan(t_2/2) \geq 1$, $ \tan(t_1/2)\tan(t_2/2) \leq 1$	$(\omega_0, \omega_\pi) = (1, 0)$	$(N_0, N_\pi) = (2, 0)$

G. Entanglement entropy

For a periodically driven system made up of free fermions, the von Neumann entanglement entropy (EE) can be obtained from the spectrum of the single-particle correlator among Floquet states. The general theory has been developed in Refs. [7–9]. In this subsection, we sketch the relevant formulas and discuss the scaling laws of EE vs the system size.

Let us consider a 1D Floquet chain of free fermions, whose Hamiltonian takes the quadratic form $H(t) = \sum_{\mu\nu} c_\mu^\dagger H_{\mu\nu}(t) c_\nu$. Here, $H(t) = H(t+T)$ and $H_{\mu\nu}(t) = H_{\mu\nu}(t+T)$ are time-periodic with the driving period T . The indices μ and ν include both spatial (lattice, sublattice) and internal (spin) degrees of freedom. They satisfy the anticommutation relations $\{c_\mu, c_\nu\} = 0$, $\{c_\mu^\dagger, c_\nu^\dagger\} = 0$ and $\{c_\mu, c_\nu^\dagger\} = \delta_{\mu\nu}$. For the SSH chain, we have $c_\nu = a_n$ for $\nu = (n, A)$ and $c_\nu = b_n$ for $\nu = (n, B)$, where n and A, B are unit cell and sublattice indices, respectively. The eigenstates $\{|\nu\rangle \equiv c_\nu^\dagger |\emptyset\rangle\}$ form an orthonormal and complete basis of the system, where $|\emptyset\rangle$ denotes the vacuum state.

The Floquet operator of the systems, which describes its evolution over a driving period T (e.g., from $t = 0 \rightarrow T$), reads $F = \hat{T} e^{-\frac{i}{\hbar} \int_0^T H(t) dt}$, where $\hat{T}$ performs time-ordering. Formally, F admits the spectral decomposition $F = e^{-i \sum_j \varepsilon_j \psi_j^\dagger \psi_j}$, where ε_j is the quasienergy and $\psi_j^\dagger$ is the creation operator of eigenstate $|\psi_j\rangle$ from the vacuum $|\emptyset\rangle$, i.e., $|\psi_j\rangle = \psi_j^\dagger |\emptyset\rangle$ and $F |\psi_j\rangle = e^{-i\varepsilon_j} |\psi_j\rangle$. The Floquet eigenstates $\{|\psi_j\rangle | j = 1, \dots, L\}$ form an orthonormal and complete basis of the system, where L denotes the dimension of the single-particle Hilbert space. For example, the SSH chain has $L = 2N$ with N being the total number of unit cells.

The two sets of eigenbasis $\{|\nu\rangle\}$ and $\{|\psi_j\rangle\}$ are related by unitary transformations. Using the orthogonality and completeness relations, we find

$$|\psi_j\rangle = \psi_j^\dagger |\emptyset\rangle = \sum_\nu |\nu\rangle \langle \nu | \psi_j\rangle = \sum_\nu a_{\nu j} c_\nu^\dagger |\emptyset\rangle, \quad (\text{S1-53})$$

$$|\nu\rangle = c_\nu^\dagger |\emptyset\rangle = \sum_j |\psi_j\rangle \langle \psi_j | \nu\rangle = \sum_j a_{\nu j}^* \psi_j^\dagger |\emptyset\rangle, \quad (\text{S1-54})$$

where $a_{\nu j} \equiv \langle \nu | \psi_j\rangle$. In terms of creation and annihilation operators, we equivalently have

$$\psi_j^\dagger = \sum_\nu a_{\nu j} c_\nu^\dagger, \quad \psi_j = \sum_\nu a_{\nu j}^* c_\nu, \quad c_\nu^\dagger = \sum_j a_{\nu j}^* \psi_j^\dagger, \quad c_\nu = \sum_j a_{\nu j} \psi_j. \quad (\text{S1-55})$$

One may also verify that the operators ψ_j and $\psi_j^\dagger$ satisfy standard anticommutation relations of Dirac fermions, i.e., $\{\psi_i, \psi_j\} = 0$, $\{\psi_i^\dagger, \psi_j^\dagger\} = 0$ and $\{\psi_i, \psi_j^\dagger\} = \delta_{ij}$.

Next, let us consider a system prepared in the Floquet state $|\Psi\rangle$, given by

$$|\Psi\rangle = \prod_{j \in \text{occ.}} \psi_j^\dagger |\emptyset\rangle, \quad (\text{S1-56})$$

where occ. includes the indices of all filled single-particle states within the eigenbasis $\{|\psi_j\rangle | j = 1, \dots, L\}$. The density matrix of the system then reads $\rho = |\Psi\rangle \langle \Psi|$. Taking a bipartition of the whole system into two subsystems X, Y, and tracing out all the degrees of freedom belonging to subsystem Y, we are left with the reduced density matrix ρ_X of subsystem X, which is given by

$$\rho_X \equiv \text{Tr}_Y \rho = \frac{1}{Z} e^{-H_X}, \quad Z \equiv \text{Tr} e^{-H_X}. \quad (\text{S1-57})$$

Here, H_X is usually referred to as the entanglement Hamiltonian of subsystem X, whose spectrum $\{\xi_j\}$ gives the entanglement spectrum (ES). Moreover, the EE between subsystems X and Y can be obtained in terms of the reduced density matrix ρ_X as

$$S = -\text{Tr}(\rho_X \ln \rho_X). \quad (\text{S1-58})$$

The ES and EE encode all information about the entanglement between subsystems X and Y of the full system described by the Floquet state $|\Psi\rangle$.

We now reveal how to obtain EE from the spectrum of the single-particle correlator. With respect to our Floquet state $|\Psi\rangle$, the matrix element of single-particle correlator in the basis $\{|\nu\rangle\}$ is given by

$$C_{\mu\nu} = \langle \Psi | c_\nu^\dagger c_\mu | \Psi \rangle. \quad (\text{S1-59})$$

Using the explicit form of $|\Psi\rangle$ in Eq. (S1-56) and the transformations in Eq. (S1-55), we can also express $C_{\mu\nu}$ as

$$C_{\mu\nu} = \langle \mu | P | \nu \rangle, \quad P = \sum_{j \in \text{occ.}} |\psi_j\rangle \langle \psi_j|, \quad (\text{S1-60})$$

where P is the projector onto occupied single-particle Floquet states in $|\Psi\rangle$. Numerically, the spectrum of the single-particle correlator C can then be obtained for a given projector P , which is further deducible from experimental measurements.

Restricting to the subsystem X , $C_{\mu\nu}$ can also be computed via the reduced density matrix ρ_X , i.e.,

$$C_{\mu\nu} = \text{Tr}(c_\mu^\dagger c_\nu \rho_X), \quad \mu, \nu \in X. \quad (\text{S1-61})$$

Using the spectral decomposition of entanglement Hamiltonian $H_X = \sum_j \xi_j \phi_j^\dagger \phi_j$ and Eq. (S1-57), we can further write

$$C_{\mu\nu} = \frac{1}{Z} \text{Tr}(c_\mu^\dagger c_\nu e^{-\sum_j \xi_j \phi_j^\dagger \phi_j}). \quad (\text{S1-62})$$

The basis $\{|\phi_j\rangle \equiv \phi_j^\dagger |\emptyset\rangle | j \in X\}$ and $\{|\nu\rangle \equiv c_\nu^\dagger |\emptyset\rangle | \nu \in X\}$ are both complete and orthonormal within X . The operators ϕ_j and $\phi_j^\dagger$ also satisfy anticommutation relations of fermions, i.e., $\{\phi_i, \phi_j\} = \{\phi_i^\dagger, \phi_j^\dagger\} = 0$ and $\{\phi_i, \phi_j^\dagger\} = \delta_{ij}$. Therefore, we have the following transformation rules between operators $\phi_j^\dagger$ and $c_\nu^\dagger$, i.e.,

$$|\phi_j\rangle = \sum_{\nu \in X} b_{\nu j} c_\nu^\dagger |\emptyset\rangle, \quad |\nu\rangle = \sum_{j \in X} b_{\nu j}^* \phi_j^\dagger |\emptyset\rangle, \quad (\text{S1-63})$$

$$\phi_j^\dagger = \sum_{\nu \in X} b_{\nu j} c_\nu^\dagger, \quad \phi_j = \sum_{\nu \in X} b_{\nu j}^* c_\nu, \quad c_\nu^\dagger = \sum_{j \in X} b_{\nu j}^* \phi_j^\dagger, \quad c_\nu = \sum_{j \in X} b_{\nu j} \phi_j, \quad (\text{S1-64})$$

where $b_{\nu j} = \langle \nu | \phi_j \rangle$. Plugging Eq. (S1-64) into Eq. (S1-62), we could obtain after straightforward calculations (for intermediate steps, see Ref. [7]) that

$$C_{\mu\nu} = \sum_{j \in X} \frac{\langle \mu | \phi_j \rangle \langle \phi_j | \nu \rangle}{e^{\xi_j} + 1}. \quad (\text{S1-65})$$

Therefore, the single-particle correlator admits the spectral decomposition

$$C = \sum_{j \in X} \zeta_j |\phi_j\rangle \langle \phi_j|, \quad \zeta_j = \frac{1}{e^{\xi_j} + 1}. \quad (\text{S1-66})$$

We have now established a one-to-one correspondence between the spectrum $\{\zeta_j\}$ of correlation matrix C and the ES $\{\xi_j\}$, i.e.,

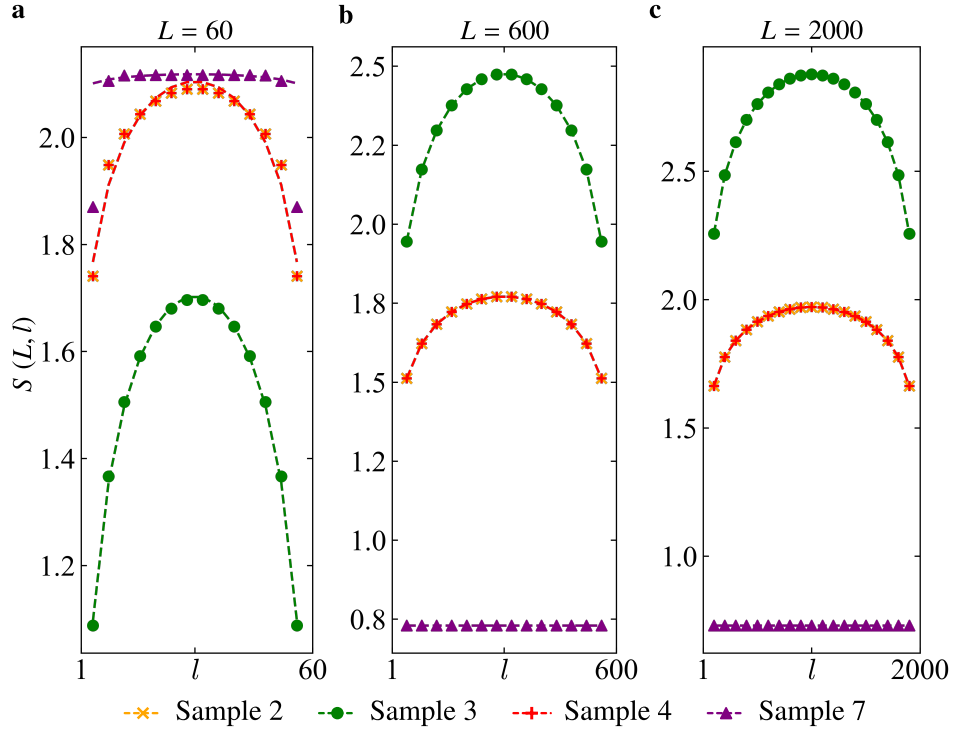
$$\xi_j = \ln(\zeta_j^{-1} - 1), \quad j \in X. \quad (\text{S1-67})$$

For a Hermitian density matrix ρ , the spectra $\xi_j \in \mathbb{R}$ and $\zeta_j \in (0, 1)$. The correlation matrix spectrum is thus bounded. We can equivalently identify $\{\zeta_j | j \in X\}$ as the ES of the system, which is also available in experimental detections.

