

Observation of critical topological phase transition

Corresponding Author: Professor Baile Zhang

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This file contains all reviewer reports in order by version, followed by all author rebuttals in order by version.

Version 0:

Reviewer comments:

Referee #1

(Remarks to the Author)

The authors have done experiment on acoustic metamaterials to demonstrate a parallel category: the critical topological phase transition, which unifies the concepts of criticality and topology. By introducing critical topological invariants at gapless phase transition boundaries, They have observed critical edge states emerging precisely at the critical boundary. The authors have further demonstrate that the critical boundary itself can undergo a topological phase transition, revealing that phase transitions can even occur along phase boundaries. These findings show that critical phenomena and topology are not mutually exclusive but deeply intertwined in the fundamental nature of phase transition.

I strongly recommend this paper for the publication in Springer-Nature but after some revision, I would like to go through the revised version of the manuscript before I finally accepted it for publication.

I am requesting authors, please notice the following points and modified the manuscript accordingly.

(1). Please reduced the abstract, only emphasize the main points of findings.

(2). Introduction has written quite nicely by the authors. But the citations are not complete. Please add some more citations on Pump probe method. Please add some more reference on Critical Topological Phase Transition.

a. Scientific Reports |
(2021) 11:1004
| <https://doi.org/10.1038/s41598-020-80337-7>

b. Scientific Reports |
(2025) 15:5916
| <https://doi.org/10.1038/s41598-025-89901-5>

(3). Critical topological phase transition: What is the speciality on the word "Criticality". There are any topological transitions which are the non-critical ? Please explain this explicitly in the revised version of the manuscript.

(4). Page-2 : First column and last paragraph: Please clarify this part of the results more explicitly.

(5). Page-3, Fig.1: Lower part of the figure, where inject of the sound wave present. Please include one more figure on more Floquet periods in the "Method" section.

(6). Page-4, Fig. 2B: Component I and V, one is four zero modes , the spectrum is gapped and the two zero mode spectrum is gapless. Why they behave differently? Similarly for the component II and IV. Please explain these incident physically in the revised version of the manuscript.

(7). Page-5, Fig.3 : I am requesting authors to present the result of imaginary part of energy also in this figure. Please mention explicitly the time scale of quasiparticle life time.

(8). Page-6, Observation of critical edge states:

Why the dynamics of wavefunction are predominantly governed by the edge mode? Why the bulk modes are inactive? Also illustrate when the bulk modes are predominant than the edge modes.

(9). Page-6: Why the evidence of π edge modes are so important? Why it is explain the physics of criticality and topology beyond the conventional paradigms of statistical and condensed matter physics?

I will strongly recommend this paper for publication after these modification. I will also go through the revised manuscript.

Referee #2

(Remarks to the Author)

Zheyu Cheng et al., reported the experimental observation of a critical topological phase transition (CTPT) in a time-periodic acoustic lattice. The authors implement a one-dimensional chiral Floquet system using a network of coupled acoustic waveguides, in which the couplings can be precisely tuned by adjusting the link geometries. The work presents an innovative approach utilizing an acoustic framework to address a long-standing conceptual problem in defining and measuring topological invariants at critical boundaries, which may have a broad scientific impact beyond acoustic studies. Yet before my recommendation of its publication in Nature, several critical issues need to be addressed first:

1. Finite-size and boundary effects: Each sample implements only four Floquet periods, leading to eigenphase deviations from $0/\pi$ "due to finite size." Without a scaling analysis, it is difficult to separate finite-size artifacts from genuine critical behavior. Simulations for 3–5 different lengths should be added to estimate eigenphase drift and to justify quantitative comparisons.
2. Loss: Experimentally, losses are non-negligible (microphone insertion, absorption, 16-day measurement). The SI includes damping in simulations, but the retrieval assumes a unitary F . The authors need to clarify whether the measured F is projected onto the nearest unitary, how loss affects the eigenphase distribution, and whether the invariants are robust to weak non-unitarity.
3. Identification of "critical" edge modes: In the SI, the authors acknowledge that small bulk admixtures can significantly affect phase, leading to apparent beating even without genuine π modes. A control experiment on a sample where π modes are theoretically absent but with similar loss/reflection would test the specificity of the diagnostic.
4. Working frequency: All results are shown at a single working frequency (9.8 kHz) defined with zero-mode wavevector. Yet it would be helpful to see the robustness of that with a small variation of the exact frequency.
5. Normally, the quantum criticality often implies a divergence of the corresponding length scale, yet in the discussion of CTPT, this point is never clearly addressed. Can the authors comment on whether there is any implication from their experiments?
6. Trivial isolated zero modes: Given that first and last guides are disconnected, a pair of trivial zero modes will always be produced. In the manuscript, it is somewhat mixed with the critical zero mode. It would be great if the authors could separate them more visually in the main figures.

Version 1:

Reviewer comments:

Referee #1

(Remarks to the Author)

Final Acceptance Report:

Title: Observation of critical topological phase transition

Authors: Zheyu Cheng , Xiuhai Zhang , Xue-Jia Yu , Longwen Zhou , Jiangbin Gong , and Baile Zhang

The authors have revised the manuscript according to my comments and suggestion. The authors have solved a new and important problem on critical topological transition.

The authors have found new and important results, which is consistent physically. I strongly recommended this paper for publication in Nature journal.

Referee #2

(Remarks to the Author)

The authors have addressed all my previous concerns in the revised manuscript. I'm now happy to recommend it for publication in Nature.

Referee #3

(Remarks to the Author)

The first and second part of the Results are not well organized, making the paper very difficult to understand. In particular, the discussion jumps frequently between figure panels in Fig. 1 and Fig. 2.

The designed structure in Fig. 1b apparently includes third-order nonlocal interaction, which has been studied and known to support topological edge states and unusual dispersion relations (see Physical Review Applied. 2023 May 1;19(5):054028; Nature Reviews Physics. 2025 Jun;7(6):299-312; Communications Physics. 2023 Nov 21;6(1):334.). The authors should clarify the difference between their study and previous work.

I suggest the authors to also plot the phonon bands of their designed structure in Fig. 1b. The transverse direction can be two different cases; one is periodic and the other is finite width. I believe band structures can already explain a lot of the wave phenomena, such as an inject wave can be localized or extending (Extended Data Fig. 1) in the transverse direction depending on the frequency. If everything can be explained by the band structure, then the advantages of the explanation with the topological critical phase transition should be more explicitly discussed.

Since the current design contains nonlocal interaction, is it a must to have nonlocal coupling to realize the critical topological phase transition? If not, can the authors provide additional examples for such topological critical phase transition.

The authors claim that the observed edge states is topological. It is better to provide experiments to show that the edge states are robust to disorder or fabrication errors.

In the main text, a period of $2L$ is used to distinguish the 0 and the π edge states. However, from the provided Fig. 4 and Extended Data Fig. 5, it is not clearly where the period of $2L$ is? Can the authors be more specific here? Additionally, is this related to the even-odd effect in some nonlocal systems (see Nature Physics. 2018 Jan 1;14(1):40-44; Nature communications. 2024 Oct 24;15(1):8882.)

In Fig. 1b, it seems that there are curved waveguides underneath the entire plate like structure. If this is needed, the authors should make these parts more visible. It is also recommended to show a unit cell of the designed structure. There are two straight waveguides at the left and the right side of the structure, which are not coupled to other waveguides. What are the purpose?

The authors should clarify whether the experiments are performed with time-harmonic excitation or transient excitation. And what are the frequencies used for experiments?

In Fig. 2b, how the 0 and π states looks like in real space? The authors should provide the Bloch modes.

It is unclear how the imaginary component of the quasi-energy band is obtained? Did the author fit the transient response with exponentially decay behaviors or derive the imaginary component in some other ways?

The "+" marker in Fig. 3 is hardly visible.

Version 2:

Reviewer comments:

Referee #3

(Remarks to the Author)

still ok to publish it.

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Response Letter for the Manuscript

“Observation of critical topological phase transition”

(Nature 2025-09-23087)

Dear Referees,

We sincerely appreciate your constructive and detailed feedback on our work. In this response letter, we will address all your questions and explain in detail how they have helped us to improve our work and manuscript accordingly.

We thank Referee #1 for the positive and encouraging evaluation of our work. In particular, Referee #1 stated that “*I strongly recommend this paper for publication in Springer-Nature,*” and highlighted that our work unifies the concepts of criticality and topology, with a clearly written Introduction. Below, we provide detailed responses to all comments and questions raised by Referee #1.

We thank Referee #2 for the careful and thoughtful evaluation of our manuscript. Referee #2 described our work as “*an innovative approach... to address a longstanding conceptual problem... which may have a broad scientific impact...*”, and raised a number of insightful questions that helped us to further clarify the conceptual nature of the criticality and strengthen the presentation of our results. We have carefully addressed all comments and suggestions raised by Referee #2 and provide detailed responses below.

In the text below, your comments are quoted in *Italics* and followed by our detailed response. We have also revised the manuscript and Supplemental Information based on your comments, and these updates are highlighted in blue in those files. In the text below, the references to these updates are highlighted in blue and by a vertical red line in the left margin.

Best regards,
All the authors

Response to Referee #1

Dear Referee #1:

We thank you for your insightful comments, which prompted us to clarify why the criticality in our work is special and why the interplay between criticality and topology goes beyond conventional frameworks.

In response, we have substantially revised the Abstract and Introduction to better articulate the conceptual significance of the work, and we have performed new experiments probing the criticality through measurements of entanglement entropy, providing an independent and quantitative diagnostic.

We hope that these revisions address your concerns regarding the role of the criticality in our system.

Referee #1 – General Comments:

The authors have done experiment on acoustic metamaterials to demonstrate a parallel category: the critical topological phase transition, which unifies the concepts of criticality and topology. By introducing critical topological invariants at gapless phase transition boundaries, they have observed critical edge states emerging precisely at the critical boundary. The authors have further demonstrated that the critical boundary itself can undergo a topological phase transition, revealing that phase transitions can even occur along phase boundaries. These findings show that critical phenomena and topology are not mutually exclusive but deeply intertwined in the fundamental nature of phase transition.

I strongly recommend this paper for the publication in Springer-Nature but after some revision, I would like to go through the revised version of the manuscript before I finally accepted it for publication.

...

I will strongly recommend this paper for publication after these modification. I will also go through the revised manuscript.

Response from authors:

Thank you very much for your highly positive evaluation of our work, including the assessment that “*These findings show that critical phenomena and topology are not mutually exclusive but deeply intertwined in the fundamental nature of phase transition...*” as well as your statement that you “*will strongly recommend this paper for publication after these modification*”.

Your comments and suggestions have been invaluable in improving the quality, clarity, and conceptual presentation of the manuscript. We greatly appreciate these insightful points. Below, we will address each of your comments and questions point by point.

Referee #1 – Comment 1:

I am requesting authors, please notice the following points and modified the manuscript accordingly.

(1) Please reduced the abstract, only emphasize the main points of findings.

Response from authors:

Thank you for this suggestion. In revising the abstract, we have streamlined its structure and sharpened its focus on the main experimental findings. Compared to the previous submission, we have removed specific textbook-style illustrative examples (such as ferromagnetic-paramagnetic transition, the quantum Hall effect, and topological insulators) and reorganized the background discussion at a more conceptual and universal level. This allows the abstract to focus more directly on the demonstration of the critical topological phase transition, the emergence of critical edge states along phase boundaries, and the topology-enforced multicritical point. In addition, **the revised abstract now explicitly highlights the newly added measurements of entanglement entropy**, which provide a quantitative and universal diagnosis of criticality and demonstrate how critical behavior is reorganized by topology.

Although the overall length remains comparable, the abstract is now more focused on key results and places clearer emphasis on the central findings. We believe this balance is appropriate given the fundamental nature of the work and the broad readership of *Nature*.

Referee #1 – Comment 2:

(2) Introduction was written quite nicely by the authors. But the citations are not complete. Please add some more citations on Pump probe method. Please add some more reference on Critical Topological Phase Transition.

a. Scientific Reports | (2021) 11:1004 | <https://doi.org/10.1038/s41598-020-80337-7>

b. Scientific Reports | (2025) 15:5916 | <https://doi.org/10.1038/s41598-025-89901-5>

Response from authors:

Thank you for this suggestion. In the revised manuscript, we have included the references suggested by the referee [*Sci. Rep.* 11, 1004 (2021), *Sci. Rep.* 15, 5916 (2025), now cited as Ref. [36] and [37]] and supplemented the Introduction with additional citations related to critical topological phase transitions. These references are now cited at appropriate locations where critical topological invariants and topology-enforced multicriticality are discussed. We believe that these revisions improve both the clarity and the organization of the manuscript.

We have reorganized the presentation of the pump–probe methodology by moving its discussion from the Introduction to the Methods section, where it is now described explicitly in the context of the experimental measurements. Specifically, the pump–probe protocol is presented in the “**Experimental measurements**” subsection of the Methods, and the relevant references [*Nat. Rev. Mat.* 7, 974 (2022), now cited as Ref. [48]] on this methodology have been added there. This restructuring allows the Introduction to remain focused on the conceptual motivation, while the technical aspects of the pump–probe implementation are

presented in a more appropriate experimental framework.

Referee #1 – Comment 3:

Critical topological phase transition: What is the speciality on the word “Criticality”. There are any topological transitions which are the non-critical? Please explain this explicitly in the revised version of the manuscript.

Response from authors:

Thank you for raising these important conceptual questions: 1) What is the specialty of “criticality” in our “critical topological phase transition”? 2) Is there any topological transition that is non-critical? These two questions are closely related. Addressing them allows us to clarify the conceptual framework underlying our “critical topological phase transition” and to more sharply articulate the novelty and significance of our work.

We begin by clarifying the conventional distinction between **criticality** and **topology**.

(1) Criticality and topology as traditionally incompatible organizing principles

The word “criticality” stems from the Landau-Ginzburg paradigm of phase transitions, which occur at critical points where a local order parameter vanishes smoothly. This vanishing is accompanied by the closing of the energy gap for collective fluctuations and the divergence of the correlation length, driving the system into a scale-invariant regime—at **criticality, microscopic details become largely irrelevant, and the behavior of the system is governed instead by universal macroscopic structures that encode information persisting across all length scales** [see detailed discussions in Landau and Lifshitz, “Statistical Physics” (Elsevier, 2013), or Sachdev, “Quantum Phase Transitions” (Cambridge University Press, 2011)]. These defining features establish criticality as an organizing principle for phase transitions.

Topology provides a fundamentally different organizing principle for phase transitions. Topological phases, such as the quantum Hall effect and topological insulators, are characterized by discrete, quantized invariants defined for gapped bulk systems. The discreteness of these invariants ties their definition intrinsically to a finite bulk energy gap and precludes any continuous interpolation between topologically distinct phases. Consequently, topological phase transitions are realized at bulk gap closings, where the invariants cease to be well defined. This intrinsic feature exposes a fundamental incompatibility between topology and criticality.

Now we answer your second question—Is there any topological transition that is non-critical? While topological phase transitions do occur at gap closings and are therefore critical in this sense, **topology and criticality do not act simultaneously at the phase boundary**. For example, in the Haldane model of a Chern insulator, the phase transition connects two topologically distinct gapped phases characterized by different Chern numbers, but, exactly at the phase boundary, the Chern number is no longer well defined. Thus, **although the phase boundary is critical, criticality and topology do not operate simultaneously**.

This conceptual distinction sets the stage for recent theoretical developments that address this incompatibility.

(2) Critical topology: overcoming the conventional incompatibility

Recent theories have introduced the concept of critical topology, in which topology acts directly on gapless critical phase boundaries rather than on gapped bulk phases [see discussions in *Phys. Rev. X* 7, 041048 (2017)] and many other following works in the past few years; we have cited an non-exhaustive list of references as Refs. [10–14, 18–40] in the revised manuscript). In this setting, topological invariants can be extended to critical phase boundaries, giving rise to critical topological invariants that protect edge states precisely at criticality. As a result, topology and criticality can coexist along gapless phase boundaries, overcoming the incompatibility implied by their conventional formulations.

Now we answer your first question—What is the specialty of “criticality” in our “critical topological phase transition”?

(3) In our “critical topological phase transition”, topology not only survives criticality, but can actively reorganize criticality

Once the conventional incompatibility between topology and criticality is lifted, topology can assume a more active role by reorganizing criticality itself. Endowing critical phase boundaries with a topological structure implies that a topological phase transition between distinct critical boundaries necessarily induces a reorganization of the underlying critical structure. At the intersection of such topologically distinct critical boundaries, this reorganization gives rise to an isolated multicritical point [see similar multicritical points in different scenario such as in *Phys. Rev. Lett.* 120, 057001 (2018), *Sci. Rep.* 11, 1004 (2021), *Sci. Rep.* 15, 5916 (2025)], at which the organization of criticality is fundamentally altered. **This defines a critical topological phase transition (CTPT), in which topology actively drives a phase transition within the critical regime.**

Revealing such a topology-driven reorganization of criticality requires a diagnostic that captures how critical correlations are organized across scales—a role naturally fulfilled by the entanglement entropy. In the revised manuscript, we therefore include new measurements of entanglement entropy, which provide a quantitative experimental diagnosis of this reorganization (see Fig. 5 and the related discussion in the main text).

Following your suggestion, we have substantially rewritten the [Introduction](#) in the revised manuscript, structuring it around the above points (1–3) in order to clearly articulate the conceptual framework, significance, and novelty of the critical topological phase transition.

Referee #1 – Comment 4:

Page-2: First column and last paragraph: Please clarify this part of the results more explicitly.

Response from authors:

Thank you for pointing this out. We have revised the last paragraph of the first column on page 2 to clarify the physical meaning of the results and to make the logical connection between the Floquet spectrum, topological invariants, and critical edge states more explicit. In particular, we now explain the bulk topology is characterized by a pair of integer winding numbers (ω_0, ω_π) , the numbers of edge states pinned at quasienergies 0 and π , denoted by N_0 and N_π , obey the bulk-edge correspondence $(N_0, N_\pi) = 2(|\omega_0|, |\omega_\pi|)$. This correspondence remains valid even when the quasienergy spectrum becomes gapless at $\varepsilon = 0$ and π [*Commun. Phys.* 8, 214 (2025)]. As a consequence, critical 0 and π edge states can emerge along gapless phase boundaries.

The revised text appears on page 2, beginning with the Paragraph 2 in the section of “[Theoretical framework and phase diagram](#)”, and now reads as follows:

“Each instantaneous Hamiltonian belongs to the BDI symmetry class and contains only short-range couplings. However, the periodic driving generates effective long-range couplings in the Floquet description, giving rise to a rich quasienergy structure and bulk topology [41]. The bulk properties are encoded in the quasienergy spectrum of the one period evolution operator F , whose eigenvalues appear in pairs $e^{\pm i\varepsilon(k)}$, symmetric about quasienergies $\varepsilon = 0$ and π .

A globally gapped Floquet phase exists only when both the 0 and π quasienergy gaps are open. Closing either gap drives the system onto a gapless critical phase boundary. The resulting phase diagram is shown in Fig. 2a, where fully gapped phases are separated by several gapless boundaries. When the bulk is globally gapped, its topology is characterized by a pair of integer winding numbers (ω_0, ω_π) associated with the 0 and π gaps, respectively (see Methods). Under open boundary conditions, the corresponding numbers of edge states (N_0, N_π) obey a bulk–edge correspondence, $(N_0, N_\pi) = 2(|\omega_0|, |\omega_\pi|)$.

Remarkably, this correspondence remains valid even when the bulk becomes gapless at $\varepsilon = 0$ or π [40]. As a consequence, critical 0 and critical π edge states can persist along gapless phase boundaries (Fig. 2b). The type of critical edge states is determined by which quasienergy gap remains open. ...”

In addition, we have expanded the discussion in the subsection of “[Topological invariants](#)” in Methods (Page 7 and 8) to provide a clearer theoretical justification for the quantization of the introduced topological invariants in both gapped and gapless regimes, and to clarify how these invariants correctly count the surviving critical edge states. We believe that these revisions significantly improve the clarity and accessibility of this part of the manuscript.

Referee #1 – Comment 5:

Page-3, Fig.1: Lower part of the figure, where inject of the sound wave present. Please include one more figure on more Floquet periods in the “Method” section.

Response from Authors:

Thank you for this helpful comment. Following your suggestion, and in accordance with *Nature's* formatting requirements, we have added additional simulations to [Extended Data Fig. E1](#) to present the Floquet response over **eight** driving periods, demonstrating the robustness and reproducibility of the Floquet characterization beyond a single period. We believe that these additional figures complement Fig. 1 and clarify the implementation of the sound-wave injection protocol.

Referee #1 – Comment 6:

Page-4, Fig. 2B: Component I and V, one is four zero modes, the spectrum is gapped and the two zero mode spectrum is gapless. Why they behave differently? Similarly for the component II and IV. Please explain these incident physically in the revised version of the manuscript.

Response from authors:

Thank you for raising this question. The different behaviors observed for the cases labelled I, II, IV, and V in Fig. 2b originate from their distinct configurations of the quasienergy gaps at $\varepsilon = 0$ and $\varepsilon = \pi$, which determine whether the Floquet spectrum is globally gapped or gapless and which topological invariants remain operative.

Specifically, case I has an open gap at $\varepsilon = 0$ but a closed gap at $\varepsilon = \pi$. As a result, although the spectrum appears locally gapped around $\varepsilon = 0$, the Floquet quasienergy spectrum is **globally gapless** on the quasienergy circle. In this critical regime, topology remains well defined only at $\varepsilon = 0$, leading to the protection of four zero-energy critical edge modes.

By contrast, cases II and IV correspond to **fully gapped Floquet phases**, in which both the $\varepsilon = 0$ and $\varepsilon = \pi$ gaps are open. In these regimes, topological invariants associated with both gaps are well defined, allowing distinct combinations of protected edge modes. In particular, case II supports both zero and π edge modes, whereas in case IV only zero-energy edge modes are present, reflecting different topological indices in the two gaps.

Case V lies at a more singular critical point where **both the $\varepsilon = 0$ and $\varepsilon = \pi$ gaps are closed**, rendering the Floquet spectrum fully gapless. In this situation, only a reduced set of critical zero-energy edge modes survives, resulting in two zero modes and no π modes.

Overall, the distinct spectral features of cases I, II, IV, and V can be uniquely traced to which of the two Floquet quasienergy gaps at $\varepsilon = 0$ and $\varepsilon = \pi$ are open or closed; this gap configuration fully determines whether the spectrum is globally gapped and which edge modes are topologically protected. We have revised the main text to explicitly clarify this physical origin.

Following your suggestion, we have revised the main text in [Introduction](#) and section [“Theoretical framework and phase diagram”](#) to explicitly clarify this physical origin, which reads as follows:

Page 2, last paragraph in Introduction:

“... Unlike static systems, where topological transitions are typically confined to isolated bulk-gap closings, the Floquet quasienergy spectrum—defined on a compact circle—supports two distinct quasienergy gaps at 0 and π , whose intertwined topological roles give rise to extended gapless critical phase boundaries....”

Page 2, Paragraph 4 in the section of “Theoretical framework and phase diagram”:

“... The type of critical edge states is determined by which quasienergy gap remains open. For example, along phase boundary I, the π gap is closed while the 0 gap remains open, hosting four 0-quasienergy edge states. Among these, two originate from the end sites that are disconnected from the main chain (Fig. 1a). These states remain fully localized at the isolated sites and persist across the entire phase diagram. We refer to them as isolated 0 edge states and mark them explicitly by “+” in the quasienergy spectra to distinguish them from other edge states whose number varies with the bulk topology. By contrast, along phase boundary III the π gap remains open and hosts two π edge states, while the 0 gap is closed. At the intersection point V (marked by a star in Fig. 2a) of these phase boundaries, where both quasienergy gaps close simultaneously, only the two isolated 0 edge states persist. A complete summary of the number and type of edge states in each phase is provided in Table I in Methods.”

More detailed derivations and technical discussions underlying this interpretation are provided in the [Methods](#) and [Supplementary Note 1](#).

Referee #1 – Comment 7:

Page-5, Fig.3: I am requesting authors to present the result of imaginary part of energy also in this figure. Please mention explicitly the time scale of quasiparticle life time.

Response from authors:

Thank you for this helpful suggestion. Following your request, we have included the imaginary part of the quasienergy in the revised manuscript. To keep the main text focused, the imaginary components are presented in [Extended Data Fig. E3](#). To enable a direct comparison, [Extended Data Fig. E3](#) is organized such that panels (a–c) reproduce the real parts of the quasienergy shown in Fig. 3 in the main text, while panels (d–e) present the corresponding imaginary parts.

The imaginary part reflects dissipation-induced decay in the acoustic system and defines the finite quasiparticle lifetime. We now explicitly state the corresponding lifetime scale in Paragraph 3 in the section of “[Quasienergy spectrum reconstruction](#)” (Page 5 left column), which reads as follows:

“The imaginary parts of the quasienergies, which quantify losses in the samples, are shown in [Extended Data Fig. E3d–e](#). From these data, we extract $\text{Im}(\epsilon T) \approx -0.5$, corresponding to a quasiparticle lifetime $\tau \approx 2T$. Since the acoustic field is sampled at intervals of $0.5T$, this lifetime is sufficient to ensure reliable reconstruction of the quasienergy spectrum.”

Referee #1 – Comment 8:

Page-6, Observation of critical edge states: Why the dynamics of wavefunction are predominantly governed by the edge mode? Why the bulk modes are inactive? Also illustrate when the bulk modes are predominant than the edge modes.

Response from authors:

Thank you for raising this important question. The predominance of edge states (whenever they exist) over bulk states in our experiment originates primarily from **mode matching between the excitation profile and the spatial structure of the edge states**.

It is well known that when the spatial profile of a source closely matches that of an eigenstate, that eigenstate is preferentially excited—a mechanism commonly referred to as **mode matching**. In our excitation scheme, the injected field is highly localized near the boundary, closely matching the localized wavefunctions of the critical edge modes. Bulk states, by contrast, are spatially extended over the entire system and therefore exhibit a much weaker overlap with the localized excitation, leading to a significantly reduced excitation efficiency.

Bulk states become predominant under different conditions. First, when the spatial overlap between the excitation source and bulk eigenstates is enhanced while that with edge states is suppressed—for example, when the source is moved deep into the bulk away from the system boundary. Second, when edge states do not exist, the excitation can only couple to bulk states, and the resulting dynamics are necessarily governed by bulk modes.

In the revised manuscript, we provide a more detailed illustration of wavefunction dynamics governed by edge and bulk states in [Extended Data Fig. E1](#). To clarify the excitation efficiency, we have revised Paragraph 3 in the section of “[Critical 0 and \$\pi\$ edge states](#)” (Page 6 right column), which now reads as follows:

“... Although this excitation may weakly overlap with bulk states, such contributions are negligible compared to those of the edge states, and the subsequent dynamics are dominated by edge-state components whenever they are present (see Extended Data Fig. E1).”

Referee #1 – Comment 9:

Page-6: Why the evidence of π edge modes are so important? Why it is explain the physics of criticality and topology beyond the conventional paradigms of statistical and condensed matter physics?

Response from authors:

Thank you for raising this important conceptual question. The observation of critical π edge modes is essential because it reveals a form of topology and criticality that cannot be captured within conventional Landau–Ginzburg descriptions or static topological classifications.

In static systems, topological edge states—when protected by chiral or related symmetries—are pinned to symmetry-defined mid-gap reference energies, most commonly zero energy. By contrast, in periodically driven (Floquet) systems, the quasienergy spectrum is defined modulo 2π , forming a compact circle that admits **two distinct symmetry-protected reference energy**

points at $\varepsilon = 0$ and $\varepsilon = \pi$. Edge states pinned at $\varepsilon = \pi$ therefore have no static counterpart and represent intrinsically dynamical topological phenomena.

In the context of critical topological phase transitions, the importance of π edge states is twofold.

(1) The survival of π edge states at gapless phase boundaries demonstrates that topological protection can persist even when the bulk spectrum is critical, thereby overcoming the conventional incompatibility between topology and criticality.

(2) The coexistence and reconfiguration of critical 0 and critical π edge states provide evidence that topology actively reorganizes criticality, rather than merely becoming ill-defined at gap closings. This reorganization culminates at the multicritical point, where distinct critical phase boundaries—characterized by different configurations of 0 and π edge modes—intersect.

For these reasons, the observation of critical π edge modes is crucial for establishing that our observed “*critical topological phase transition*” goes beyond both conventional statistical-mechanical criticality and standard topological phase transitions.

In the revised manuscript, we have explicitly highlighted the role of critical π edge states in Paragraph 2 in the section of “[Critical 0 and \$\pi\$ edge states](#)” (Page 6 left column):

“... The emergence of such critical π edge states—unique to Floquet systems—highlights the suitability of the Floquet platform for realizing and diagnosing the CTPT.”

Response to Referee #2

Dear Referee #2:

We would like to express our gratitude for your positive comments and suggestions. Your comments have enabled us to explain the underlying mechanism of our work more clearly, expand the reach of our study and to clarify many intricate details.

Referee #2 – General Comments:

Zheyu Cheng et al., reported the experimental observation of a critical topological phase transition (CTPT) in a time-periodic acoustic lattice. The authors implement a one-dimensional chiral Floquet system using a network of coupled acoustic waveguides, in which the couplings can be precisely tuned by adjusting the link geometries. The work presents an innovative approach utilizing an acoustic framework to address a long-standing conceptual problem in defining and measuring topological invariants at critical boundaries, which may have a broad scientific impact beyond acoustic studies. Yet before my recommendation of its publication in Nature, several critical issues need to be addressed first:

Response from authors:

Thank you for your highly positive comments, in particular your recognition that our work addresses a long-standing conceptual problem with potential impact beyond acoustic studies. We will address your concerns point by point below.