Finally, using the definition of ρ_X in Eq. (S1-57), the expression of S in Eq. (S1-58), and the spectral identity in Eq. (S1-67), we can find EE from the correlation matrix spectrum as

$$S = - \sum_j [\zeta_j \ln \zeta_j + (1 - \zeta_j) \ln (1 - \zeta_j)]. \quad (\text{S1-68})$$

This result looks formally the same as that of static systems [10]. The key difference is that the underlying state $|\Psi\rangle$ is given by the product of Floquet eigenstates $|\psi_j\rangle$ at a given filling fraction, thereby reflecting the Floquet nature of the system.



Supplementary Fig. 1 | Bipartite entanglement entropy $S(L, l)$ vs subsystem size l at half-filling and under OBC for the Floquet SSH α -chain with $(\alpha, \alpha') = (1, 2)$. (a–c) Results for three distinct total system sizes $L = 60, 600, 2000$ ($N = 30, 300, 1000$ unit cells). In each panel, the curves with $\times$, $\bullet$, $+$, and $\blacktriangle$ are obtained at system parameters $(J_1 T_2/\pi, J_2 T_1/\pi) = (0.35, 0.65), (0.5, 0.5), (0.65, 0.65),$ and $(0.5, 0.73)$, which correspond to the representative samples 2, 3, 4, 7 of the phase diagram Fig. 2a in the main text. The four samples with system size $L = 60$ are fabricated in experiments.

Supplementary Table 6 | Scaling coefficients γ of $S(L, l)$ vs subsystem size l at different total system sizes L . The scaling function $S(L, l) \sim \gamma \ln[\sin(\pi l/L)]$ is used for curve fittings. The numerical error of γ is set at $O(10^{-4})$ for its values shown in the second to fourth columns of the Table. In the first column, 2, 3, 4, 7 label the four representative samples in the phase diagram Fig. 2a of the main text. In the fifth column, the theoretically expected value of γ in the thermodynamic limit ($L \rightarrow \infty$) is given for each representative point, leading to the entanglement coefficients $c = 1, 2, 1, 0$ for the four samples. The last two columns show the topological invariants and the number of $0/\pi$ edge states.

Label	$L = 60$	$L = 600$	$L = 2000$	$L \rightarrow \infty$	c	(ω_0, ω_π)	(N_0, N_π)
2	0.1670	0.1655	0.1661	$\gamma = 1/6$	1	(2, 0)	(4, 0)
3	0.3755	0.3371	0.3348	$\gamma = 1/3$	2	(1, 0)	(2, 0)
4	0.1670	0.1655	0.1661	$\gamma = 1/6$	1	(1, 1)	(2, 2)
7	0.0446	0	0	$\gamma = 0$	0	(2, 1)	(4, 2)

We are now ready to provide certain numerical evidence, which would help to establish connections between our theoretical predictions and experimental observations. We consider the scaling law of EE versus the size of subsystem X . The state $|\Psi\rangle$ of the system is assumed to be prepared at half-filling. For our Floquet SSH α -chain, it means the uniform filling of the first N Floquet eigenstates whose quasienergies ε_j are inside the range of $[-\pi, 0]$, where N denotes the total number of unit cells and the total system size $L = 2N$. The resulting EE versus subsystem size l for three different L are shown in [Supplementary Fig. 1](#). We also conduct curve fittings for the EE $S(L, l)$ vs l via the scaling function [11]

$$S(L, l) \sim \gamma \ln \left(\sin \frac{\pi l}{L} \right) = \frac{c}{6} \ln \left(\sin \frac{\pi l}{L} \right). \quad (\text{S1-69})$$

The scaling coefficients γ for representative samples are collected in [Supplementary Table 6](#).

Each panel of [Supplementary Fig. 1](#) presents the results at two distinct critical phase boundaries, at a multicritical point, and in a gapped phase. Based on their shapes, we can classify the results into three groups.

- The first group, corresponding to the lines with “▲” in [Supplementary Fig. 1a–c](#), is evaluated at the sample “7” in the phase diagram Fig. 2a of the main text. As the system size L increases, the EE $S(L, l)$ rapidly saturates to a constant with respect to the subsystem size l , exhibiting area-law scaling expected for gapped phases (insulator).
- The second group, including the lines with “x” and “+” in [Supplementary Fig. 1a–c](#) are evaluated at the samples “2” and “4” in the phase diagram Fig. 2a of the main text, respectively. As the system size L increases, the EE $S(L, l)$ gradually collapses to the same logarithmic shape $\sim \gamma \ln[\sin(\pi l/L)]$ vs the subsystem size l , exhibiting log-law scaling as expected for critical phase boundaries. Specifically, the scaling coefficient γ approaches $1/6$ in the limit $L \rightarrow \infty$, suggesting an entanglement coefficient $c = 1$ for these two curves.
- The third group, referring to the lines with “•” in [Supplementary Fig. 1a–c](#), are evaluated at the sample “3” in the phase diagram Fig. 2a of the main text. With the increase of system size L , the EE $S(L, l)$ gradually converges to the logarithmic form $\sim \gamma \ln[\sin(\pi l/L)]$ versus the subsystem size l , showing a behavior of log-law scaling that is also expected at multicritical points. Being different from the samples “2” and “4”, the scaling coefficient γ approaches $1/3$ in the limit $L \rightarrow \infty$, implying an entanglement coefficient $c = 2$ at this multicritical point.

Finally, we sketch how to extract the EE from our acoustic simulation or experimental data:

1. Use the pump-probe method [12] to measure the matrix elements of an $L \times L$ Floquet operator for a lattice of length L ;
2. Reconstruct the Floquet operator of the system from the experimental data;
3. Diagonalize the Floquet operator and find its eigensystem;
4. Determine a specific value of ε_0 such that the number of bulk states with $\text{Re}(\varepsilon_j) \in [\varepsilon_0 - \pi, \varepsilon_0]$ equals that with $\text{Re}(\varepsilon_j) \in [\varepsilon_0, \varepsilon_0 + \pi]$;
5. Determine the set containing half of the edge states, $\{|\psi_{j,\text{edge}}\rangle\}$, including all types of edge states;
6. Find out $N = L/2$ Floquet eigenstates $\{|\psi_j\rangle\} = \{|\psi_{j,\text{bulk}}\rangle | \text{Re}(\varepsilon_j) \in [\varepsilon_0 - \pi, \varepsilon_0]\} \cup \{|\psi_{j,\text{edge}}\rangle\}$;
7. Construct the projector P in Eq. (S1-60);
8. Take the upper $l \times l$ block of P and obtain its spectrum as the ES $\{\zeta_j\}$;
9. By taking $\zeta_j = \text{Re}(\zeta_j)$ [13], use the ES to compute the EE $S(L, l)$ according to Eq. (S1-68).
10. Repeat steps 8 and 9 with different subsystem sizes l .

In experiments, we fabricate samples with $N = 30$ unit cells and $L = 60$ lattice sites. We note in passing that, although the experimental platform employed here constitutes a classical wave system—where genuine quantum entanglement is absent—entanglement-entropy-based measurements using acoustic waves have already been implemented to verify gapped topological phases [14]. The key point is that the theoretical derivations in this section, which map the spectrum of an entanglement Hamiltonian of free fermions (obtained from the reduced density matrix of a many-particle Floquet Gaussian state) to that of a single-particle correlator is generic for noninteracting systems, and the latter can be well-defined and measured in classical wave dynamics.

Supplementary Note 2. DESIGN OF THE ACOUSTIC STRUCTURE

We begin by deriving the paraxial equation from the wave equation using the slowly varying amplitude (SVA) approximation. The wave equation is given by

$$\left(\frac{\partial^2}{\partial t^2} - v^2 \nabla^2 \right) p(\mathbf{r}, t) = 0. \quad (\text{S2-1})$$

Here, v is the wave speed. Separating the variable $p(\mathbf{r}, t) = \bar{p}(\mathbf{r}) e^{-i\omega t}$ and denoting $\mathbf{k}^2 = \omega^2/v^2$, we get the Helmholtz equation

$$(\nabla^2 + \mathbf{k}^2) \bar{p}(\mathbf{r}) = 0. \quad (\text{S2-2})$$

Taking $\bar{p}(\mathbf{r}) = \bar{\bar{p}}(\mathbf{r}) e^{ik_x x}$, we have

$$\left(\nabla_{\perp}^2 + \frac{\partial^2}{\partial x^2} + 2ik_x \frac{\partial}{\partial x} + \mathbf{k}^2 - k_x^2 \right) \bar{\bar{p}}(\mathbf{r}) = 0. \quad (\text{S2-3})$$

with $\nabla_{\perp}^2 = \partial_y^2 + \partial_z^2$ as the transverse Laplacian operator. SVA approximation means that

$$\left| \frac{\partial^2 \bar{\bar{p}}}{\partial x^2} \right| \ll k_x \left| \frac{\partial \bar{\bar{p}}}{\partial x} \right|. \quad (\text{S2-4})$$

From Eqs. (S2-3)(S2-4), we have

$$i \frac{\partial \bar{\bar{p}}}{\partial x} = H_{\text{eff}} \bar{\bar{p}} = \left(-\frac{\nabla_{\perp}^2}{2k_x} - \frac{\mathbf{k}^2 - k_x^2}{2k_x} \right) \bar{\bar{p}}. \quad (\text{S2-5})$$

Equation (S2-5) is the paraxial equation, which shares the same mathematical form as the Schrödinger equation.

In Ref. [15], we demonstrate that the paraxial equation can be mapped onto a tight-binding model, indicating that an acoustic lattice can be employed to realize the lattice model discussed in the main text. In this section, we present the design principles of the corresponding acoustic structures.

A. Determination of the working frequency

Let us first consider the 0 state in an isolated, straight, rectangular waveguide. The pressure field of the lowest propagating state can be expressed as

$$p \propto e^{-i2\pi f t} e^{ik_x x}. \quad (\text{S2-6})$$

For the 0 state, the wavevector is

$$k = \frac{2\pi n}{L}, \quad (\text{S2-7})$$

where n is an integer. The corresponding frequency is then given by

$$f = \frac{nv}{L}, \quad (\text{S2-8})$$

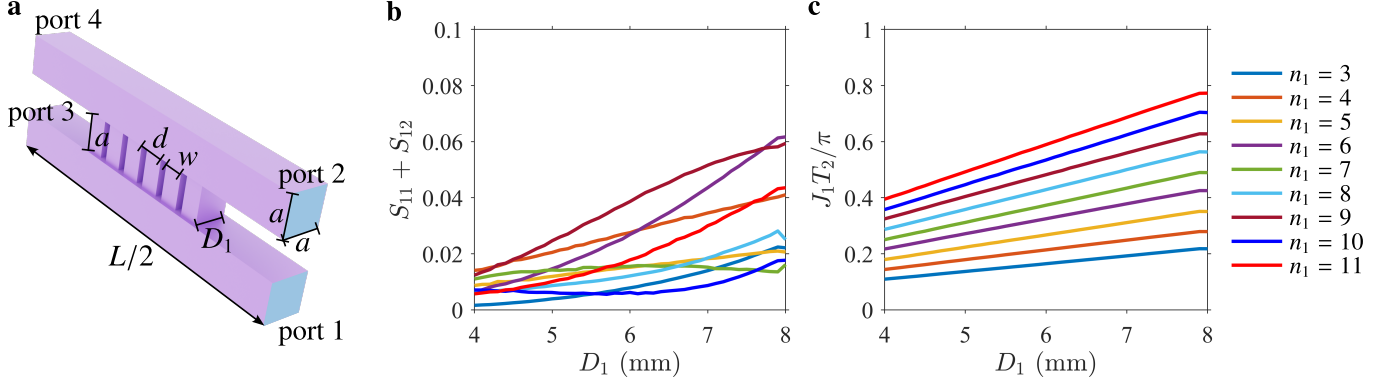
where $v = 343\text{m/s}$ is the speed of sound in air. Choosing $n = 4$, $L = 14\text{ cm}$, we obtain a working frequency of $f = 9.8\text{ kHz}$.