Referee #2 – Comment 1:

Finite-size and boundary effects: Each sample implements only four Floquet periods, leading to eigenphase deviations from $0/\pi$ “due to finite size.” Without a scaling analysis, it is difficult to separate finite-size artifacts from genuine critical behavior. Simulations for 3–5 different lengths should be added to estimate eigenphase drift and to justify quantitative comparisons.

Response from authors:

Thank you for raising the issue of finite-size and boundary effects. We address this issue by separating it into two aspects: (1) whether the number of Floquet periods is sufficient to reliably extract the quasienergy spectrum, and (2) whether the lattice size along the transverse (y) direction is sufficient. We further clarify the interpretation of the small deviations of the 0 and π edge-state quasienergies from their ideal values.

(1) Number of Floquet periods

The simulated Floquet spectra in Fig. 3b in the main text are not obtained from a single Floquet operator. Instead, for each sample, we construct seven Floquet operators by choosing different starting positions of the one-period Floquet evolution, $x = 0, 0.5L, \dots, 2.5L, 3L$, and Fig. 3b shows the spectrum averaged over these seven Floquet operators. The corresponding

quasienergy spectra from the seven Floquet operators are nearly identical, indicating that the implemented number of Floquet periods is sufficient and does not introduce noticeable finite-size artifacts.

As an explicit example, Fig. R1 below shows the real and imaginary parts of the quasienergy spectra for the seven Floquet operators corresponding to sample 1, demonstrating negligible discrepancies among them. Figure R2 further summarizes this consistency for all eight samples labelled in Fig. 2a in the main text, where the error bars represent the standard deviation among the seven Floquet operators.

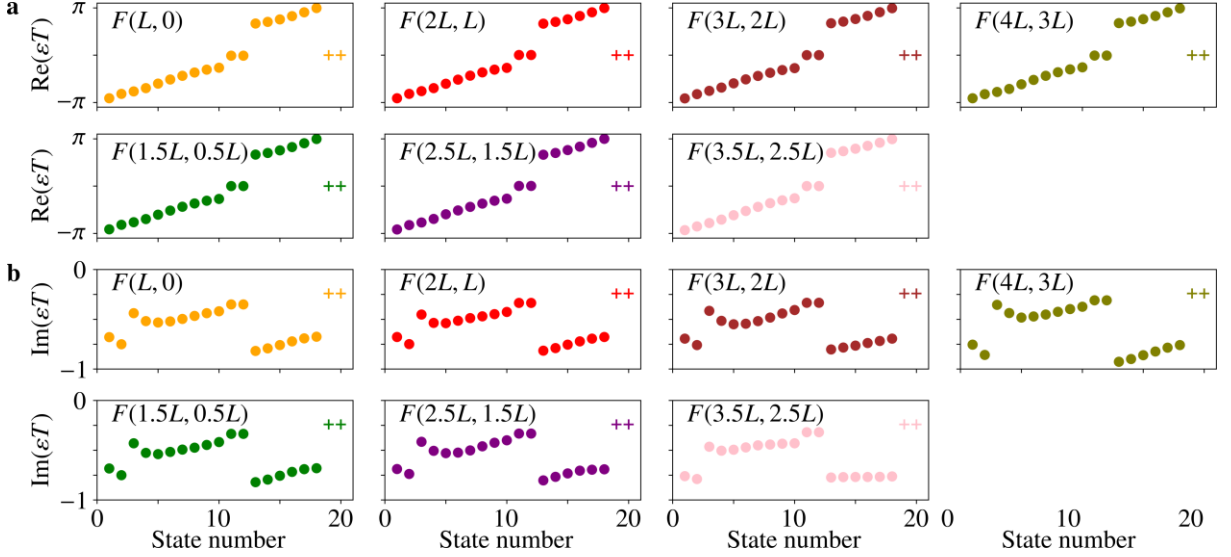


Fig. R1 | The seven eigenspectra corresponding to the seven Floquet operators for sample 1. (a) Real part of the quasienergy spectrum. (b) Imaginary part of the quasienergy spectrum.

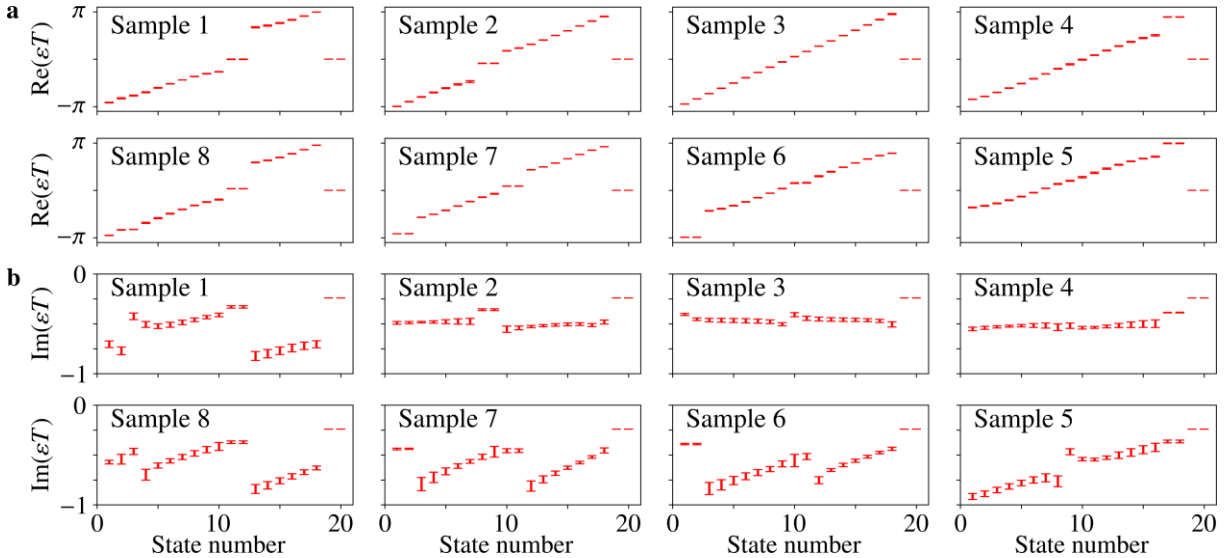


Fig. R2 | The averaged eigenspectra for eight samples. Red error bars indicate the standard deviation obtained from the seven eigenspectra. (a) Real part of the quasienergy spectrum. (b) Imaginary part of the quasienergy spectrum.

We further performed simulations over eight Floquet periods, as shown in [Extended Data Fig. E1](#). In particular, we analyze the Floquet operator constructed from the evolution between $x =$

$7L$ and $x = 8L$, with the corresponding quasienergy spectra shown in Fig. R3 below. The resulting spectra are fully consistent with those obtained in Fig. 3b of the main text for Samples 1–8.

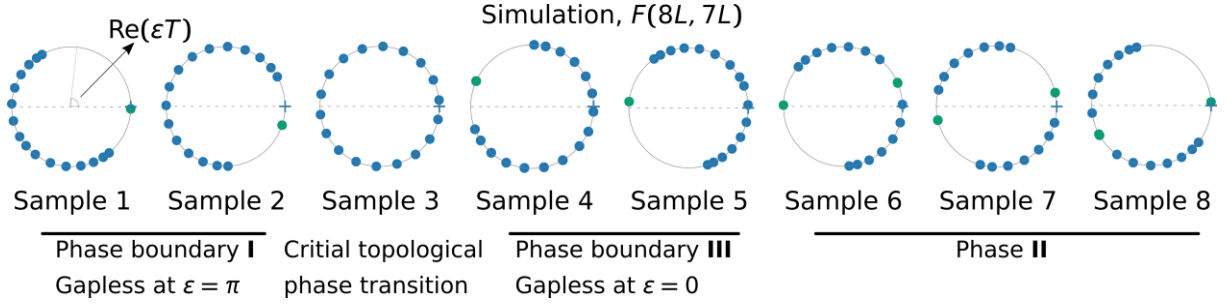


Fig. R3 | The eigenspectra from the Floquet operator $F(8L, 7L)$ for eight samples.

In the revised manuscript, we have included the above Fig. R1 and Fig. R2 as [Supplementary Fig. 10](#) and [Supplementary Fig. 11](#) in Supplementary Information. We have also added a clarification in Paragraph 2 in the subsection of “[Numerical simulations](#)” in Methods noting that Fig. 3b displays the averaged spectrum from these seven Floquet operators, which read as follows (Page 9 right column):

“... Figure 3b displays the averaged spectrum from the seven Floquet operators for each sample in the acoustic simulation, with negligible discrepancies observed among the seven eigenspectral (see Supplementary Note 4).”

(2) Lattice size along the y direction

We further examine whether the transverse lattice size is sufficient by performing numerical simulations with a larger system size along the y direction. The resulting quasienergy spectra for the eight representative samples are shown in Fig. R4 below. Compared with Fig. 3b in the main text, the positions of the edge states remain unchanged when the lattice size is increased. This confirms that the experimental lattice size of $N = 10$ unit cells is sufficient to resolve the spectral features of quasienergy spectrum.

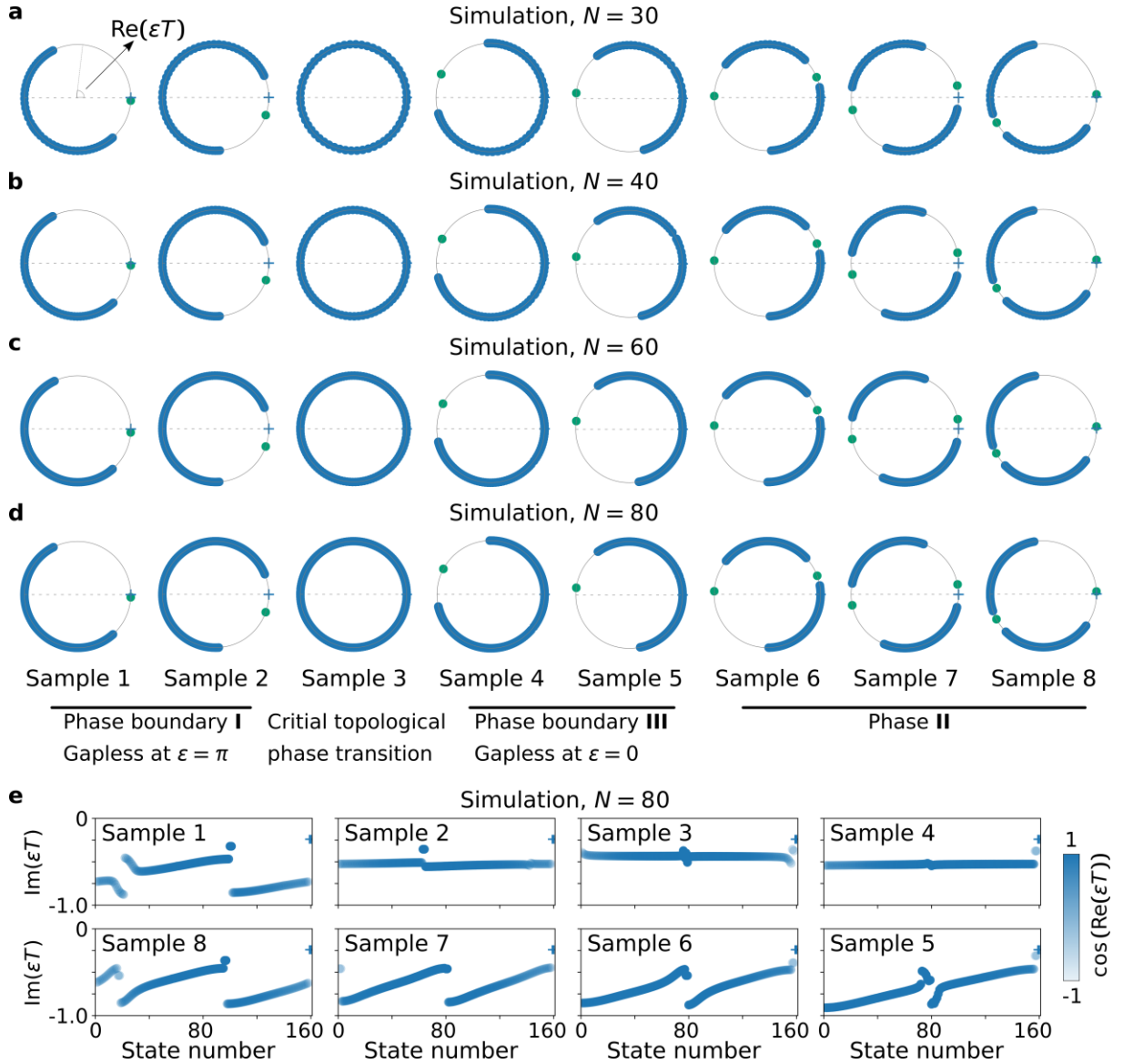


Fig. R4 | Floquet operator spectra for the numerical simulations with a larger lattice size. (a, b, c, d) Real parts of the quasienergies for eight representative samples from the phase diagram in Fig. 2a, plotted on the unit circle, where the angular coordinate corresponds to the real quasienergy. All samples host two isolated 0-quasienergy edge states (marked by “+”). Green markers indicate the location of other edge states. (e) Imaginary parts of the quasienergies for eight representative samples from the phase diagram in Fig. 2a. The lighter (denser) the color, the closer the real part of the quasienergy is to π (0).

In the revised manuscript, we have included the above Fig. R4 as [Extended Data Fig. E2](#) in the main text.

(3) Clarification of the 0 and π edge-state quasienergies deviated from exactly 0 and π

Finally, we clarify the interpretation of the 0 and π edge states observed in the simulations and experiments. In an ideal Floquet system with perfect tight-binding parameters, such edge states are strictly pinned at quasienergies 0 or π . In realistic acoustic implementations, however, unavoidable residual reflections in the link tubes lead to slight deviations from the ideal

coupling parameters, which can shift the edge-state quasienergies away from the exact values. Despite that, importantly, these states remain boundary-localized and reside within the corresponding 0-gap or π -gap, and the bulk quasienergy spectrum preserves chiral symmetry, as evidenced by the pairs of quasienergies $\varepsilon_0 \pm \varepsilon_j$ observed in both simulations and experiments. The identification of the 0 and π edge states therefore rely on their gap association, boundary localization, and robustness at the critical phase boundaries, rather than on exact quasienergy pinning.