B. Relationship between the acoustic lattice geometry and tight-binding couplings

In this section, we present the procedure for extracting the coupling strengths between two acoustic waveguides from numerical simulations.

1. Nearest neighbour couplings

The diagram of two nearest neighbour coupled waveguides is displayed in [Supplementary Fig. 2a](#), with four ports marked 1, 2, 3, 4. The coupling strengths can be tuned by the number of link tubes n_1 and the tube width D_1 . The plane wave with pressure p_i is launched at port 1. And we measure the pressure at four ports (p_1, p_2, p_3, p_4). The scattering parameters are determined by $s_{11} = (p_1 - p_i)/p_i$, $s_{12} = p_2/p_i$, $s_{13} = p_3/p_i$, $s_{14} = p_4/p_i$. The reflection from port 1 and 2 is $S_{11} + S_{12} = |s_{11}|^2 + |s_{12}|^2$.



Supplementary Fig. 2 | The structural details and nearest neighbor coupling strengths. (a) Illustration of the structural details (6 tubes as an example) used to realize nearest neighbor coupling. The coupling strength can be tuned by adjusting the number of link tubes n_1 and the tube width D_1 . The other structural parameters are $L = 14$ cm, $a = 8$ mm, $d = 6$ mm, and $w = 4$ mm. (b) Dependence of the reflection coefficient on D_1 and n_1 at 9.8 kHz. (c) Dependence of the coupling strength on D_1 and n_1 at 9.8 kHz. Here $T_2 = T/2$ and T is the Floquet period.

[Supplementary Fig. 2b](#) displays the relationship between the reflection coefficient and the number of link tubes n_1 and the tube width D_1 . We notice that reflection is small for $4 \text{ mm} \leq D_1 \leq 8 \text{ mm}$ and $n_1 = 3, \dots, 11$. In this scenario, we can apply the slowly varying amplitude (SVA) approximation and treat the waveguides as a tight-binding model with Hamiltonian $\mathcal{H} = J_1 \sigma_1$ [16]. By noting that [Supplementary Fig. 2a](#) represents a Hamiltonian over half Floquet period ($T_2 = T/2$), the pressure (p_1, p_2, p_3, p_4) are related by

$$\begin{pmatrix} p_3 \\ p_4 \end{pmatrix} = e^{-iJ_1 \sigma_1 T_2} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \approx e^{-iJ_1 \sigma_1 T_2} \begin{pmatrix} p_i \\ 0 \end{pmatrix}. \quad (\text{S2-9})$$

Considering that p_3 and p_4 are complex in the simulation result, we have

$$\cos\left(J_1 \frac{T}{2}\right) = \text{sgn}\left[\text{Re}\left(\frac{p_3}{p_i}\right)\right] \frac{|p_3|}{\sqrt{|p_3|^2 + |p_4|^2}}. \quad (\text{S2-10})$$

The coupling strengths for different numbers of link tubes and tube width D_1 are displayed in [Supplementary Fig. 2c](#). The fitted results between $J_1 T_2 / \pi$ and D_1 for different n_1 are

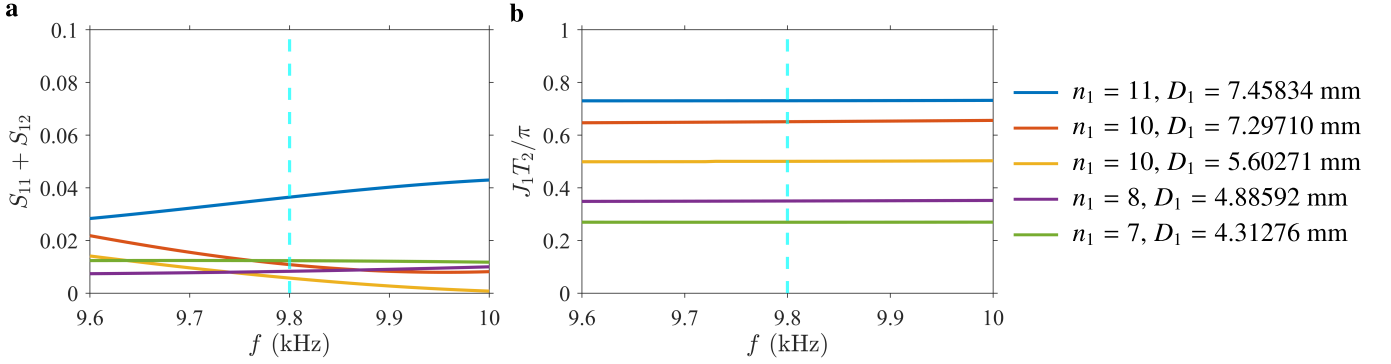
$$\begin{aligned} n_1 = 3 : & \quad J_1 T_2 / \pi = 0.0277817 D_1 - 0.00189749 \\ n_1 = 4 : & \quad J_1 T_2 / \pi = 0.0345901 D_1 + 0.00619646 \\ n_1 = 5 : & \quad J_1 T_2 / \pi = 0.0438355 D_1 + 0.00432496 \\ n_1 = 6 : & \quad J_1 T_2 / \pi = 0.0532422 D_1 + 0.00511620 \\ n_1 = 7 : & \quad J_1 T_2 / \pi = 0.0610943 D_1 + 0.00651471 \\ n_1 = 8 : & \quad J_1 T_2 / \pi = 0.0706920 D_1 + 0.00460430 \\ n_1 = 9 : & \quad J_1 T_2 / \pi = 0.0775444 D_1 + 0.0161492 \\ n_1 = 10 : & \quad J_1 T_2 / \pi = 0.0885274 D_1 + 0.00400641 \\ n_1 = 11 : & \quad J_1 T_2 / \pi = 0.0967201 D_1 + 0.00862881 \end{aligned}$$

For a given value of $J_1 T_2 / \pi$, there may be multiple combinations of D_1 and n_1 that satisfy the condition. Among these, we select the one that minimizes $S_{11} + S_{12}$. The five selected values of $J_1 T_2 / \pi$ in the experiment, along with the corresponding values of n_1 , D_1 and $S_{11} + S_{12}$, are listed below.

Supplementary Table 7 | $J_1 T_2 / \pi$, along with the corresponding values of n_1 , D_1 and $S_{11} + S_{12}$ in the experiment.

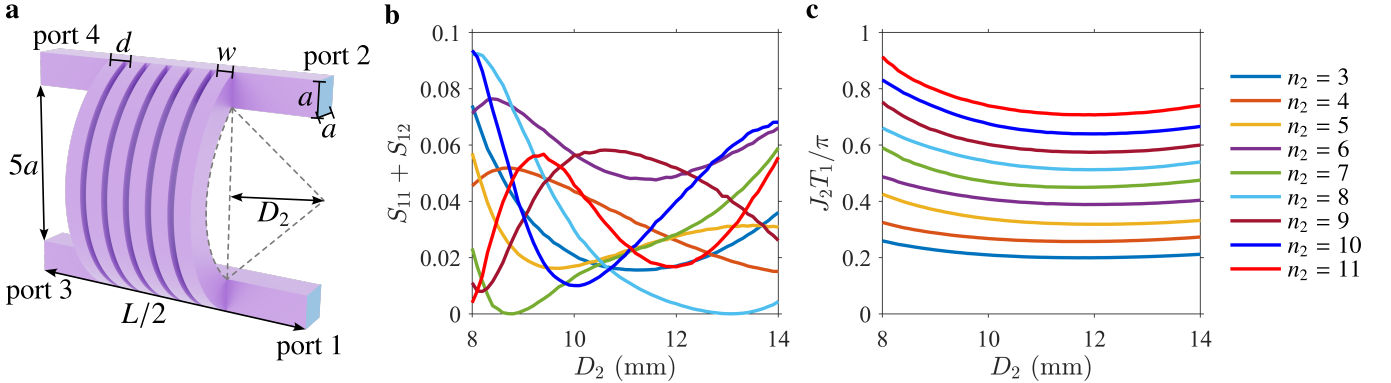
$J_1 T_2 / \pi$	n_1	D_1 (mm)	$S_{11} + S_{12}$
0.73	11	7.45834	0.0360566
0.65	10	7.29710	0.0109033
0.50	10	5.60271	5.75416e-3
0.35	8	4.88592	8.25345e-3
0.27	7	4.31276	0.0126093

Supplementary Fig. 3 demonstrates that the coupling strength exhibits minimal variation with small shifts in frequency around 9.8 kHz.



Supplementary Fig. 3 | **The relationship between the nearest neighbor reflection coefficient, coupling strength, and frequency.** (a) Dependence of the reflection coefficient on frequency. (b) Dependence of the coupling strength on frequency.

2. Next-nearest neighbour couplings



Supplementary Fig. 4 | **The structural details and next-nearest neighbor coupling strengths.** (a) Illustration of the structural details (6 tubes as an example) used to realize next-nearest neighbor coupling. The coupling strength can be tuned by adjusting the number of link tubes n_2 and D_2 . The other structural parameters are $L = 14$ cm, $a = 8$ mm, $d = 6$ mm, and $w = 4$ mm. (b) Dependence of the reflection coefficient on D_2 and n_2 at 9.8 kHz. (c) Dependence of the coupling strength on D_2 and n_2 at 9.8 kHz. Here $T_1 = T/2$ and T is the Floquet period.

The diagram of two next-nearest neighbour coupled waveguides is displayed in **Supplementary Fig. 4a**, with four ports marked 1, 2, 3, 4. The link tubes form part of a circular arc, with inner and outer radii given by

$$R_{in} = \sqrt{D_2^2 + \left(\frac{5}{2}a\right)^2}, \quad R_{out} = \sqrt{D_2^2 + \left(\frac{7}{2}a\right)^2}, \quad (\text{S2-11})$$

respectively. The coupling strength can be tuned by adjusting the number of link tubes n_2 and D_2 . The subsequent procedures follow the same approach as described previously in [Supplementary Note 2 B 1](#).

The coupling strengths for different numbers of link tubes and tube width D_2 are displayed in [Supplementary Fig. 4c](#). The fitted results between J_2T_1/π and D_2 for different n_2 are

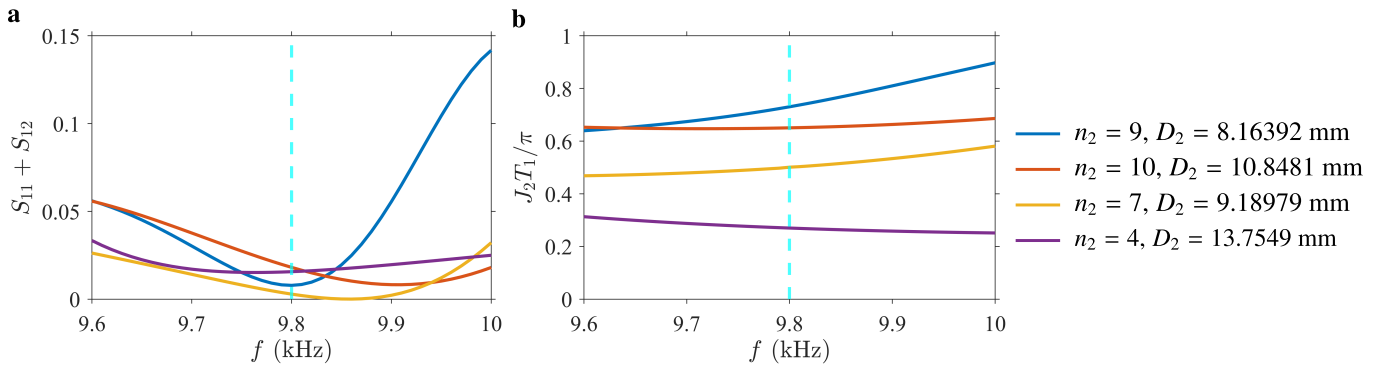
$$\begin{aligned}
 n_2 = 3 : \quad & J_2T_1/\pi = 0.0000565041D_2^4 - 0.0028562D_2^3 + 0.0565462D_2^2 - 0.512549D_2 + 1.97152 \\
 n_2 = 4 : \quad & J_2T_1/\pi = 0.0000952521D_2^4 - 0.0046188D_2^3 + 0.0868514D_2^2 - 0.745844D_2 + 2.70728 \\
 n_2 = 5 : \quad & J_2T_1/\pi = 0.000117688D_2^4 - 0.00590475D_2^3 + 0.114562D_2^2 - 1.01245D_2 + 3.73417 \\
 n_2 = 6 : \quad & J_2T_1/\pi = 0.0000759723D_2^4 - 0.00389838D_2^3 + 0.0786607D_2^2 - 0.728829D_2 + 2.96869 \\
 n_2 = 7 : \quad & J_2T_1/\pi = 0.000200588D_2^4 - 0.0100278D_2^3 + 0.192377D_2^2 - 1.66832D_2 + 5.93672 \\
 n_2 = 8 : \quad & J_2T_1/\pi = 0.000103083D_2^4 - 0.0052094D_2^3 + 0.105205D_2^2 - 0.986215D_2 + 4.06184 \\
 n_2 = 9 : \quad & J_2T_1/\pi = 0.000350036D_2^4 - 0.0168284D_2^3 + 0.30866D_2^2 - 2.55744D_2 + 8.63849 \\
 n_2 = 10 : \quad & J_2T_1/\pi = 0.0000742642D_2^4 - 0.00443172D_2^3 + 0.102942D_2^2 - 1.06856D_2 + 4.75663 \\
 n_2 = 11 : \quad & J_2T_1/\pi = 0.000360736D_2^4 - 0.0173756D_2^3 + 0.3206D_2^2 - 2.68052D_2 + 9.25296
 \end{aligned}$$

For a given value of J_2T_1/π , there may be multiple combinations of D_2 and n_2 that satisfy the condition. Among these, we select the one that minimizes $S_{11} + S_{12}$. The four selected values of J_2T_1/π in the experiment, along with the corresponding values of n_2 , D_2 and $S_{11} + S_{12}$, are listed below.