To avoid confusion, we have added the explanation in Paragraph 4 in the subsection of “Topological invariants” in Methods (Page 8 right column):

“... Note that throughout this work, the terms “0” and “ π ” edge states refer to boundary-localized states residing in the corresponding quasienergy gaps, rather than to exact quasienergy pinning at 0 or π .”

Furthermore, we have clarified in Paragraph 1 in the section of “Quasienergy spectrum reconstruction” (Page 3 right column):

“... Each sample consists of $N = 10$ unit cells (20 waveguides). Numerical simulations with much larger lattice sizes confirm that the finite system size does not alter the main conclusions (see Extended Data Figs. E1 and E2).”

Revisions are also added in Paragraph 2 in the subsection of “Numerical simulations” in Methods (Page 9 right column):

“... Though in theory, the edge states should have quasienergies of 0 or π , in our acoustic simulations and experiments, the link tubes cannot perfectly reproduce the tight-binding parameters, which naturally causes the reconstructed quasienergies to deviate from their ideal values, as shown in Fig. 3b–c, Extended Data Fig. E2a–d and Fig. E3b–c.”

Referee #2 – Comment 2:

Loss: Experimentally, losses are non-negligible (microphone insertion, absorption, 16-day measurement). The SI includes damping in simulations, but the retrieval assumes a unitary F . The authors need to clarify whether the measured F is projected onto the nearest unitary, how loss affects the eigenphase distribution, and whether the invariants are robust to weak non-unitarity.

Response from authors:

Thank you for raising the issue of loss. We emphasize that we do **not** assume the Floquet operator F to be unitary when reconstructing or plotting the quasienergy spectra.

In the experiment, the Floquet operator is reconstructed directly from the measured acoustic pressure fields. As described in the Supplementary Information, Eq. (S4-4) reads:

$$F(x + L, x) = \mathbf{p}(x + L, x)\mathbf{p}^{-1}(x) \quad (\text{R1})$$

Here $\mathbf{p}(x) = [p^1(x), p^2(x), \dots, p^{20}(x)]$, and $p^i(x)$ are measured acoustic pressure field.

Importantly, the vectors $p^i(x)$ are **not normalized** in this reconstruction, and therefore the resulting Floquet operator F is generally non-unitary due to loss. The real and imaginary parts of the quasienergy spectrum are both retained, where the imaginary parts directly encode the decay rates of the eigenstates. To avoid confusion, we have added an explicit clarification following Eq. (S4-4) in the Supplementary Information (Page 21).

“Here, it is unnecessary to normalize $p^i(x)$ to $\|p^i(x)\| = 1$ as we directly plot the real and imaginary parts of the Floquet spectrum. The imaginary parts correspond to the decay rates of the eigenstates.”

Following the suggestion of Referee #1, we now explicitly include the imaginary part of the quasienergy spectrum in [Extended Data Fig. E3](#), which directly visualizes the effect of loss in both simulations and experiments. From [Extended Data Fig. E3d–e](#), we extract $\text{Im}(\varepsilon T) \approx -0.5$, corresponding to a quasiparticle lifetime of order $\tau \approx 2T$. Since the acoustic field is sampled at intervals of $0.5T$, this lifetime is sufficient to reliably reconstruct the quasienergy spectrum within the experimentally relevant time window.

Despite the presence of loss, the quasienergy spectra obtained from simulations and experiments remain consistent with the tight-binding predictions in terms of the edge-state structure, as shown in Fig. 3 of the main text. In particular, the critical edge states are clearly resolved in Fig. 4. This demonstrates that the topological features are robust against weak to moderate non-unitarity and are not obscured by loss.

In the revised manuscript, we have indicated the loss and lifetime scale in Paragraph 3 in the section of “[Quasienergy spectrum reconstruction](#)” (Page 5 left column), which reads as follows:

“... The imaginary parts of the quasienergies, which quantify losses in the samples, are shown in [Extended Data Fig. E3d–e](#). From these data, we extract $\text{Im}(\varepsilon T) \approx -0.5$, corresponding to a quasiparticle lifetime $\tau \approx 2T$. Since the acoustic field is sampled at intervals of $0.5T$, this lifetime is sufficient to ensure reliable reconstruction of the quasienergy spectrum.”

Referee #2 – Comment 3:

Identification of “critical” edge modes: In the SI, the authors acknowledge that small bulk admixtures can significantly affect phase, leading to apparent beating even without genuine π modes. A control experiment on a sample where π modes are theoretically absent but with similar loss/reflection would test the specificity of the diagnostic.

Response from authors:

Thank you for raising this important point regarding the identification and specificity of the critical π edge modes. Indeed, the bulk admixtures can affect the exact phase values extracted from the acoustic fields. For this reason, our identification of edge states does not rely on precise phase measurements alone.

Instead, our analysis proceeds in two steps.

Step 1. Identification of edge states

In the first step, we determine whether edge states are present by examining the spatial intensity profile of the acoustic field, $|p(x)|^2$, under edge excitation, as shown in Fig. R5 below. For samples 1, 2, 4, and 5, the intensity distributions clearly indicate the presence of edge-localized states; however, intensity measurements alone cannot distinguish between zero- and π -quasienergy edge states. Only sample 7 exhibits a clear double-period beating pattern, as it hosts both zero- and π -quasienergy edge states. By contrast, Sample 3 does not exhibit any edge localization, indicating the absence of edge states.

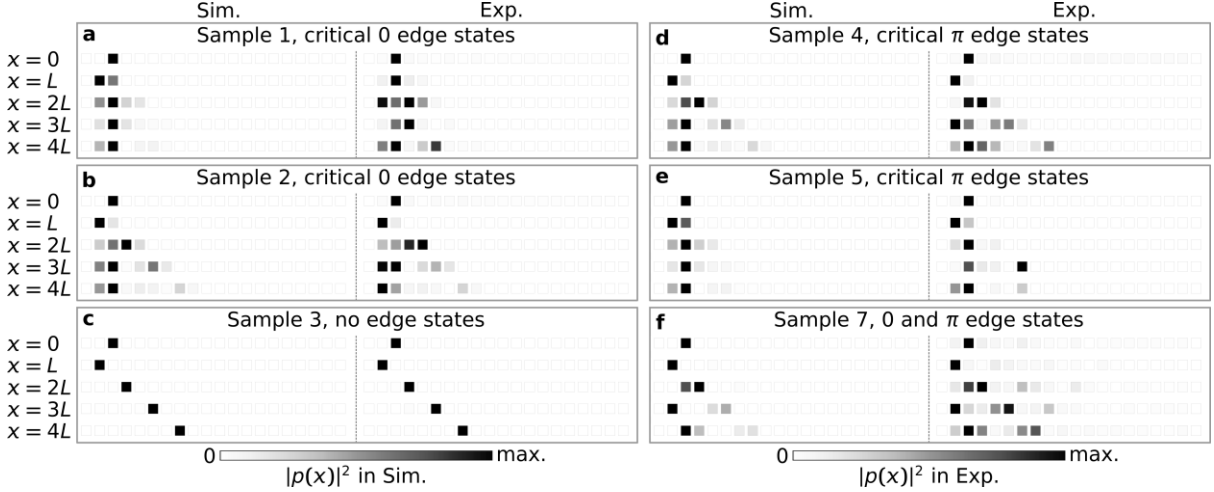


Fig. R5 | Wave intensity measurements revealing the presence or absence of edge states. The isolated 0 edge states have been excluded. While the types of edge states are labeled for reference, the intensity $|p(x)|^2$ alone can only determine whether edge states exist.

Step 2. Distinguishing zero- and π -quasienergy edge states

To distinguish between zero- and π -quasienergy edge states, we employ the interference-based approach described in the main text and analyze the corresponding interference signal $I(x)$, as shown in Fig. R6 below (duplicated from Fig. 4 in the main text). In this step, π -quasienergy edge states exhibit a characteristic double-period beating, whereas zero-quasienergy edge states do not. This clear qualitative difference provides an unambiguous criterion for identifying π edge states.

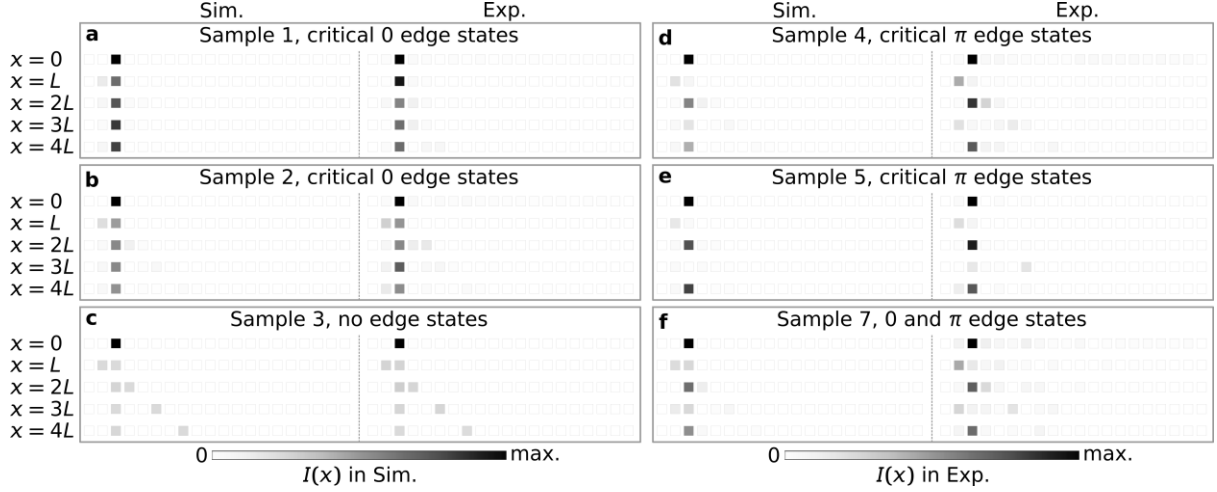


Fig. R6 | Interference measurements distinguishing critical zero- and π -quasienergy edge states (duplicate of Fig. 4 in the main text). The isolated 0 edge states have been excluded. (a,b) Samples 1 and 2 on phase boundary I host critical 0 edge states. (c) Sample 3 at phase boundary intersection V does not host edge states. (d,e) Samples 4 and 5 on phase boundary III host critical π edge states. (f) Sample 7 in gapped phase II hosts both 0 and π edge states.

Regarding your suggestion of performing a control experiment on a sample where π edge states are theoretically absent but with comparable loss and reflection, sample 3 in Figs. R5 and R6 already serves as such a control. As shown, sample 3 exhibits neither edge localization nor double-period beating in the interference signal.

As sample 3 corresponds to a special case located at a multicritical point, we further provide an additional control experiment using sample 9, shown in Fig. R7 below. Sample 9 lies on critical phase boundary VII, distinct from the eight samples presented in the main text. As shown in Fig. R7(b), this sample hosts no edge states (apart from the isolated 0 states). Consistently, neither edge localization is observed in $|p(x)|^2$ [Fig. R7(c)] nor double-period beating in the interference intensity $I(x)$ [Fig. R7(d)].

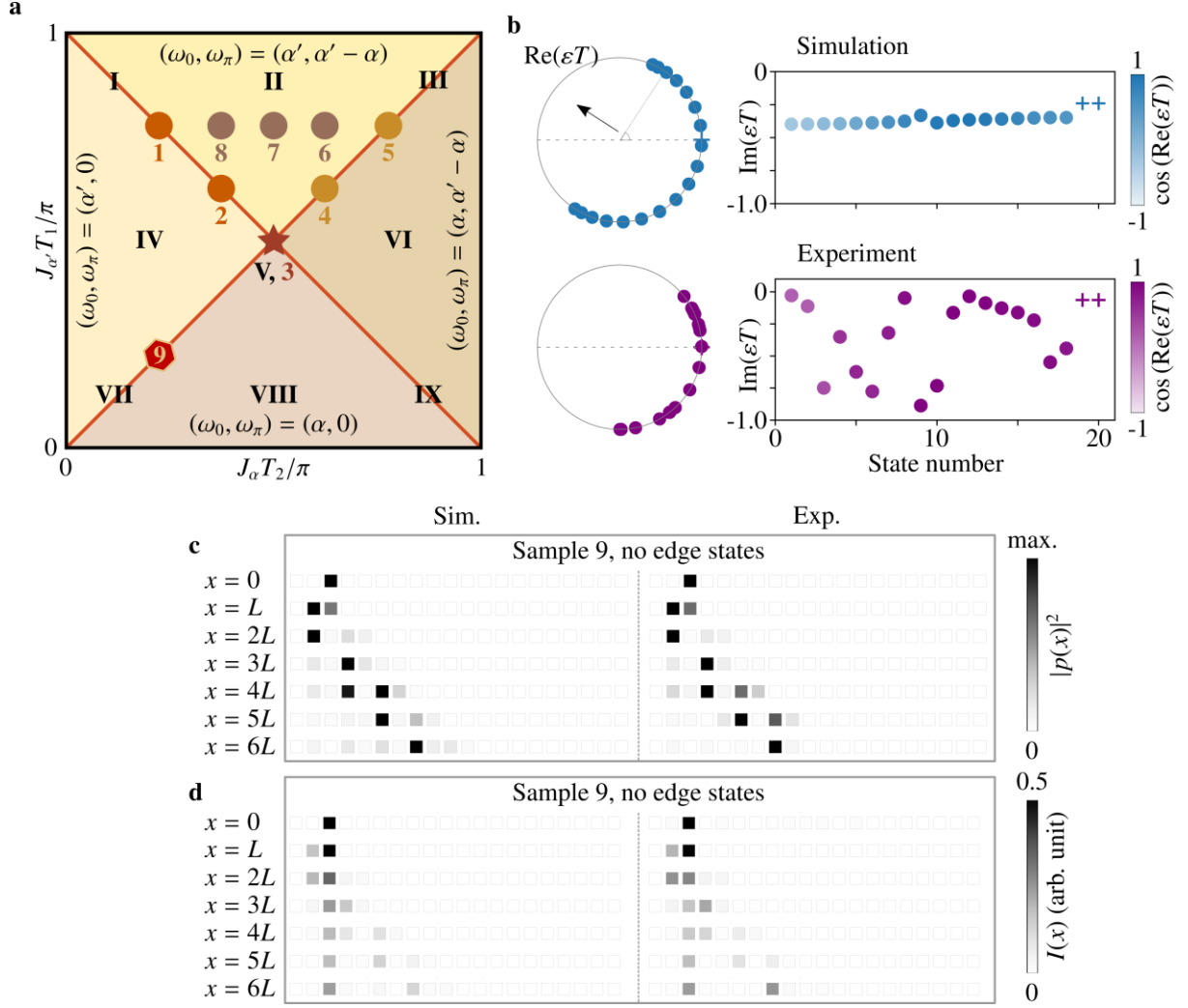


Fig. R7 | $|p(x)|^2$ and $I(x)$ measurements for Sample 9. (a) Phase diagram of the Floquet SSH α -chain, reproduced from Fig. 2a of the main text, with sample 9 located on phase boundary VII. (b) Real and imaginary parts of the quasienergies. Blue and purple markers represent acoustic simulations and experiments, respectively. Two isolated 0-quasienergy edge states are marked by “+”. (c) Wave intensity profile $|p(x)|^2$. (d) Interference intensity $I(x)$.