Supplementary Table 8 | J_2T_1/π , along with the corresponding values of n_2 , D_2 and $S_{11} + S_{12}$ in the experiment.

J_2T_1/π	n_2	D_2 (mm)	$S_{11} + S_{12}$
0.73	9	8.16392	7.80654e-3
0.65	10	10.8481	0.0183388
0.50	7	9.18979	2.81413e-3
0.27	4	13.7549	0.0156811

[Supplementary Fig. 5](#) demonstrates that the coupling strength exhibits minimal variation with small shifts in frequency around 9.8 kHz.



Supplementary Fig. 5 | The relationship between the next-nearest neighbor reflection coefficient, coupling strength, and frequency. (a) Dependence of the reflection coefficient on frequency. (b) Dependence of the coupling strength on frequency.

C. Structure parameters for different samples

We provide detailed structural parameters for all samples. Samples 1–8 are labeled in Fig. 2a of the main text. Sample 9, fabricated for the control experiment described in [Supplementary Note 6](#), is highlighted in Supplementary Fig. 13a.

Supplementary Table 9 | Structural parameters for different samples.

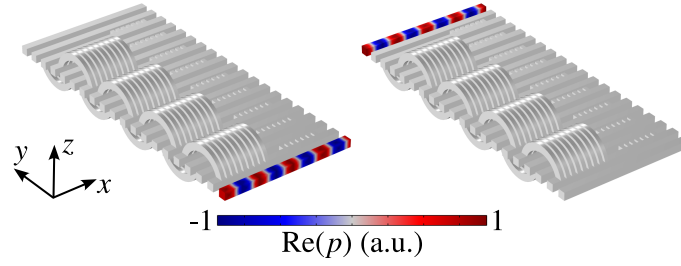
Sample	$J_1 T_2 / \pi$	$J_2 T_1 / \pi$	n_1	D_1 (mm)	n_2	D_2 (mm)
1	0.27	0.73	7	4.31276	9	8.16392
2	0.35	0.65	8	4.88592	10	10.8481
3	0.50	0.50	10	5.60271	7	9.18979
4	0.65	0.65	10	7.29710	10	10.8481
5	0.73	0.73	11	7.45834	9	8.16392
6	0.65	0.73	10	7.29710	9	8.16392
7	0.50	0.73	10	5.60271	9	8.16392
8	0.35	0.73	8	4.88592	9	8.16392
9	0.27	0.27	7	4.31276	4	13.7549

Supplementary Note 3. EIGENSTATES FOR DIFFERENT SAMPLES

In this section, we provide the eigenstate calculation for the practical acoustic structure with Floquet periodic boundary conditions along x . In our system, the x plays the role of time, so the quasimomentum k_x corresponds to the quasienergy. To visualize the edge localization of 0 and π states in the acoustic system, we plot the real part of the acoustic pressure.

A. Isolated 0 edge states

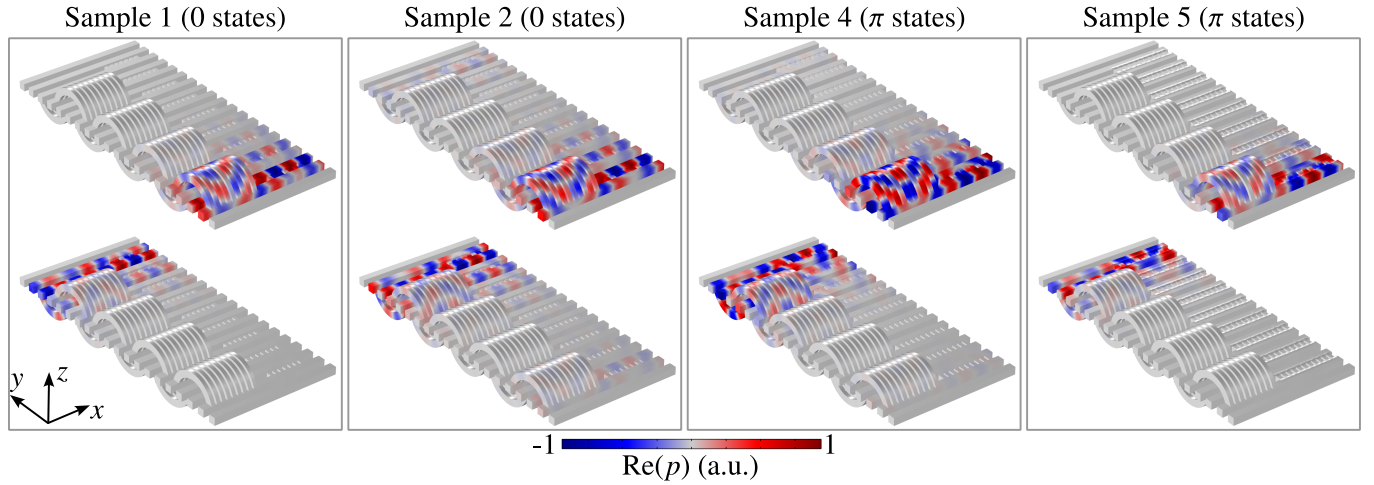
From Fig. 1 in the main text, we observe that the 1st and 20th waveguides are disconnected from the main structure, resulting in two isolated 0 edge states in each sample. For simplicity, we present the isolated 0 edge state profiles only for sample 1.



Supplementary Fig. 6 | Simulated isolated 0 edge state profiles. All states are doubly degenerate.

B. 0 and π states for gapless Floquet phase boundaries I, III

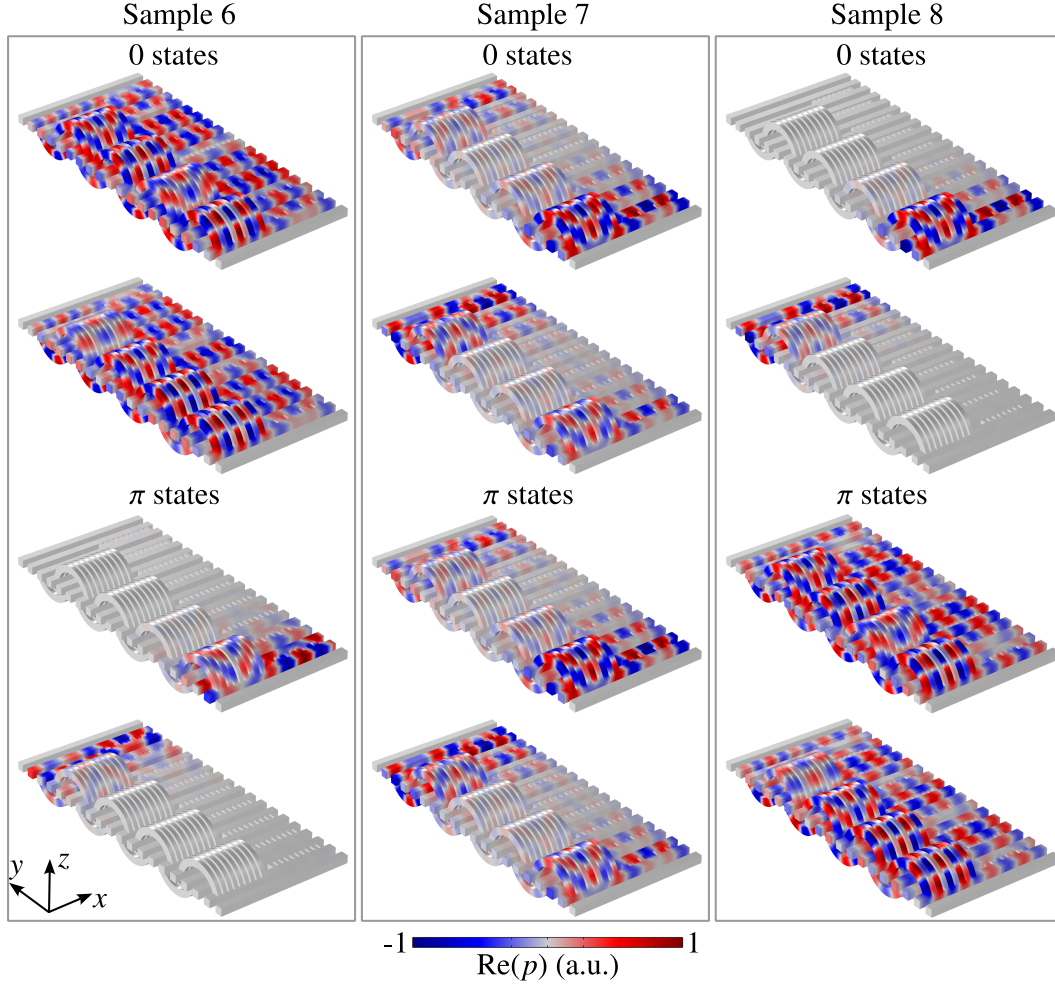
In this section, we present 0 and π states for gapless Floquet phase boundaries I (samples 1, 2), III (samples 4, 5).



Supplementary Fig. 7 | Critical 0 and π edge states for gapless Floquet phase boundaries I (samples 1, 2), III (samples 4, 5). All states are doubly degenerate.

C. 0 and π states for gapped Floquet phase II

In this section, we present 0 and π states for gapped Floquet phases II (samples 6, 7, 8). In sample 6, the 0 edge states extend into the bulk due to the small bulk 0 -gap near phase boundary III (Fig. 3a,b in the main text). Similarly, in sample 8, the π edge states penetrate the bulk because the bulk π -gap is small near boundary I.



Supplementary Fig. 8 | Simulated 0 and π edge states for gapped Floquet phase II (samples 6, 7, 8). All states are doubly degenerate.

From [Supplementary Figs. 7 and 8](#), we observe that the edge state is most strongly excited when the source is placed at port 3 or 17. This is why port 3 is chosen as the source location for calculating the interference intensity function (Fig. 4 in the main text).

Supplementary Note 4. RETRIEVAL OF FLOQUET OPERATORS FROM MEASURED WAVEFUNCTIONS

In this section, we detail the procedure for retrieving the Floquet operator from the measured wavefunctions, applicable to both simulations and experiments.