In summary, these results demonstrate that the observed double-period beating in Fig. 4 of the main text is a robust and distinctive hallmark of critical π -quasienergy edge states, rather than an artifact of loss, reflection, or bulk admixture.

We have included the above Fig. R5 as [Extended Data Fig. E6](#) in the main text, and Fig. R7 as [Supplementary Fig. 12](#) in Supplementary Informaion.

Referee #2 – Comment 4:

Working frequency: All results are shown at a single working frequency (9.8 kHz) defined with zero-mode wavevector. Yet it would be helpful to see the robustness of that with a small variation of the exact frequency.

Response from authors:

Thank you for this helpful suggestion. In Figs. R8 and R9 below, we present the simulated and experimentally measured quasienergy spectra and interference intensities, respectively, at a slightly detuned working frequency of 9.85 kHz. This corresponds to a relative wavevector variation of approximately 0.5% compared to the original working frequency of 9.80 kHz. We note that the frequency detuning is considered within a small range, as larger shifts would modify the effective next-nearest-neighbor hopping strength in the acoustic implementation

As shown in Figs. R8 and R9, the quasienergy spectra, the presence of the 0 and π edge states, and the identification of the phase boundaries remain unchanged under this small frequency detuning. This demonstrates that the observed topological phases and the associated critical phase boundaries are robust against moderate variations of the working frequency. We have added Figs. R8 and R9 as [Extended Data Figs. E4 and E5](#) in the main text.

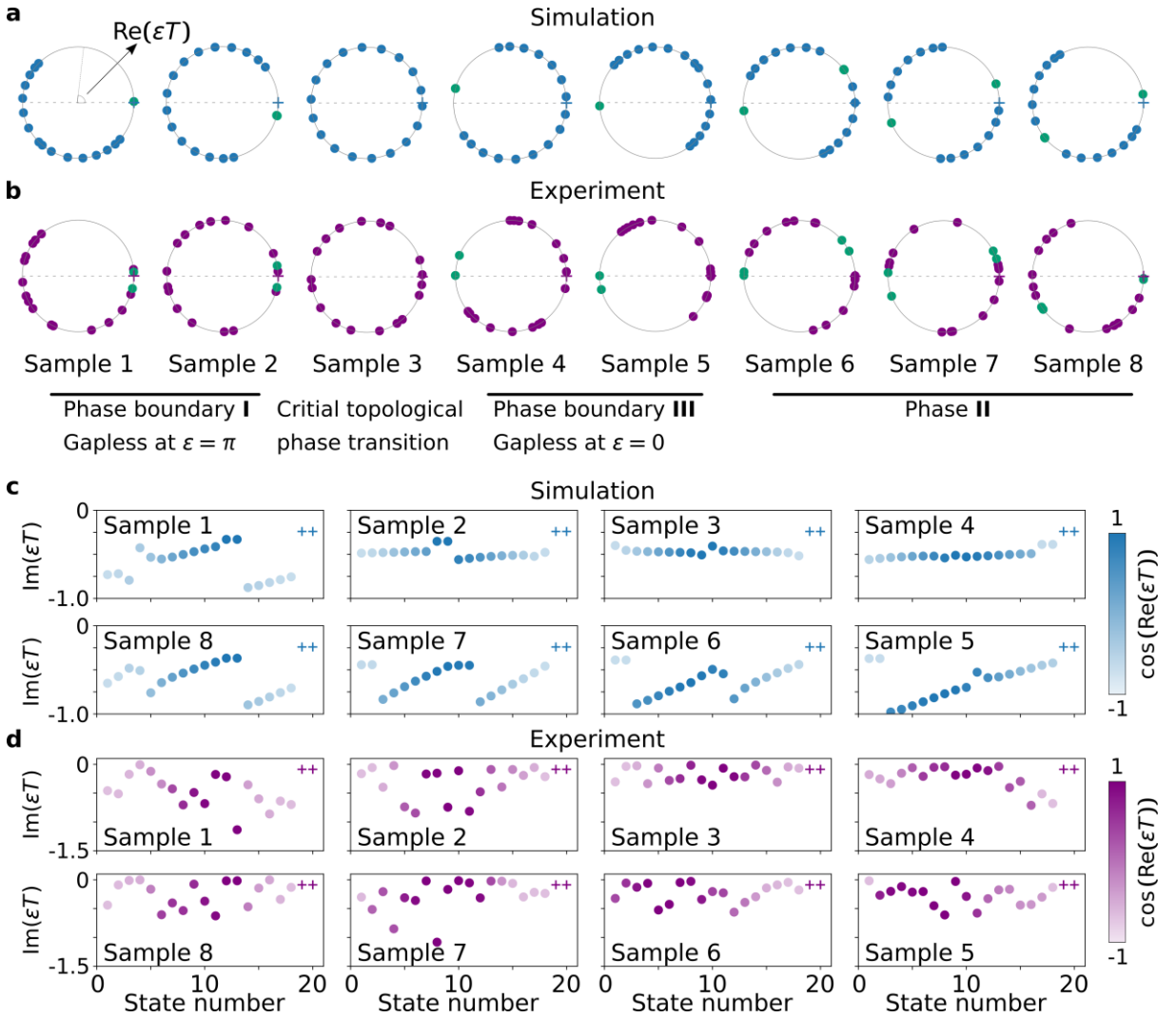


Fig. R8 | Floquet operator spectra for the numerical simulations and experiments at 9.85 kHz. (a, b) Real parts of the quasienergies for eight representative samples from the phase diagram in Fig. 2a are plotted on the unit circle, where the angular coordinate corresponds to the real quasienergy. Blue and purple markers represent acoustic simulations and acoustic experiments, respectively. All samples host two isolated 0-quasienergy edge states (marked by “+”). Green markers indicate the locations of other edge states. (c,d) Imaginary parts of the quasienergies for eight representative samples from the phase diagram in Fig. 2a. Blue and

purple markers represent acoustic simulations and acoustic experiments, respectively. The lighter (denser) the color, the closer the real part of the quasienergy is to π (0).

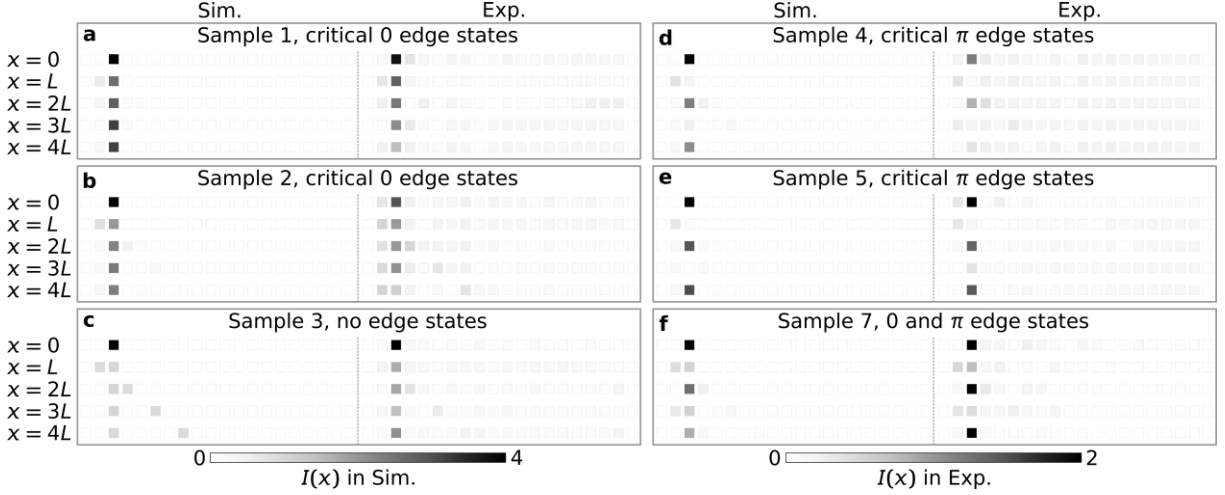


Fig. R9 | Interference measurements distinguishing critical 0 and critical π edge states at 9.85 kHz. The isolated 0 edge states have been excluded. (a,b) Samples 1 and 2 on phase boundary I host critical 0 edge states. (c) Sample 3 at phase boundary intersection V does not host edge states. (d,e) Samples 4 and 5 on phase boundary III host critical π edge states. (f) Sample 7 in gapped phase II hosts both 0 and π edge states.

Referee #2 – Comment 5:

Normally, the quantum criticality often implies a divergence of the corresponding length scale, yet in the discussion of CTPT, this point is never clearly addressed. Can the authors comment on whether there is any implication from their experiments?

Response from authors:

Thank you for this insightful comment. We fully agree that, in conventional quantum criticality, a diverging correlation length is a defining hallmark, and that its experimental manifestation is a central issue in the context of a critical topological phase transition.

Motivated by your comment, we have introduced new experiments to probe directly the criticality based on entanglement entropy (see Fig.R10 below). For a finite one-dimensional system of N unit cells with open boundaries, partitioned into a contiguous subsystem X of n unit cells and its complement Y , the entanglement entropy is expected to follow [J. Stat. Mech. 2004, P06002 (2004)]

$$S_X \sim \frac{c}{6} \ln \left(\frac{2N}{\pi} \sin \frac{\pi n}{N} \right) \quad (\text{R2})$$

Here, the logarithmic scaling with subsystem size is a hallmark of criticality, reflecting scale-invariant correlations. The slope c in Eq. (R2) defines an effective entanglement coefficient that quantifies how much scale-invariant critical information is encoded across length scales. We fabricated four samples located at critical phase boundaries (samples 2 and 4), at the multicritical point (sample 3), and in a gapped phase (sample 7). In Fig. R10, we experimentally measured and analyzed the reduced entanglement entropy ΔS_X , where non-scale-invariant constant contributions are subtracted. We observe a clear logarithmic scaling behavior at the

phase boundaries, which is a well-established signature of criticality. This provides direct experimental evidence that the system is indeed in a critical regime. By contrast, in the gapped phase, the entanglement entropy saturates to a constant as the subsystem size increases, consistent with area-law scaling. The calculation details can be found in the subsection “Entanglement entropy” of Methods (Page 10) and [Supplementary Note 1](#).

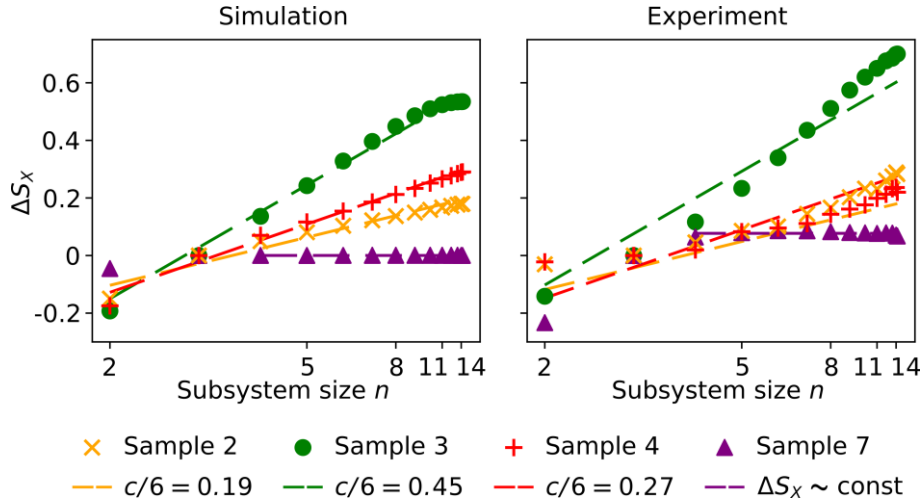


Fig. R10 | Entanglement entropy experiments (duplicate of Fig. 5 in the main text). The full system size is $N = 30$, and the horizontal axis is scaled as $\ln[\sin(\pi n/N)]$. Along the gapless phase boundaries (samples 2, 3, and 4), the reduced entanglement entropy ΔS_X follows a logarithmic scaling as in Eq. (R2). By contrast, for the gapped phase (sample 7), ΔS_X rapidly approaches a constant with increasing subsystem size n . Dashed lines indicate the entanglement coefficients simulated for a much larger system with $N = 80$.

Importantly, in our setting the critical phase boundaries themselves carry distinct topological character. This enables a topological phase transition *within* the critical regime, in which topology reorganizes the underlying critical behavior. At the intersection of such topologically distinct critical boundaries, we observe a topology-enforced multicritical point (sample 3), where the entanglement entropy exhibits an **additive logarithmic scaling** equal to the sum of the contributions from the adjacent boundaries ($0.19 + 0.27 \approx 0.45$).

In the revised manuscript (Page 7, section “[Entanglement-entropy scaling and topology-enforced multicriticality](#)”, as well as [Supplementary Note 6](#) “[Entanglement entropy in simulation](#)”), we have included the above Fig. R10 as the new Fig. 5, and presented the detailed entanglement-entropy measurements and simulations, including the deviation of the exact values of the entanglement coefficients that does not affect their robust additivity—the direct evidence of topology-enforced multicriticality.

Together with the spectral and edge-state measurements, this entanglement-entropy experiment completes the experimental evidence required to establish the critical topological phase transition, forming a coherent and self-consistent chain of evidence for the central advance of this work.

Referee #2 – Comment 6:

Trivial isolated zero modes: Given that first and last guides are disconnected, a pair of trivial zero modes will always be produced. In the manuscript, it is somewhat mixed with the critical zero mode. It would be great if the authors could separate them more visually in the main figures.

Response from authors:

Thank you for this helpful suggestion. Indeed, the two isolated 0 states arising from the disconnected end waveguides are distinct from the critical zero modes discussed in the main text. These isolated 0 states are fully localized at the two disconnected end sites and are present for all parameter choices.

Following your suggestion, these isolated 0 edge states are now consistently marked by “+” symbols in the quasienergy spectra, while all other states are marked by dots. This notation has been updated in all related figures in the main text and figure captions. Furthermore, we have revised Paragraph 4 in the section of “[Theoretical framework and phase diagram](#)” (Page 2 right column) to explicitly distinguish these isolated 0 states from other states:

“... Among these, two originate from the end sites that are disconnected from the main chain (Fig. 1a). These states remain fully localized at the isolated sites and persist across the entire phase diagram. We refer to them as isolated 0 edge states and mark them explicitly by “+” in the quasienergy spectra to distinguish them from other edge states whose number varies with the bulk topology...”

Response Letter for the Manuscript “Observation of critical topological phase transition” (Nature 2025-09-23087A)

Dear Referees,

We thank Referees #1 and #2 for recommending the publication of our work in *Nature*.

We are grateful to Referee #3 for the careful evaluation of our manuscript. We have followed your suggestions to streamline the flow of the Results section and other comments to improve the overall presentation of the manuscript.

Regarding your question of whether “*everything can be explained by band structure*”, we would like to clarify that, while conventional topology is indeed encoded in momentum-space band structure, **criticality is fundamentally characterized in scale space through scale invariance and universality emerging under Renormalization-Group (RG) flow**—the conceptual framework underlying the 1982 Nobel Prize awarded to Kenneth G. Wilson “for his theory for critical phenomena in connection with phase transitions”. The novelty and significance of our work lie in **combining topology and criticality, two concepts that have traditionally been regarded as incompatible, into a unified framework**, and in demonstrating that topology can actively reorganize criticality through a critical topological phase transition, thereby revealing **a fundamentally new type of phase transition** in the critical regime. These findings are beyond the band structure descriptions.

In the text below, all comments are quoted in *Italics* and followed by our detailed response. We have also revised the manuscript and Supplemental Information based on your comments, and these updates are highlighted in **blue** in those files. In the text below, the references to these updates are highlighted in **blue** and by a vertical **red** line in the left margin.

Best regards,
All the authors

Response to Referee #1

Referee #1 – Remarks to the Author:

Final Acceptance Report:

Title: Observation of critical topological phase transition

Authors: Zheyu Cheng , Xiuhai Zhang , Xue-Jia Yu , Longwen Zhou , Jiangbin Gong , and Baile Zhang

The authors have revised the manuscript according to my comments and suggestion. The authors have solved a new and important problem on critical topological transition.

The authors have found new and important results, which is consistent physically. I strongly recommended this paper for publication in Nature journal.

Response from authors:

Thank you for the positive evaluation and strong recommendation for publication in *Nature*. We are particularly encouraged by your comment that “*The authors have solved a new and important problem on critical topological transition.*”

Response to Referee #2

Referee #2 – Remarks to the Author:

The authors have addressed all my previous concerns in the revised manuscript. I’m now happy to recommend it for publication in Nature.

Response from authors:

Thank you for the positive assessment of our revised manuscript. We are pleased that you find all previous concerns have been addressed and now recommend our work for publication in *Nature*.

Response to Referee #3

Dear Referee #3,

We thank you for your careful evaluation of our manuscript and for raising a number of important conceptual and technical questions. Your comments prompted us to clarify several key concepts of our work, particularly the distinctions between conventional band topology, critical topology, and their relationship to band structures.

We first clarify the distinction between topology and criticality from the perspective of band structures.

In conventional topological systems, topology is typically formulated in terms of gapped band structures and momentum-space invariants. By contrast, criticality can arise only in gapless systems. However, **gaplessness itself does not necessarily imply criticality**. The defining physics of criticality lies in scale invariance and universality, which are characterized through Renormalization-Group flow and entanglement-entropy scaling, rather than through momentum-space band structures.

We then clarify how the conventional incompatibility between topology and criticality is resolved in recent theories of critical topology.

In gapless systems, conventional band topology generally becomes ill-defined. Recent theories overcome this difficulty by incorporating critical theory to redefine topology in the critical regime, through critical theoretical structures (e.g., symmetry flux, and conformal sectors), rather than through conventional band structures alone. For example, in the Flouquet system studied in our work, the critical topology is **not defined directly from the gapless band structure**, but instead defined through the analytical continuation of topological wavefunctions from the gapped regime into the gapless critical regime. In this sense, critical descriptions are essential for defining topology at criticality.

Finally, we would like to clarify that our work is not to merely realize a specific nonlocal acoustic structure, but to demonstrate a new type of phase transition, in which **topology actively reorganizes criticality**.

Following your comments, we have substantially revised the manuscript to improve the presentation and further clarify these conceptual distinctions, including reorganizing the Results section, adding the phonon-band analysis, supplementing additional experiments demonstrating the robustness of topological edge states, and providing further comparisons with previous studies on nonlocal acoustic structures.

Below, we respond to your comments point by point.

Referee #3 – Comment 1:

The first and second part of the Results are not well organized, making the paper very difficult to understand. In particular, the discussion jumps frequently between figure panels in Fig. 1 and Fig. 2.

Response from authors:

Thank you for this valuable comment. We agree that the presentation in the original version could be improved for clarity, particularly due to the “discussion jumps” between Fig. 1 and Fig. 2.

Following your suggestion, in the revised manuscript we have reorganized the Results section to establish a clearer and more sequential narrative. Specifically, the first part of the Results is now dedicated to Fig. 1 with the sub-section title of “[Floquet tight-binding model and its mapping to acoustic implementation](#)”, without referring forward to Fig. 2. The second part is dedicated to Fig. 2, with the sub-section title of “[Phase diagram and theoretical characterization](#)”.

In addition, we have carefully revised the text to eliminate cross-referencing between Fig. 1 and Fig. 2 within the same discussion, ensuring that each figure is introduced and explained in a self-contained and continuous manner. As a result, the previous back-and-forth “discussion jumps” between figure panels have been avoided.

We believe that these changes significantly improve the logical flow and readability of the manuscript.

Referee #3 – Comment 2:

The designed structure in Fig. 1b apparently includes third-order nonlocal interaction, which has been studied and known to support topological edge states and unusual dispersion relations (see Physical Review Applied. 2023 May 1;19(5):054028; Nature Reviews Physics. 2025 Jun;7(6):299-312; Communications Physics. 2023 Nov 21;6(1):334.). The authors should clarify the difference between their study and previous work.

Response from authors:

Thank you for pointing out these important works, now cited as Refs. [41–43] in the revised manuscript:

- “Acoustic topological metamaterials of large winding number”, *Physical Review Applied*, 19, 054028 (2023).
- “Breakdown of conventional winding number calculation in one-dimensional lattices with interactions beyond nearest neighbors”, *Communications Physics*, 6, 334 (2023)
- “Nonlocal metamaterials and metasurfaces”, *Nature Reviews Physics*, 7, 299-312 (2025)

These important works help clarify the conceptual distinction between previous studies of nonlocal topological systems and the critical topological phase transition investigated in our work.

We would like to emphasize that the central physics of our work is not the mere presence of nonlocal coupling, but rather the realization of topology at criticality and the reorganization of criticality by topology. In our system, nonlocal coupling serves only as an approach to enrich the phase diagram of accessing the topological critical phase boundaries.

The fundamental differences between our work and the previous works are summarized below.

(1) Difference in the underlying physical regime: previous works focus on gapped topological phases, whereas our work concerns topology directly on gapless critical boundaries.

In the previous nonlocal topological systems, topology is defined through bulk band gaps and characterized by conventional momentum-space invariants such as winding numbers and Zak phases. By contrast, our work investigates critical systems in which the bulk spectrum becomes gapless, where conventional band-topological descriptions generally cease to apply. We demonstrate that topology can nevertheless remain well-defined directly at criticality through critical topological invariants defined on gapless phase boundaries.

(2) Difference in the role of topology: topology in our work does not merely support edge states, but also actively reorganizes criticality.

In the above-mentioned nonlocal topological systems, topology primarily manifests through edge states and unusual dispersion relations. In contrast, in our work topology not only protects critical edge states, but also reorganizes criticality, leading to topology-enforced multicriticality and a critical topological phase transition occurring entirely within the critical regime.

(3) Difference in physically observable quantities: our work goes beyond spectral band descriptions and probes universal critical behaviour.

Previous works are mainly characterized through spectral quantities such as dispersion relations and edge-state spectra. By contrast, our work further probes universal critical behaviour through entanglement-entropy scaling and additive entanglement coefficients (“central charge” in the context of conformal field theory) at the multicritical point. These quantities characterize **scale-invariant critical correlations emerging under Renormalization-Group flow** and cannot be captured by band structures.

To explicitly clarify the distinction between our work and previous works, we have added a sentence in the main text emphasizing that the nonlocal coupling in our system serves as an enabling mechanism to enrich the landscape for the interplay between topology and criticality, rather than being the primary focus of investigation (Page 2, left column, at the end of the first paragraph in Results):

“...Here, the $\alpha' = 2$ term corresponds to next-nearest-neighbour (nonlocal) coupling. Previous studies employed such nonlocal couplings in gapped topological systems hosting edge states [41–43], whereas here we use them to enrich the landscape for the interplay between topology and criticality.”

Referee #3 – Comment 3:

I suggest the authors to also plot the phonon bands of their designed structure in Fig. 1b. The transverse direction can be two different cases; one is periodic and the other is finite width. I believe band structures can already explain a lot of the wave phenomena, such as an inject wave can be localized or extending (Extended Data Fig. 1) in the transverse direction depending on the frequency. If everything can be explained by the band structure, then the advantages of the explanation with the topological critical phase transition should be more explicitly discussed.

Response from authors:

Thank you for this valuable suggestion and for raising the important question of whether the phenomena observed in our work can be fully explained by the phononic band structures.

Following your suggestion, we have calculated the phononic band structures for representative samples (samples 1, 3, and 5) in our work, as shown in Fig. R1 below. As you specified, the x direction is periodic while the y direction is finite. The red dashed line indicates the experimental operating frequency of 9.8 kHz used in our measurements. For completeness, these phononic band structures have also been included in the revised Supplementary Information as Supplementary Note 5 (“Phononic bands of the designed structure”) and Supplementary Fig. 12.

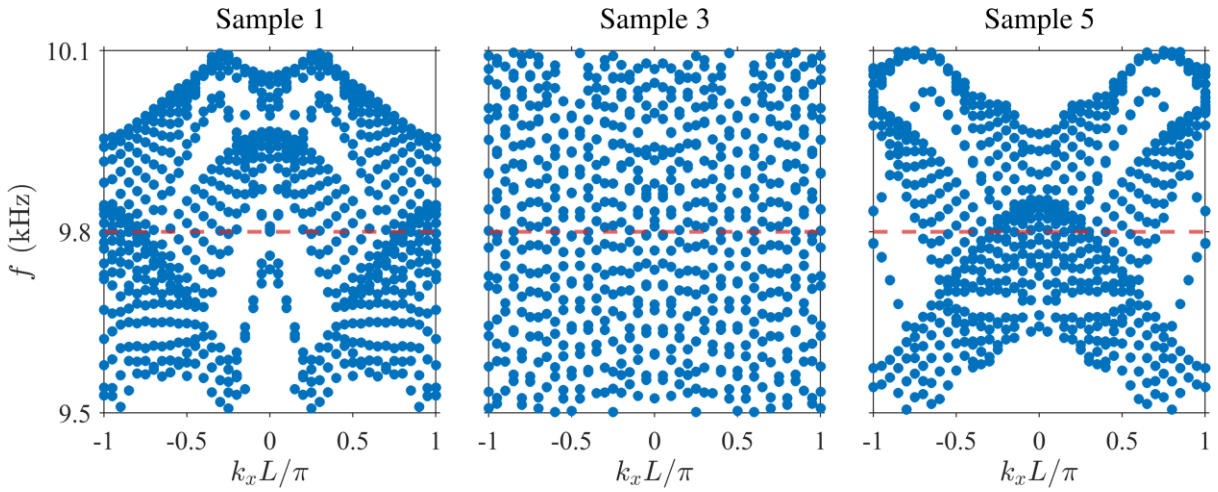


Fig. R1 | Phononic band structures of representative samples (samples 1, 3 and 5 in the main text).

We agree that in general the phononic band structure can explain certain spectral and localization-related features. However, all the phononic band structures shown in Fig. R1 are gapless and do not provide the conventional band-gap-based characterization typically used to identify localized states in gapped topological systems.

Regarding your comment that “*an inject wave can be localized or extending... depending on the frequency*”, we would like to clarify that all experiments in the main text are performed at a fixed frequency of 9.8 kHz. Therefore, the localized or extended behaviours observed in Extended Data Fig. 1 do not arise from varying the frequency, but arise from a different mechanism working at a fixed frequency.

At this frequency, the acoustic structure can be mapped to a time-modulated Floquet system, in which the propagation direction x plays the role of an effective time dimension, following a framework previously established in photonic and acoustic systems [e.g., *Nature* 496, 196-200 (2013), *Nature Communications* 13, 11 (2022), and many others]. This mapping enables the definition of a Floquet quasienergy spectrum and, more importantly, with the Floquet quasi-energy spectrum and the closure of Floquet quasi-energy bands, we are able to identify a class of systems, in the nonequilibrium context, when gap-based topological characterization ceases to work and where we may do analytical continuation of topological invariants with quasi-energy gap closing. Without this mapping, everything is buried in the 2D phonon bands and our analysis on how criticality organizes topology will not be feasible. It further enables the

experimental probing of criticality through entanglement-entropy scaling, features that cannot be captured from the phononic band structures in Fig. R1 alone.

We therefore emphasize that the key physics of our work lies not simply in the phononic band structures themselves, but in **the realization of topology directly at gapless critical boundaries and its interplay with critical behaviour.**

Referee #3 – Comment 4:

Since the current design contains nonlocal interaction, is it a must to have nonlocal coupling to realize the critical topological phase transition? If not, can the authors provide additional examples for such topological critical phase transition.

Response from authors:

No, nonlocal coupling is **not** a must to realize the critical topological phase transition. We thank you for raising this important question regarding the generality of the phenomenon.

To facilitate the discussion, we first reproduce the phase diagram of the Floquet SSH α -chain in Fig. R2 below (adopted from Fig. 2a in the main text), governed by the Hamiltonian

$$H(k, t) = \begin{cases} J_{\alpha'} [\sigma_x \cos(\alpha' k) + \sigma_y \sin(\alpha' k)], & t \in [\ell T, \ell T + T_1] \\ J_{\alpha} [\sigma_x \cos(\alpha k) + \sigma_y \sin(\alpha k)], & t \in [\ell T + T_1, \ell T + T_1 + T_2] \end{cases}$$

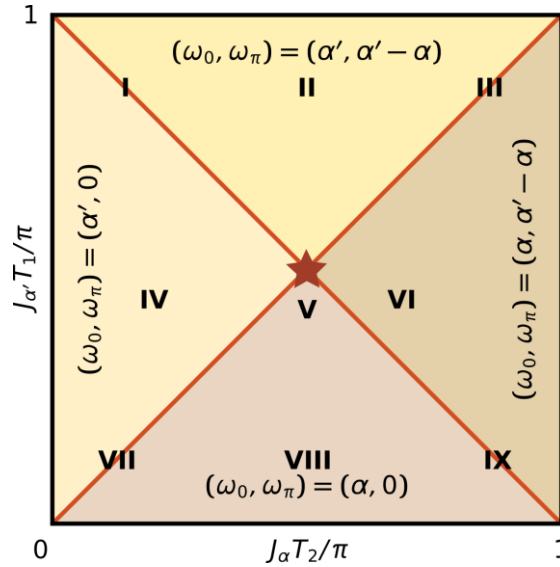


Fig. R2 | Phase diagram of the Floquet SSH α -chain. Figure adopted from Fig. 2a in the main text. For simplicity, the Arabic numerals labelling the experimentally realized samples 1-8 in the original Fig. 2a have been removed.