The time axis t is mapped to the synthetic dimension x , which corresponds to the wave propagation direction. Similarly, the driving period T is replaced by the Floquet period L . Specifically, by placing the source at port i , as shown in [Supplementary Fig. 9](#), we measure the pressure field at both integer and half-integer Floquet periods. In other words, we obtain $p^i(x)$ for $x = 0, 0.5L, \dots, 3.5L, 4L$. And $p^i(x)$ is a 20-row and 1-column vector. We have the relationship

$$p^i(x + L) = F(x + L, x) p^i(x), \quad (\text{S4-1})$$

Here, $F(x + L, x)$ denotes the Floquet operator. To retrieve $F(x + L, x)$, we require 20 linearly independent wavefunctions $p^i(x)$ and their corresponding $p^i(x + L)$. This can be readily achieved by repeating the measurement of $p^i(x)$, $x = 0, 0.5L, \dots, 3.5L, 4L$ with the source placed at ports $i = 1, 2, \dots, 20$. We denote

$$\mathbf{p}(x) = [p^1(x), p^2(x), \dots, p^{20}(x)]. \quad (\text{S4-2})$$

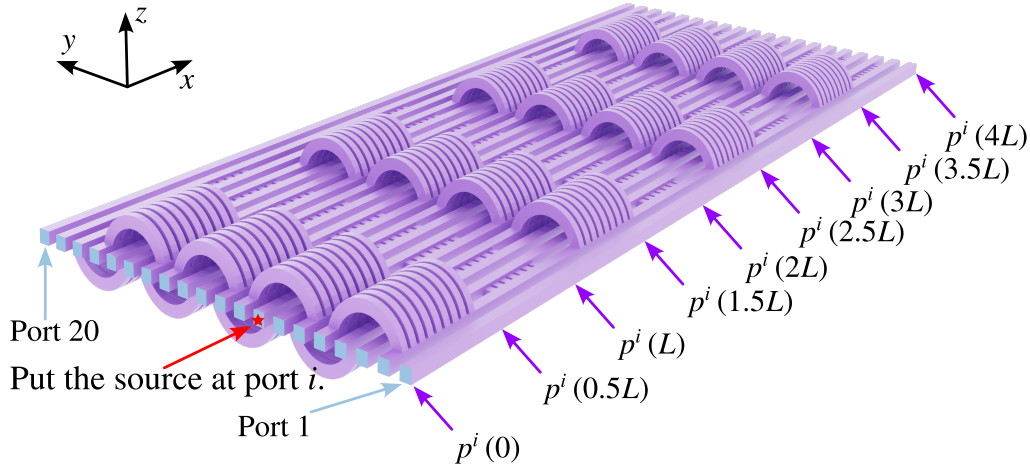
From Eqs. (S4-1)(S4-2), we have

$$\mathbf{p}(x + L) = F(x + L, x) \mathbf{p}(x), \quad (\text{S4-3})$$

and the Floquet operator can be written as

$$F(x + L, x) = \mathbf{p}(x + L) \mathbf{p}^{-1}(x). \quad (\text{S4-4})$$

Here, it is unnecessary to normalize $p^i(x)$ to $\|p^i(x)\| = 1$ as we directly plot the real and imaginary parts of the Floquet spectrum. The imaginary parts correspond to the decay rates of the eigenstates.



Supplementary Fig. 9 | Retrieval of Floquet operators from measured wavefunctions. The entire structure, consisting of four Floquet periods, is composed of 20 waveguides. The purple regions indicate hard boundaries, while the light blue areas represent air-filled hollow regions. Curved tubes are used to implement next-nearest neighbor couplings, and straight tubes realize nearest neighbor couplings. The source is placed at port i , where $i = 1, 2, \dots, 20$.

From Eq. (S4-4) and [Supplementary Fig. 9](#), we can obtain 7 Floquet operators— $F(L, 0)$, $F(1.5L, 0.5L)$, $\dots$, $F(4L, 3L)$ —corresponding to 7 eigen-spectra. For the simulation results shown in Fig. 3b of the main text, we plot the average of these 7 eigen-spectra. For the experimental results in Fig. 3c, we present the Floquet spectrum corresponding to $F(2.5L, 1.5L)$.

For quasienergy spectrum experiment, in principle, for each sample, 20 experiments should be conducted—one for each port. In each experiment, both the sound pressure amplitude and phase need to be measured at $20 \times 9 = 180$ detection points. In other words, we should measure $20 \times 180 = 3600$ detection points for each sample. And we have 9 samples.

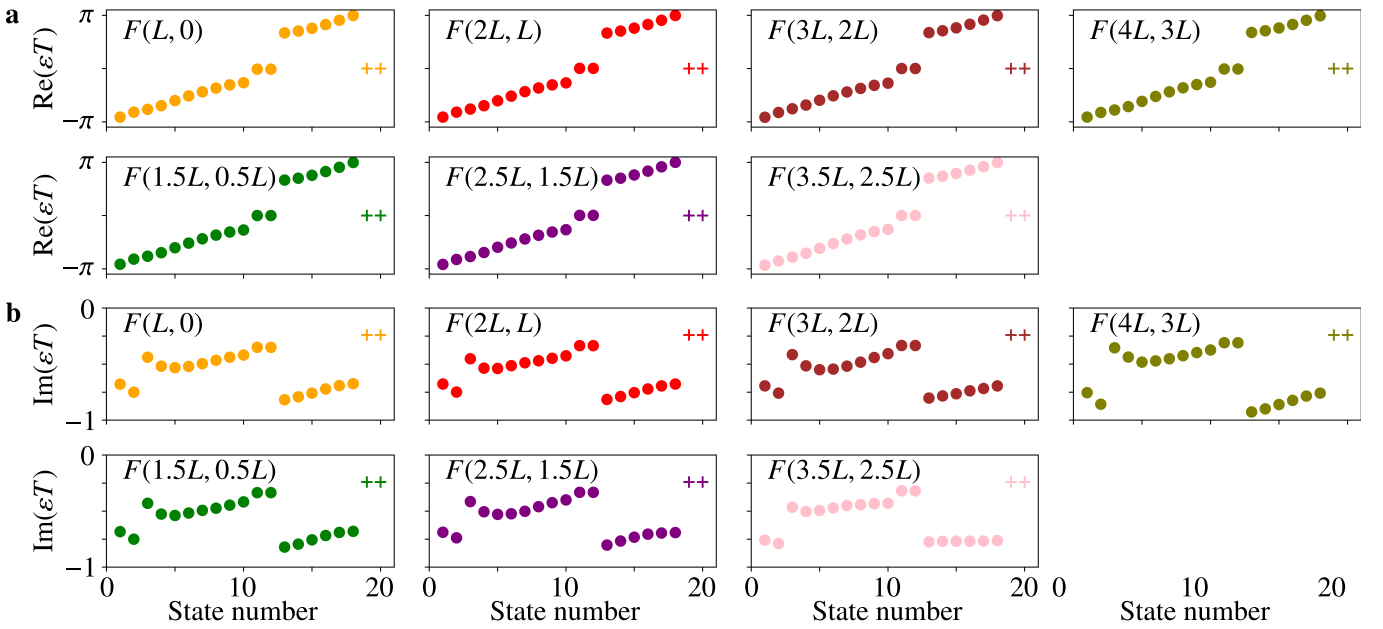
However, due to the chiral symmetry of the structure, we have

$$p^i(x, y) = p^{(21-i)}(x, -y). \quad (\text{S4-5})$$

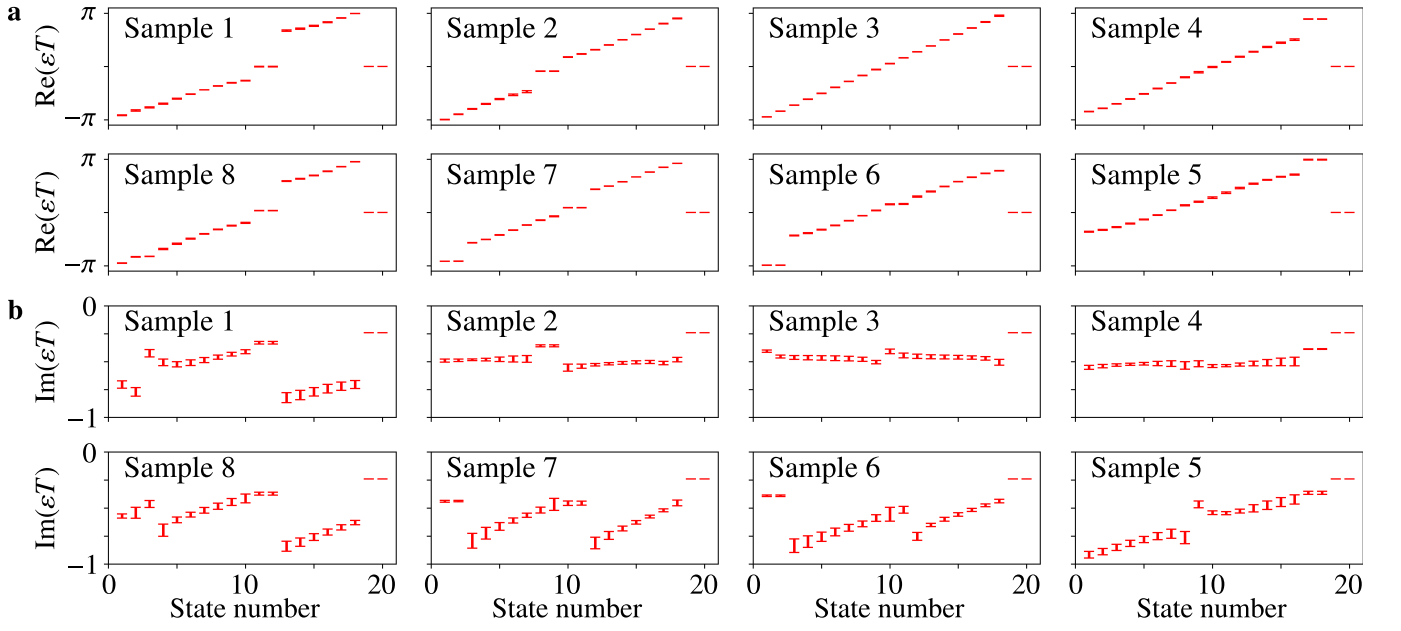
Additionally, ports 1 and 20 are disconnected from the main structure, so the detection points located at these ports can be excluded. As a result of these two factors, only 9 experiments are required for each sample, with the source placed at ports 2 through 10. In each experiment, both the sound pressure amplitude and phase are measured at $18 \times 9 = 162$ detection points. In other words, a total of $9 \times 162 = 1458$ detection points are measured per sample. Since there are 9 samples, the total number of measurements is $9 \times 1458 = 13122$. For each measurement point, data is collected after 34 seconds to ensure stable and precise pressure readings.

For the entanglement-entropy measurement, it is also necessary to measure the quasienergy spectrum. Each sample has 60 ports, and taking symmetry into account, 29 experiments are performed per sample (from port 2 to port 30). The acoustic field is measured only at $x = 0$ and $x = L$; thus, in each experiment, both the sound pressure amplitude and phase are recorded at $58 \times 2 = 116$ detection points. In total, this corresponds to $29 \times 116 = 3364$ detection points per sample. With 4 samples, the total number of measurements is $4 \times 3364 = 13456$. For each measurement point, data is collected after 34 seconds to ensure stable and precise pressure readings.

In [Supplementary Fig. 10](#), we present the real and imaginary parts of the quasienergy spectrum for sample 1. It can be observed that the discrepancies among the seven eigenspectra—corresponding to the seven Floquet operators—are negligible. In [Supplementary Fig. 11](#), we present the average eigenspectra for eight samples labeled in Fig. 2a of the main text, with error bars indicating the standard deviation obtained from the seven eigenspectra. The sound speed is set to $v = 343 \times (1 + 0.01j)$ m/s, and the working frequency is 9.8 kHz.



Supplementary Fig. 10 | The seven eigenspectra corresponding to the seven Floquet operators for sample 1 at 9.8 kHz. (a) Real part of the quasienergy spectrum. (b) Imaginary part of the quasienergy spectrum.

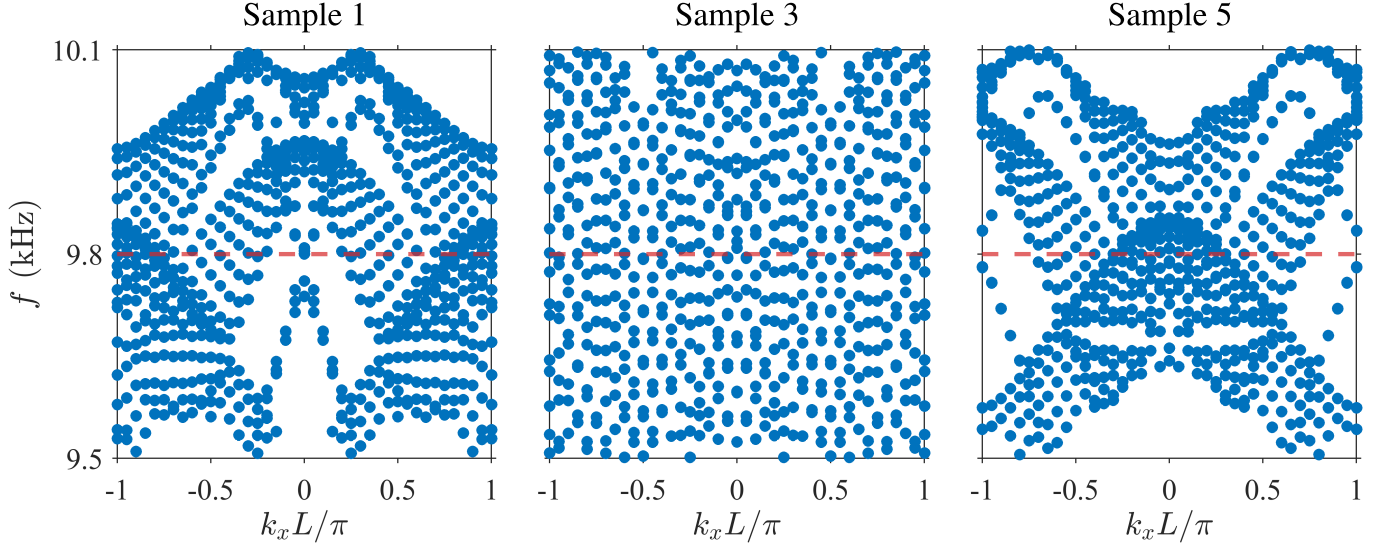


Supplementary Fig. 11 | The averaged eigenspectra for eight samples at 9.8 kHz. Red error bars indicate the standard deviation obtained from the seven eigenspectra. (a) Real part of the quasienergy spectrum. (b) Imaginary part of the quasienergy spectrum.

Supplementary Note 5. PHONONIC BANDS OF THE DESIGNED STRUCTURE

In this section, we present the phononic band structures for representative samples 1, 3, and 5, as shown in [Supplementary Fig. 12](#). The x direction is periodic while the y direction is finite. The red dashed line indicates the experimental working frequency of 9.8 kHz in our measurement. In general the phononic band structure can explain certain spectral and localization-related features. However, all these phononic band structures in [Supplementary Fig. 12](#) are gapless and do not provide the conventional band-gap-based characterization typically used to identify localized states in gapped topological systems.

For the present Floquet systems, the appropriate description should instead be based on the quasienergy spectrum, as presented in [Supplementary Figs. 10 and 11](#), as well as Fig. 3 in the main text.



Supplementary Fig. 12 | Phononic band structures of representative samples 1, 3 and 5.

Supplementary Note 6. INTERFERENCE INTENSITY CONSTRUCTION

A. Interference for previous work

In previous works, such as Refs. [15, 16], both 0 and π states can be excited simultaneously. In that situation, the wavefunction at $t = 0$ can be expressed as

$$\psi(0) = \sum_{\varepsilon} c_{\varepsilon} \phi_{\varepsilon}, \quad (\text{S6-1})$$

where ϕ_{ε} is the eigenstate at quasienergy ε with $\|\phi_{\varepsilon}\| = 1$. After evolving for a time t , the wavefunction can be expressed as

$$\psi(t) = \sum_{\varepsilon} c_{\varepsilon} e^{-i\varepsilon t - \gamma_{\varepsilon} t} \phi_{\varepsilon} \quad (\text{S6-2})$$

Here, γ_{ε} denotes the decay rate accounting for energy loss during evolution. We approximate the decay rate of all states (see [Supplementary Fig. 11b](#)) by their average value γ . Therefore, the wavefunction can be simplified as

$$\psi(t) \approx e^{-\gamma t} \sum_{\varepsilon} c_{\varepsilon} e^{-i\varepsilon t} \phi_{\varepsilon}, \quad (\text{S6-3})$$

Eq. (S6-3) is not normalized, we use the normalized one

$$\tilde{\psi}(t) \approx \sum_{\varepsilon} \tilde{c}_{\varepsilon} e^{-i\varepsilon t} \phi_{\varepsilon} \quad \text{with } \tilde{c}_{\varepsilon} = \frac{c_{\varepsilon}}{\sqrt{\sum_{\varepsilon} |c_{\varepsilon}|^2}}. \quad (\text{S6-4})$$

Once we put the source at the edge, then

$$|c_0|, |c_{\pi/T}| \gg |c_{\varepsilon}| \quad \text{for } \varepsilon \neq 0, \pi/T, \quad (\text{S6-5})$$

or

$$|\tilde{c}_0|, |\tilde{c}_{\pi/T}| \gg |\tilde{c}_{\varepsilon}| \quad \text{for } \varepsilon \neq 0, \pi/T. \quad (\text{S6-6})$$

From Eq. (S6-4)(S6-6), we have

$$\tilde{\psi}(t) \approx \sum_{\varepsilon} \tilde{c}_{\varepsilon} e^{-i\varepsilon t} \phi_{\varepsilon} = \tilde{c}_0 \phi_0 + \tilde{c}_{\pi/T} e^{-i\pi t/T} \phi_{\pi/T} + \sum_{\varepsilon \setminus \{0, \pi/T\}} \tilde{c}_{\varepsilon} e^{-i\varepsilon t} \phi_{\varepsilon} \quad (\text{S6-7})$$

$$\approx \tilde{c}_0 \phi_0 + \tilde{c}_{\pi/T} e^{-i\pi t/T} \phi_{\pi/T}. \quad (\text{S6-8})$$

From Eq. (S6-8), we have

$$\tilde{\psi}(0) \approx \tilde{c}_0 \phi_0 + \tilde{c}_{\pi/T} \phi_{\pi/T} \quad (\text{S6-9})$$

$$\tilde{\psi}(T) \approx \tilde{c}_0 \phi_0 - \tilde{c}_{\pi/T} \phi_{\pi/T} \quad (\text{S6-10})$$

$$\tilde{\psi}(2T) \approx \tilde{c}_0 \phi_0 + \tilde{c}_{\pi/T} \phi_{\pi/T} \quad (\text{S6-11})$$

Thus,

$$|\tilde{\psi}(0)| \approx |\tilde{\psi}(2T)| \neq |\tilde{\psi}(T)|. \quad (\text{S6-12})$$

Therefore, the quantity $|\tilde{\psi}(t)|$ is a reliable indicator for the presence of π states.

B. Interference in our work

As illustrated in Extended Data Fig. E6 in the main text, the wave intensity $|p(x)|^2$ can only diagnose the existence of edge states, but not their type. For a π edge state, the phase returns to its original value after $2T$. However, because the excitation of bulk states is unavoidable, the phase itself is not a reliable indicator for identifying π edge states. To overcome this issue, we define the interference intensity function

$$I(t) = |\tilde{\psi}(t) + \tilde{\psi}(0)|^2. \quad (\text{S6-13})$$

Here, we continue to use $\tilde{\psi}(t)$ rather than $\tilde{p}(x)$ for consistency. Intuitively, Eq. (S6-13) corresponds to interfering $\tilde{\psi}(t)$ with a reference “0 state” $\tilde{\psi}(0)$. From Eqs. (S6-7)(S6-13), we have

$$I(t) = 2 \sum_{\varepsilon} (1 + \cos \varepsilon t) |\tilde{c}_{\varepsilon}|^2 |\phi_{\varepsilon}|^2. \quad (\text{S6-14})$$

- 0 state exists and it is maximum excited, i.e. $|\tilde{c}_0| \gg |\tilde{c}_{\varepsilon}|$ for $\varepsilon \neq 0$, then $I(t) \approx 4|\tilde{c}_0|^2 |\phi_0|^2$. And

$$I(0) \approx I(T) \approx I(2T) \quad (\text{S6-15})$$

- π state exists and it is maximum excited, i.e. $|\tilde{c}_{\pi/T}| \gg |\tilde{c}_{\varepsilon}|$ for $\varepsilon \neq \pi/T$, then $I(t) \approx 2 \left(1 + \cos \pi \frac{t}{T}\right) |\tilde{c}_{\pi/T}|^2 |\phi_{\pi/T}|^2$. And

$$I(0) \approx I(2T) \neq I(T) \quad (\text{S6-16})$$

- Both 0 and π states exist and they are excited with comparable efficiency, i.e. $|\tilde{c}_0|, |\tilde{c}_{\pi/T}| \gg |\tilde{c}_{\varepsilon}|$ for $\varepsilon \neq 0, \pi/T$, then $I(t) \approx 4|\tilde{c}_0|^2 |\phi_0|^2 + 2 \left(1 + \cos \pi \frac{t}{T}\right) |\tilde{c}_{\pi/T}|^2 |\phi_{\pi/T}|^2$. And

$$I(0) \approx I(2T) \neq I(T) \quad (\text{S6-17})$$

- When there are no 0 or π states, Eq. (S6-13) shows that $I(t)$ becomes either a function with a longer period or an aperiodic function.

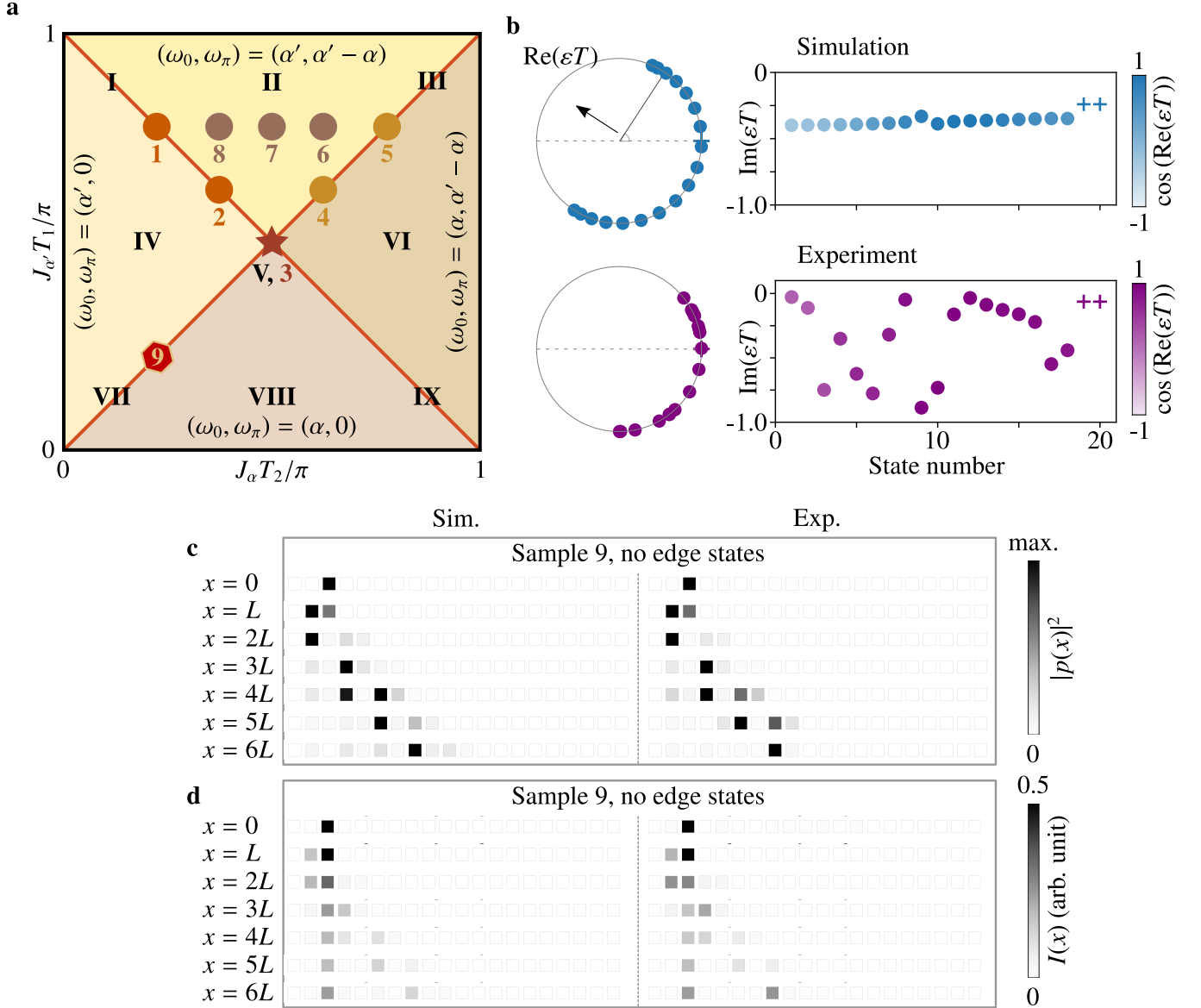
For the acoustic samples, we simply replace t with x , $\psi(t)$ with $p(x)$, and $\tilde{\psi}(t)$ with $\tilde{p}(x)$, leading to the definition:

$$I(x) = |\tilde{p}(x) + \tilde{p}(0)|^2. \quad (\text{S6-18})$$

In the main text, we present $|p(x)|^2$ and $I(x)$ for samples 1, 2, 3, 4, 5, and 7. Sample 3 does not host edge states, apart from two isolated 0 states; consequently, the wave evolves into the bulk. This behavior is observed in Extended Data Fig. E6c, where $|p(x)|^2$ becomes bulk-dominated after the evolution. Consistently, Fig. 4c in the main text shows that $I(x)$ is also dominated by bulk contributions after the evolution.

To further demonstrate that the type of edge states can be resolved by analyzing the evolution of $|p(x)|^2$ and $I(x)$, we perform an additional control experiment using the sample 9 highlight as a hexagon in [Supplementary Fig. 13a](#). From its quasienergy spectrum (see [Supplementary Fig. 13b](#)), we find that sample 9 does not host edge states, apart from two isolated 0 states. Consequently, placing the source at site 3 excites only bulk states. This phenomenon is observed in [Supplementary Fig. 13c](#), where $|p(x)|^2$ becomes dominant in the bulk after the evolution. $I(x)$ also evolves into the bulk and there is no double-period beating. In [Supplementary Fig. 13d](#), $I(x)$ exhibits a finite value at site 3, which arises from the interference between $\tilde{p}(x)$ and $\tilde{p}(0)$; the latter takes a finite value at site 3. From [Supplementary Fig. 13c,d](#), we therefore conclude that sample 9 does not host edge states, and hence there is no need to further discuss the type of edge states.

In conclusion, the interference intensity $I(t)$ is not only a reliable indicator of the presence of π states, but also serves as a more general diagnostic tool than directly analyzing $\tilde{\psi}(t)$, particularly in samples where 0 and π states do not coexist.



Supplementary Fig. 13 $|p(x)|^2$ and $I(x)$ measurements for sample 9 at 9.8 kHz. (a) Phase diagram of the Floquet SSH α -chain, reproduced from Fig. 2a of the main text, with sample 9 located on phase boundary VII. (b) Real and imaginary parts of the quasienergies. Blue and purple markers represent acoustic simulations and experiments, respectively. Two isolated 0-quasienergy edge states are marked by “+”. (c) Wave intensity profile $|p(x)|^2$. (d) Interference intensity $I(x)$.

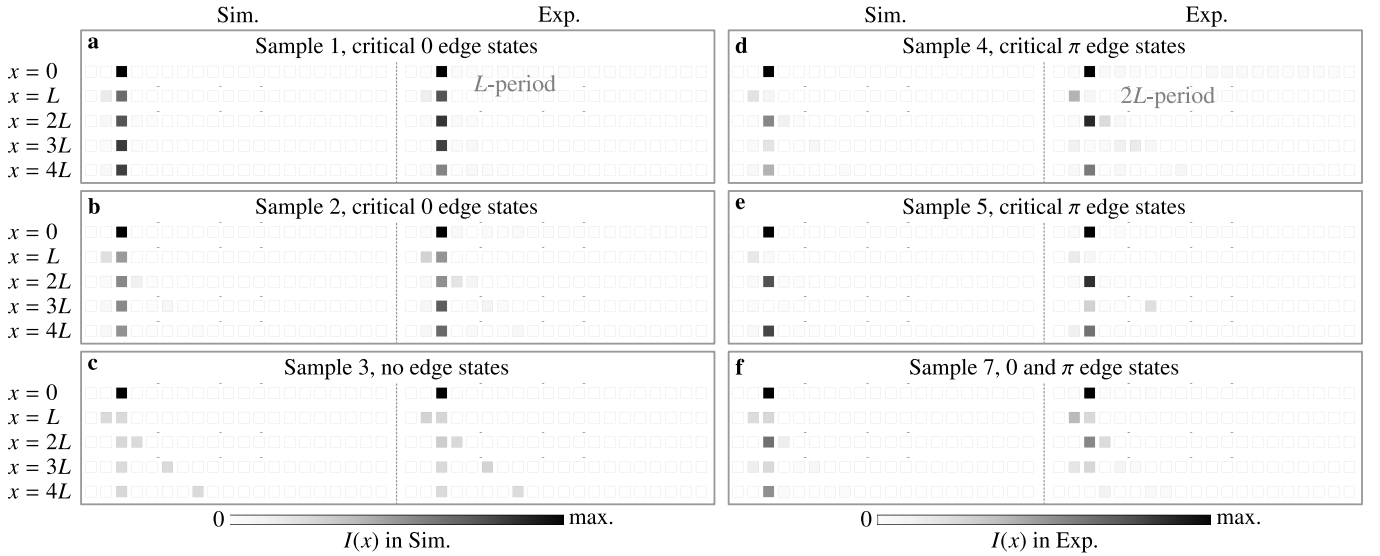
Supplementary Note 7. ROBUSTNESS OF TOPOLOGICAL EDGE STATES AGAINST STRUCTURE DEFECTS

In this section, we investigate the robustness of the topological edge states against structure defects through both numerical simulations and experimental measurements. Robustness against disorder and fabrication errors is an important hallmark of topological edge states. In our Floquet system, the topological signatures are encoded in the existence of the critical 0 and π edge states, particularly the doubled spatial periodicity associated with the π state. To examine this directly, we investigate the effect of substantial structural perturbations through both numerical simulations and experimental measurements. Two representative types of defects are considered: (i) dissipative (absorbing) defects and (ii) rigid (hard-wall) defects. In both cases, the defects are randomly distributed in the samples.

A. Dissipative (absorbing) defects

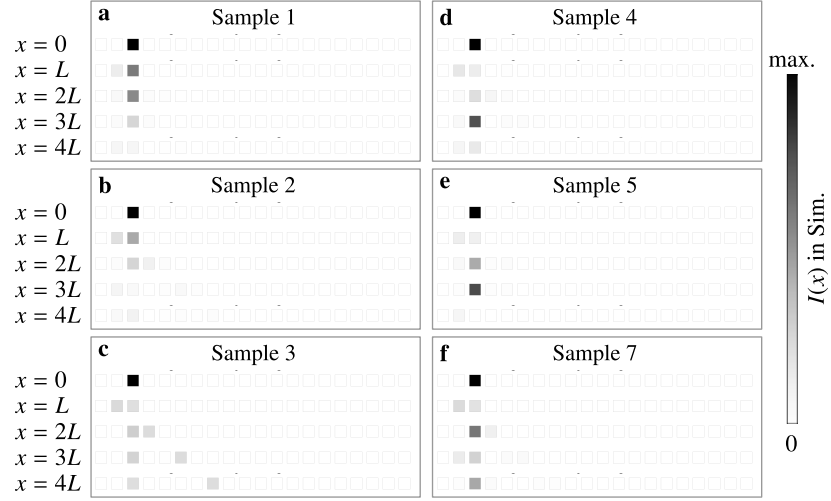
We randomly place 20 absorbing elements into 20 waveguides, with each waveguide containing one absorbing element at a random position along the x (effective time) direction. Each absorbing element has dimensions of $2\text{ mm} \times 2\text{ mm} \times 10\text{ cm}$, while the waveguides have dimensions of $8\text{ mm} \times 8\text{ mm} \times 56\text{ cm}$. In the simulations, the sound speed in air is set to $v = 343 \times (1 + 0.01j)\text{ m/s}$, while that in the absorbing material is taken as $v = 343 \times (1 + 0.1j)\text{ m/s}$ to model dissipation.

The corresponding simulation and experimental results are summarized in [Supplementary Fig. 14](#). We observe that $I(x)$ retains the characteristic doubled spatial periodicity $2L$ in the presence of critical π edge states, consistent with the Floquet π -state signature discussed in the main text. By contrast, sample 3, located at the phase-boundary intersection V and not supporting edge states, still does not carry edge states in the presence of defects. These results show that the characteristic signatures of the critical Floquet edge states arise from critical topology, rather than the introduced defects.



Supplementary Fig. 14 | Interference measurements distinguishing critical 0 and critical π edge states in presence of dissipative (absorbing) defects at 9.8 kHz. The isolated 0 edge states have been excluded. (a,b) Samples 1 and 2 on phase boundary I host critical 0 edge states. (c) Sample 3 at phase boundary intersection V does not host edge states. (d,e) Samples 4 and 5 on phase boundary III host critical π edge states. (f) Sample 7 in gapped phase II hosts both 0 and π edge states. The oscillation periods L and $2L$, corresponding to the critical 0 and π edge states respectively, are explicitly labelled in panels a and d.

We emphasize that the edge states are robust against moderate defects. As an extreme example, we model dissipation in the absorbing material by setting the sound velocity to $v = 343 \times (1 + 10j)$ m/s, corresponding to an extremely strong defect. The simulated results are shown in [Supplementary Fig. 15](#). By comparing [Supplementary Figs. 14 and 15](#), we observe that under such strong dissipation, the Floquet topological edge states are destroyed.

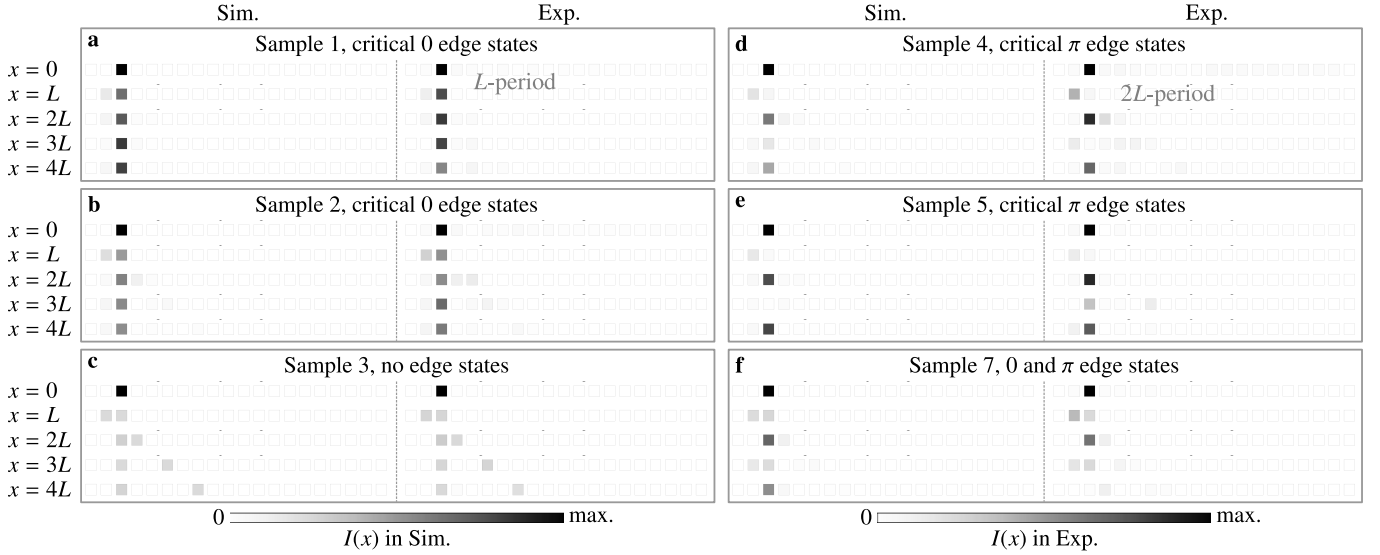


Supplementary Fig. 15 | Simulations showing interference in presence of strong dissipative (absorbing) defects at 9.8 kHz.