For completeness, we also explicitly list the topological invariants defined at the gapless phase boundaries in Table R1 below.

Table R1 | Topological invariants at the gapless phase boundaries.

Phase boundaries	Topological invariants (ω_0, ω_π)
I	$(\alpha', 0)$
III	$(\alpha, \alpha' - \alpha)$
V, VII, and IX	$(\alpha, 0)$

The essential ingredient for the critical topological phase transition is **not the nonlocality itself, but the existence of distinct topologies (i.e., different nontrivial topological invariants) defined on adjacent gapless critical boundaries**. In other words, the transition originates from **a change of topology along the gapless critical boundaries**, rather than from the spatial range of coupling.

Indeed, a minimal model containing only nearest-neighbour coupling already supports such a critical topological phase transition. Specifically, by choosing $\alpha' = 1$ (inter-cell coupling) and $\alpha = 0$ (intra-cell coupling), the system reduces to a nearest-neighbour Floquet SSH model. In this minimal case, the critical topological phase transition along the path $I \rightarrow V \rightarrow III$ remains present, and the multicritical point V still separates distinct topological critical phase boundaries I and III . **This demonstrates that nonlocal coupling is not fundamentally required for realizing the critical topological phase transition.**

The role of nonlocal coupling is instead to enrich the topological structure of the critical phase diagram.

Under the minimal nearest-neighbour choice $(\alpha', \alpha) = (1, 0)$, the phase boundaries V , VII , and IX carry trivial invariants $(0, 0)$. By contrast, introducing nonlocal coupling allows all gapless critical boundaries to become topologically nontrivial. The minimal parameter choice achieving this is $(\alpha', \alpha) = (2, 1)$, which is precisely the configuration adopted in our manuscript.

Therefore, **nonlocal coupling is not the origin of the critical topological phase transition**, but rather serves as **an enabling mechanism** that provides a richer and experimentally accessible landscape of topological critical phase boundaries.

Referee #3 – Comment 5:

The authors claim that the observed edge states is topological. It is better to provide experiments to show that the edge states are robust to disorder or fabrication errors.

Response from authors:

Thank you for raising this important point regarding the robustness of the observed edge states. Robustness against disorder or fabrication errors is indeed an important hallmark of topological edge states. In our Floquet system, the topological signatures are encoded in the existence of the critical 0 and π edge states, particularly the doubled periodicity associated with the π state. To examine this directly, we investigate the effect of substantial structural perturbations through both numerical simulations and experimental measurements. Two representative types of defects are considered: (i) dissipative (absorbing) defects and (ii) rigid (hard-wall) defects. In both cases, the defects are randomly distributed in the samples.

(1) Dissipative (absorbing) defects:

We randomly place 20 absorbing elements into 20 waveguides, with each waveguide containing one absorbing element at a random position along the x (effective time) direction. Each absorbing element has dimensions of $2 \text{ mm} \times 2 \text{ mm} \times 10 \text{ cm}$, while the waveguides have dimensions of $8 \text{ mm} \times 8 \text{ mm} \times 56 \text{ cm}$. In the simulations, the sound speed in air is set to $v = 343 \times (1 + 0.01j) \text{ m/s}$, while that in the absorbing material is taken as $v = 343 \times$

$(1 + 0.1j)$ m/s to model dissipation.

The corresponding simulation and experimental results are summarized in Fig. R3. We observe that $I(x)$ retains the characteristic doubled spatial periodicity $2L$ in the presence of critical π edge states, consistent with the Floquet π -state signature discussed in the main text. By contrast, sample 3, located at the phase-boundary intersection V and not supporting edge states, still does not carry edge states in the presence of defects. These results show that the characteristic signatures of the critical Floquet edge states arise from critical topology, rather than the introduced defects.

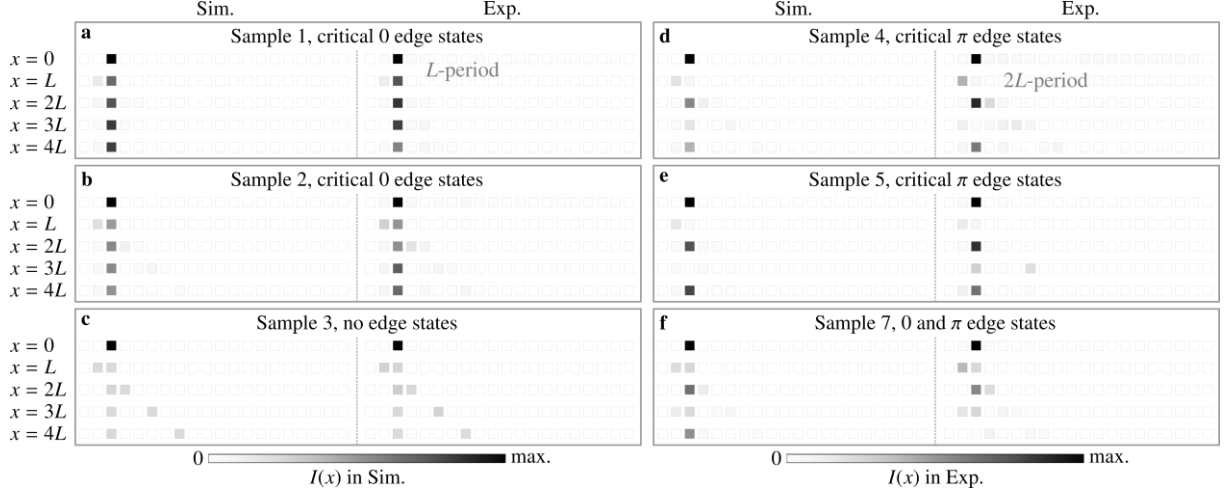


Fig. R3 | Interference measurements distinguishing critical 0 and critical π edge states in presence of dissipative (absorbing) defects at 9.8 kHz. The isolated 0 edge states have been excluded. (a,b) Samples 1 and 2 on phase boundary I host critical 0 edge states. (c) Sample 3 at phase boundary intersection V does not host edge states. (d,e) Samples 4 and 5 on phase boundary III host critical π edge states. (f) Sample 7 in gapped phase II hosts both 0 and π edge states. The oscillation periods L and $2L$, corresponding to the critical 0 and π edge states respectively, are explicitly labelled in panels a and d.

We emphasize that the edge states are robust against moderate defects. As an extreme example, we model dissipation in the absorbing material by setting the sound velocity to $v = 343 \times (1 + 10j)$ m/s, corresponding to an extremely strong defect. The simulated results are shown in Fig. R4. By comparing Figs. R3 and R4, we observe that under such strong dissipation, the Floquet topological edge states are destroyed.

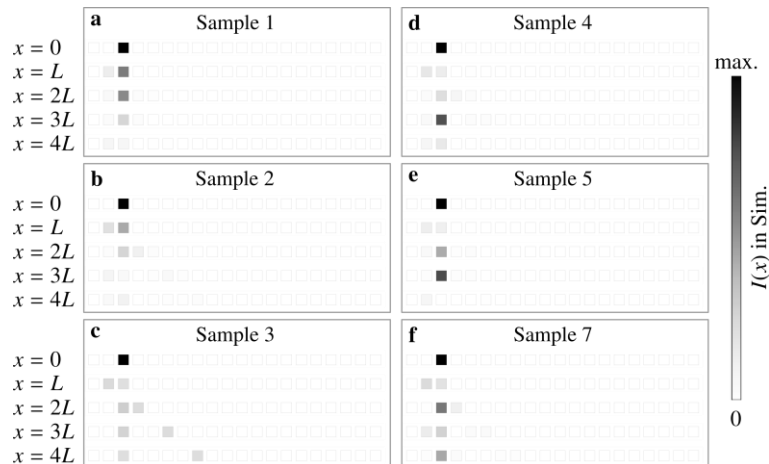


Fig. R4 | Simulations showing interference in presence of strong dissipative (absorbing) defects at 9.8 kHz.

(2) Rigid (Hard-wall) defects:

We randomly place 20 rigid elements fabricated from PLA (polylactic acid) into 20 waveguides, with each waveguide containing one rigid element at a random position along the x (effective time) direction. Each rigid element has dimensions of $2 \text{ mm} \times 2 \text{ mm} \times 10 \text{ cm}$, while the waveguides have dimensions of $8 \text{ mm} \times 8 \text{ mm} \times 56 \text{ cm}$. In the COMSOL simulations, these elements are modeled as hard-wall boundaries, as PLA can be well approximated as acoustic rigid. The sound speed in air is set to $v = 343 \times (1 + 0.01j) \text{ m/s}$.

The corresponding simulation and experimental results are summarized in Fig. R5. We observe that $I(x)$ exhibits a spatial period of $2L$ in the presence of π edge states, consistent with the characteristic doubled periodicity discussed in the main text. These results demonstrate that the characteristic signatures of the critical Floquet edge states remain observable in the presence of rigid (hard-wall) defects.

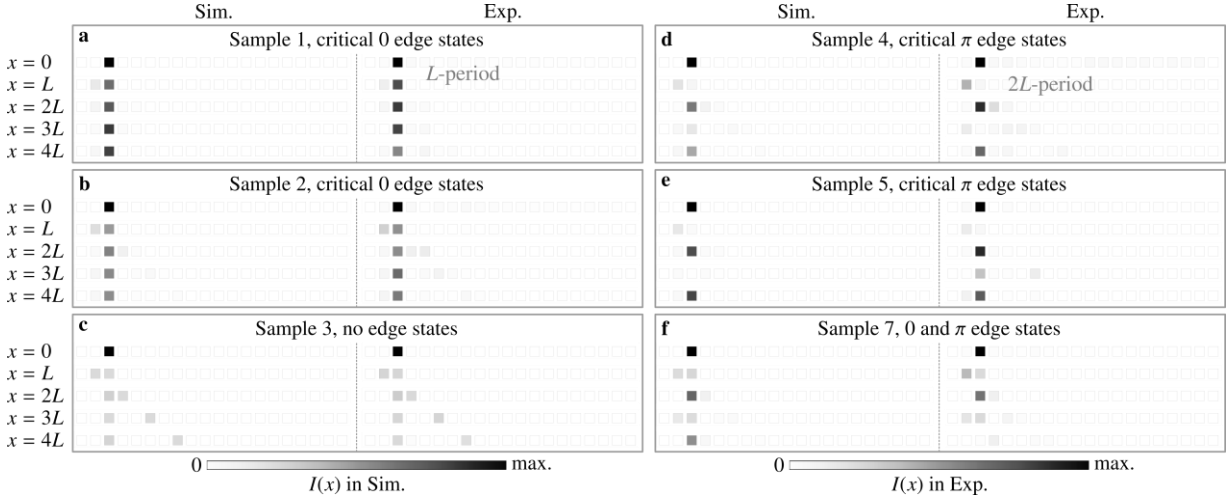


Fig. R5 | Interference measurements distinguishing critical 0 and critical π edge states in presence of rigid (hard-wall) defects at 9.8 kHz. The isolated 0 edge states have been excluded. (a,b) Samples 1 and 2 on phase boundary I host critical 0 edge states. (c) Sample 3 at phase boundary intersection V does not host edge states. (d,e) Samples 4 and 5 on phase boundary III host critical π edge states. (f) Sample 7 in gapped phase II hosts both 0 and π edge states. The oscillation periods L and $2L$, corresponding to the critical 0 and π edge states respectively, are explicitly labelled in panels a and d.

We emphasize that the edge states are robust against moderate defects. As an extreme example, we introduce rigid defects with dimensions of $7 \text{ mm} \times 7 \text{ mm} \times 10 \text{ cm}$, corresponding to extremely strong perturbations to the system. The results are shown in Fig. R6. By comparing Figs. R5 and R6, we observe that under such strong rigid defects, the Floquet topological edge states are destroyed.

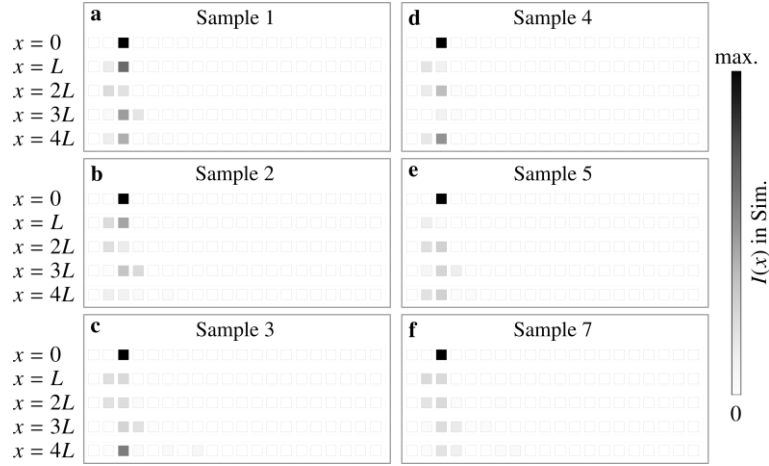


Fig. R6 | Simulations showing interference in presence of strong rigid (hard-wall) defects at 9.8 kHz.

In summary, both the numerical and experimental results show that the characteristic signatures of the critical Floquet edge states persist under moderate dissipative and rigid structural perturbations, confirming that the observed edge-state phenomena originate from critical topology.

We have incorporated Figs. R3–R6 into the Supplementary Information as [Supplementary Figs. 14–17](#) and [Supplementary Note 7](#), entitled “Robustness of topological edge states against structure defects”.

Referee #3 – Comment 6:

In the main text, a period of $2L$ is used to distinguish the 0 and the π edge states. However, from the provided Fig. 4 and Extended Data Fig. 5, it is not clearly where the period of $2L$ is? Can the authors be more specific here? Additionally, is this related to the even-odd effect in some nonlocal systems (see Nature Physics. 2018 Jan 1;14(1):40-44; Nature communications. 2024 Oct 24;15(1):8882.)

Response from authors:

To be more specific, we have explicitly labelled the oscillation periods L and $2L$ associated with the critical 0 and critical π edge states in panels a and d of Fig. 4 and Extended Data Fig. 5, which are now reproduced as Fig. R7 and Fig. R8 below.