B. Rigid (Hard-wall) defects

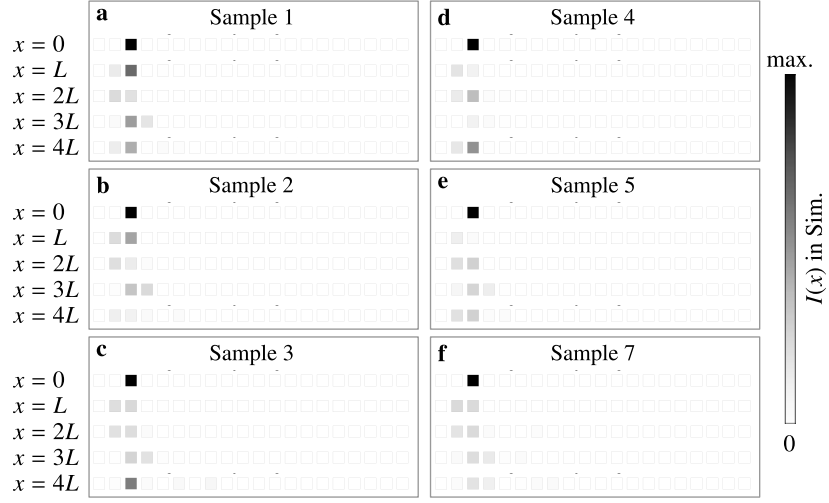
We randomly place 20 rigid elements fabricated from PLA (polylactic acid) into 20 waveguides, with each waveguide containing one rigid element at a random position along the x (effective time) direction. Each rigid element has dimensions of $2 \text{ mm} \times 2 \text{ mm} \times 10 \text{ cm}$, while the waveguides have dimensions of $8 \text{ mm} \times 8 \text{ mm} \times 56 \text{ cm}$. In the COMSOL simulations, these elements are modeled as hard-wall boundaries, as PLA can be well approximated as acoustic rigid. The sound speed in air is set to $v = 343 \times (1 + 0.01j) \text{ m/s}$.

The corresponding simulation and experimental results are summarized in [Supplementary Fig. 16](#). We observe that $I(x)$ exhibits a spatial period of $2L$ in the presence of π edge states, consistent with the characteristic doubled periodicity discussed in the main text. These results demonstrate that the characteristic signatures of the critical Floquet edge states remain observable in the presence of rigid (hard-wall) defects.



Supplementary Fig. 16 | Interference measurements distinguishing critical 0 and critical π edge states in presence of rigid (hard-wall) defects at 9.8 kHz. The isolated 0 edge states have been excluded. (a,b) Samples 1 and 2 on phase boundary I host critical 0 edge states. (c) Sample 3 at phase boundary intersection V does not host edge states. (d,e) Samples 4 and 5 on phase boundary III host critical π edge states. (f) Sample 7 in gapped phase II hosts both 0 and π edge states. The oscillation periods L and $2L$, corresponding to the critical 0 and π edge states respectively, are explicitly labelled in panels a and d.

We emphasize that the edge states are robust against moderate defects. As an extreme example, we introduce rigid defects with dimensions of $7\text{ mm} \times 7\text{ mm} \times 10\text{ cm}$, corresponding to extremely strong perturbations to the system. The results are shown in [Supplementary Fig. 17](#). By comparing [Supplementary Figs. 16 and 17](#), we observe that under such strong rigid defects, the Floquet topological edge states are destroyed.



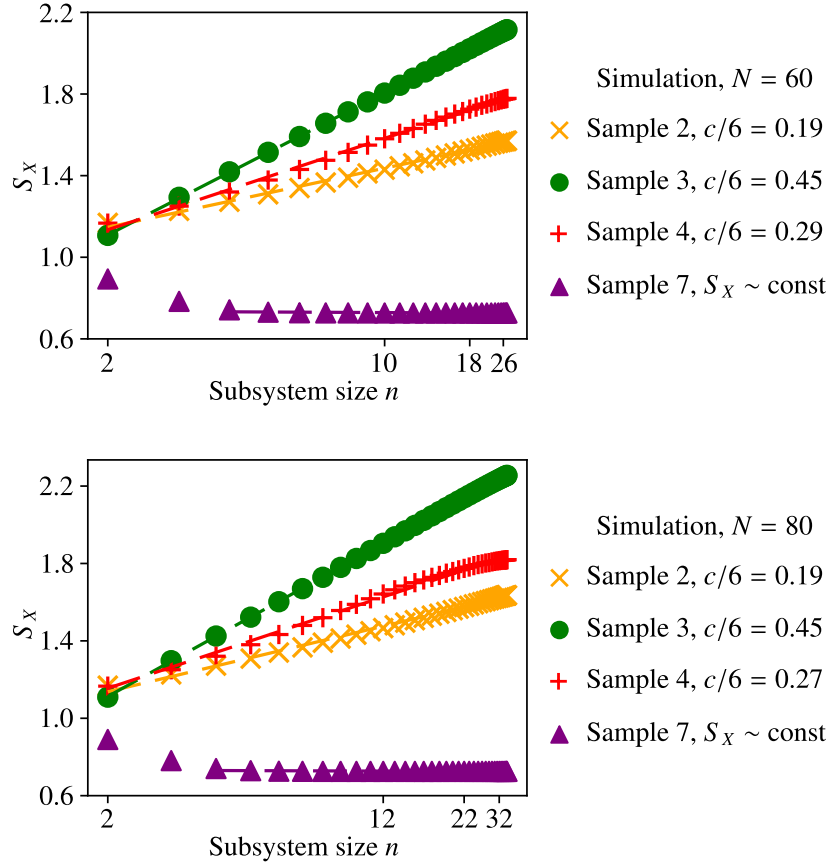
Supplementary Fig. 17 | Simulations showing interference in presence of strong rigid (hard-wall) defects at 9.8 kHz.

Supplementary Note 8. ENTANGLEMENT ENTROPY IN SIMULATION

In this section, we demonstrate that a lattice size of $N = 30$ is sufficient to observe the logarithmic scaling behavior of the entanglement entropy, and detail the reasons why the entanglement coefficient obtained from simulations differs from its tight-binding counterpart.

A. Finite size effect

In [Supplementary Fig. 18](#), we present the entanglement entropy for lattice sizes $N = 60$ and $N = 80$ obtained from simulations. Owing to the small difference in $c/6$ between $N = 60$ and $N = 80$, we infer that $c/6$ converges to 0.19, 0.45, and 0.27 for samples 2, 3, and 4, respectively, in the thermodynamic limit. These values are close to the simulation results shown in Fig. 5 of the main text. We therefore conclude that a lattice size of $N = 30$ is sufficient for the experimental observation.



Supplementary Fig. 18 | Entanglement entropy with different lattice size N in simulation. $N = 60$ and $N = 80$ lattice sizes are used. The working frequency is 9.8 kHz. The sound velocity is $v = 343\text{m/s}$.

Despite finite-size effects, the upward deviation of the entanglement coefficients from their theoretically expected values can arise from strong critical fluctuations of the Floquet band structure at criticality, induced by long-range coupling perturbations in both simulations and experimental samples. In particular, the quasienergy scale of the original short-range coupling model becomes vanishingly small near critical (Dirac) points, such that even weak kinetic perturbations can substantially reshape the touching structure of the Floquet bands. This, in turn, leads to enhanced entanglement coefficients as the effective coupling range increases. This mechanism will be demonstrated numerically in the next subsection.

Meanwhile, we note that a robust sum rule, $c_2 + c_4 \simeq c_3$, holds among the entanglement coefficients of samples 2, 3, and 4. This relation is enforced by the nontrivial topology at the multicritical point (sample 3), thereby providing strong evidence for the existence of a critical topological phase transition in our system.

B. Long-range couplings

As listed in [Supplementary Table 6](#), $c/6$ converges to $1/6$ or $1/3$ in the tight-binding model as the lattice size $L = 2N$ becomes large. In [Supplementary Note 2](#), we use acoustic structure to realize the Hamiltonian

$$H(k, t) = \begin{cases} H_2(k), & t \in [\ell T, \ell T + T_1) \\ H_1(k), & t \in [\ell T + T_1, \ell T + T_1 + T_2) \end{cases}, \quad (\text{S8-1})$$

where t denotes time, $\ell \in \mathbb{Z}$, $T = T_1 + T_2$ is the driving period, $k \in [-\pi, \pi)$ is the quasimomentum with $\sigma_{x,y,z}$ denoting the Pauli matrices. We assume

$$H_1(k) = J_1 (\cos k\sigma_x + \sin k\sigma_y), \quad (\text{S8-2})$$

$$H_2(k) = J_2 (\cos 2k\sigma_x + \sin 2k\sigma_y). \quad (\text{S8-3})$$

and J_1, J_2 are listed in [Supplementary Table 9](#). However, longer-range couplings inevitably arise once link tubes are introduced between the waveguides. In other words, the realized system might be approximately modeled as

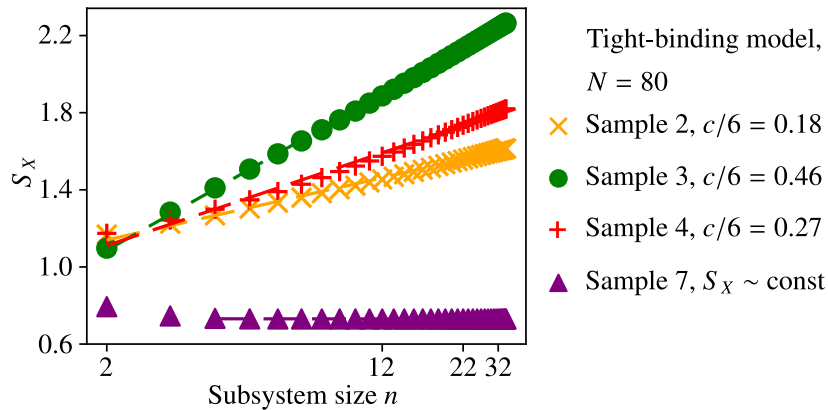
$$H_1(k) = J_1 (\cos k\sigma_x + \sin k\sigma_y) + \sum_{n=3} J_n (\cos nk\sigma_x + \sin nk\sigma_y), \quad (\text{S8-4})$$

$$H_2(k) = J_2 (\cos 2k\sigma_x + \sin 2k\sigma_y) + \sum_{n=3} J_n (\cos nk\sigma_x + \sin nk\sigma_y). \quad (\text{S8-5})$$

Here, we show that introducing small J_3 and J_4 into the tight-binding model causes $c/6$ to deviate from $1/6$ and $1/3$, consistent with the results shown in [Supplementary Fig. 18](#). The parameters are listed in [Supplementary Table 10](#), and the entanglement entropy calculated from the tight-binding model with lattice size $N = 80$ is shown in [Supplementary Fig. 19](#), which is nearly identical to that in [Supplementary Fig. 18](#).

Supplementary Table 10 | Long-range couplings J_3 and J_4 .

Sample	$J_1 T_2 / \pi$	$J_2 T_1 / \pi$	$J_3 T / 2\pi$	$J_4 T / 2\pi$
2	0.35	0.65	0.001	0
3	0.50	0.50	-0.002	0.002
4	0.65	0.65	-0.005	0
7	0.50	0.73	0.001	0



Supplementary Fig. 19 | Entanglement coefficient with lattice size $N = 80$ in tight-binding model. The coupling values are listed in [Supplementary Table 10](#).

We observe that, once longer-range coupling terms J_n are present at the gapless phase boundary, the asymptotic behavior of the entanglement entropy changes abruptly.

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