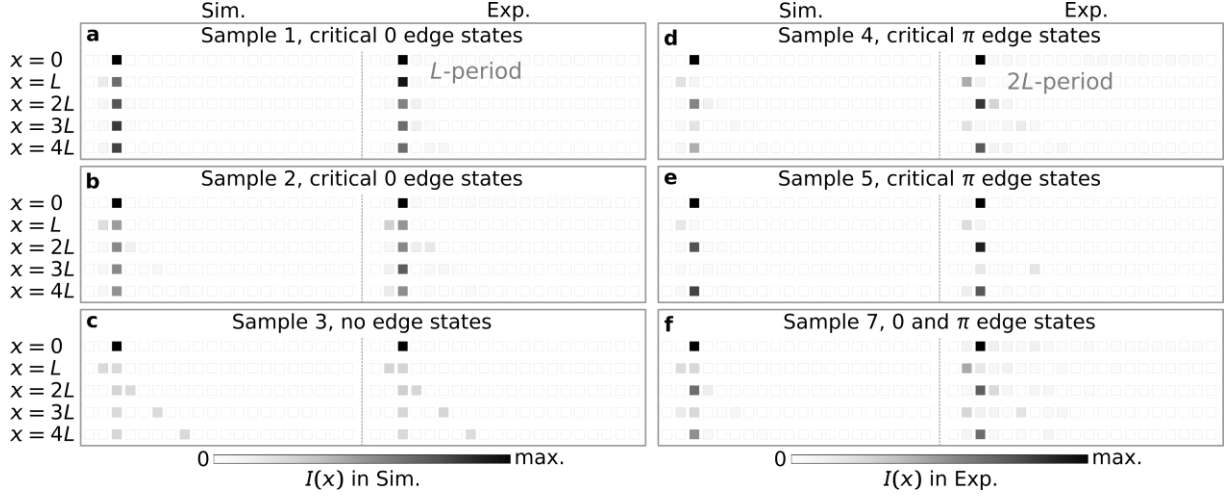


Fig. R7 | Interference measurements distinguishing critical 0 and critical π edge states at 9.8 kHz. The isolated 0 edge states have been excluded. (a,b) Samples 1 and 2 on phase boundary I host critical 0 edge states. (c) Sample 3 at phase boundary intersection V does not host edge states. (d,e) Samples 4 and 5 on phase boundary III host critical π edge states. (f) Sample 7 in gapped phase II hosts both 0 and π edge states. The oscillation periods L and $2L$, corresponding to the critical 0 and π edge states respectively, are explicitly labelled in panels a and d.

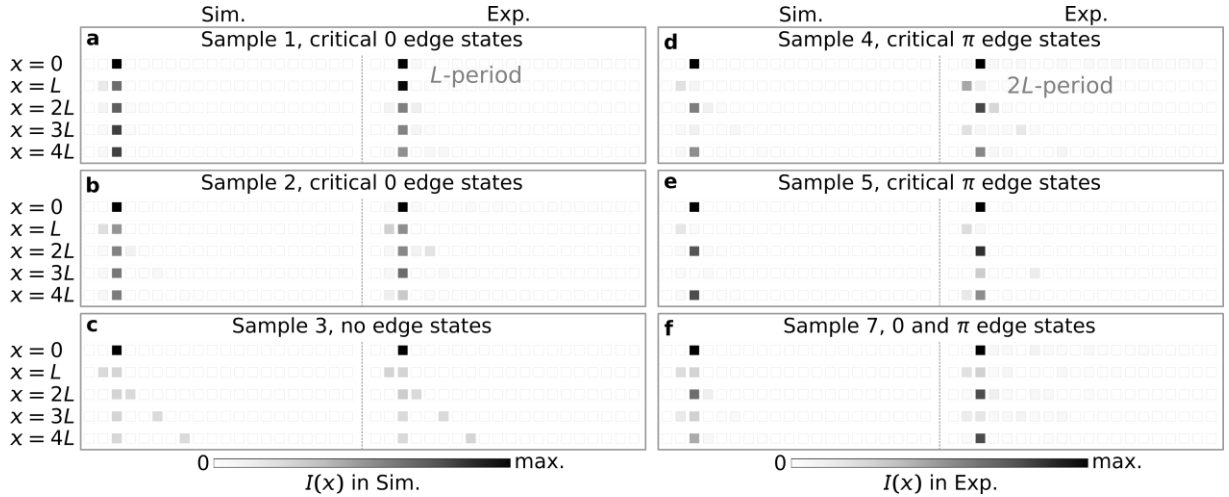


Fig. R8 | Interference measurements distinguishing critical 0 and critical π edge states at 9.85 kHz. The isolated 0 edge states have been excluded. (a,b) Samples 1 and 2 on phase boundary I host critical 0 edge states. (c) Sample 3 at phase boundary intersection V does not host edge states. (d,e) Samples 4 and 5 on phase boundary III host critical π edge states. (f) Sample 7 in gapped phase II hosts both 0 and π edge states. The oscillation periods L and $2L$, corresponding to the critical 0 and π edge states respectively, are explicitly labelled in panels a and d.

Regarding your question about a possible connection to the even-odd effect reported in systems with nonlocal coupling, we would like to emphasize that the underlying mechanisms are fundamentally different.

In our work, the critical π edge states satisfy $F|\psi_\pi\rangle = -|\psi_\pi\rangle$ where F is the Floquet operator. As a consequence, the π edge state recovers only after two Floquet periods, $F^2|\psi_\pi\rangle = |\psi_\pi\rangle$, leading to the observed effective spatial periodicity of $2L$. Importantly, this mechanism is a

generic property of Floquet π states and does not rely on nonlocal coupling. Floquet 0 and π edge states can arise in systems with only local couplings. Therefore, despite Floquet 0 and π edge states carry some similar features with previously reported even-odd effect in nonlocal mechanical systems, their physical origins are fundamentally different.

Thank you for pointing out these important works, now cited as Refs. [48–49] in the revised manuscript (Page 7 left column):

- “A characteristic length scale causes anomalous size effects and boundary programmability in mechanical metamaterials”, *Nature Physics*, 14, 40 (2018).
- “Anomalous frozen evanescent phonons”, *Nature Communications*, 15, 8882 (2024)

We have explicitly compared them with our work by adding the following clarification (Page 7, left column, the end of first paragraph):

“...Although the resulting $2L$ oscillation resembles the odd-even effect reported in certain nonlocal metamaterials [48, 49], its physical origin is fundamentally different: here the period doubling arises from the Floquet π quasienergy of the edge state, rather than from nonlocal coupling effects.”

Referee #3 – Comment 7:

In Fig. 1b, it seems that there are curved waveguides underneath the entire plate like structure. If this is needed, the authors should make these parts more visible. It is also recommended to show a unit cell of the designed structure. There are two straight waveguides at the left and the right side of the structure, which are not coupled to other waveguides. What are the purpose?

Response from authors:

Thank you for this helpful suggestion. We agree that the original Fig. 1b did not sufficiently illustrate the detailed structure of the designed acoustic lattice. Following your suggestion, we have substantially revised Fig. 1b in the manuscript to provide a clearer visualization of the experimental structure and its correspondence to the Floquet tight-binding model. For your reference, we duplicate the updated Fig. 1 in the main text as Fig. R9 below.

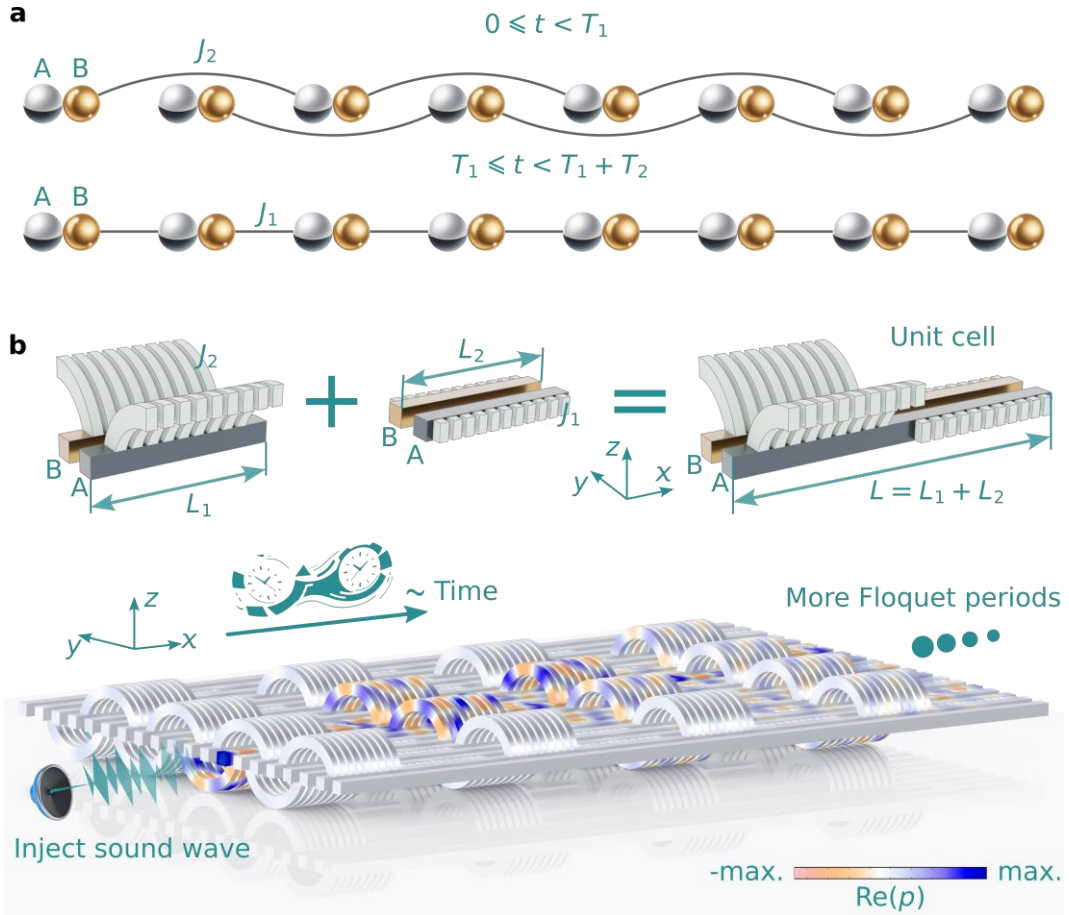


Fig. R9 | Illustration of a periodically driven one-dimensional dimerized lattice model (the Floquet SSH α -chain) and its acoustic implementation. (a) Schematic of the Floquet SSH α -chain composed of two distinct half periods. The two half periods alternately host only next-nearest-neighbor coupling (J_2) and nearest-neighbor coupling (J_1), respectively. (b) Acoustic implementation in a waveguide array, where the excitation is injected through the input port, and the propagation direction x is mapped onto an effective time coordinate. The couplings are engineered via connecting tubes: straight tubes implement nearest-neighbor coupling J_1 , whereas curved tubes implement next-nearest-neighbor coupling J_2 .

Firstly, we have redrawn the structure with improved visibility of the curved waveguides. In the revised figure, the lower curved waveguides underneath the plate-like structure are partially indicated through mirror reflection rendering, such that readers can have a better understanding of the whole structure.

Secondly, as you suggested, we now explicitly show how a complete unit cell is constructed. In the revised figure, the two coupling sections corresponding to the effective couplings J_1 and J_2 are separately illustrated and then combined to form one Floquet unit cell with total length $L = L_1 + L_2$. This revised presentation makes the correspondence between the experimental structure and the Floquet SSH tight-binding model much clearer.

Regarding your questions about the two straight waveguides at the left and right sides of the structure, these straight waveguides are intentionally introduced to preserve the completeness of the unit-cell termination at the boundaries. Without them, the structure would terminate with only a partial unit cell, which would break the direct correspondence between the experimental

realization and the theoretical Floquet lattice model. In sum, they only serve to maintain a complete boundary termination of the Floquet unit cell.

Referee #3 – Comment 8:

The authors should clarify whether the experiments are performed with time-harmonic excitation or transient excitation. And what are the frequencies used for experiments?

Response from authors:

We thank you for this important question. In our experiments, we employ time-harmonic excitation at 9.8 kHz. To avoid any ambiguity, we have revised the “Experimental Measurements” section in the Methods (Page 10, left column) as follows:

“For all experimental characterizations in the main text, we use a fixed-frequency acoustic source at 9.8 kHz. For the measurement in Extended Data Fig. E4, to show the stability of acoustic measurement, the frequency is slightly increased to 9.85 kHz.”

In addition, to clearly specify the frequencies used in the plotted results, we have explicitly indicated that the data shown correspond to 9.8 kHz in Figs. 3–5 and Extended Data Figs. E1–E3 and E6–E7 in the main text, as well as Supplementary Figs. 10, 11, 13–17 in the Supplementary Information.

Referee #3 – Comment 9:

In Fig. 2b, how the 0 and π states looks like in real space? The authors should provide the Bloch modes.

Response from authors:

We thank you for this helpful suggestion. The analytical form of the edge-state wavefunctions is provided in the “Topological Invariants” section of the Methods (Page 8). The form of the 0 and π edge states does not depend on whether the system is gapped or gapless. To explicitly illustrate their real-space structure, we present the wavefunctions for Sample 2, which hosts two isolated 0 states, two 0 edge states, and two π edge states. The corresponding real-space profiles (i.e., Bloch-mode representations) are shown below.

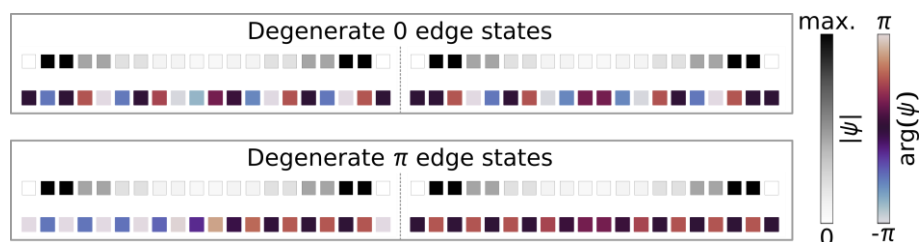


Fig. R10 | Tight-binding wavefunctions corresponding to the edge states in Sample 2. The isolated 0 edge states are excluded.

In the revised manuscript, we have incorporated Fig. R10 as Extended Data Fig. E8 in the main text. In addition, we have added the following description in the “Topological Invariants” section of the Methods (Page 9 left column):

“The wavefunctions corresponding to the edge states in Sample 2 are shown in Extended Data Fig. E8.”

Referee #3 – Comment 10:

It is unclear how the imaginary component of the quasi-energy band is obtained? Did the author fit the transient response with exponentially decay behaviors or derive the imaginary component in some other ways?

Response from authors:

We thank you for raising this important question. The imaginary component of the quasienergy is not obtained by fitting transient responses. Instead, it is directly extracted from the Floquet operator. Specifically, in the tight-binding model, acoustic simulations, and experiments, we first reconstruct the Floquet operator F , and then solve the eigenvalue equation

$$F|\psi\rangle = e^{-i\varepsilon}|\psi\rangle,$$

where ε denotes the quasienergy.

In the tight-binding model, the Floquet operator is unitary, and therefore ε is purely real. In contrast, in acoustic simulations and experiments, the presence of loss renders F non-unitary, leading to a finite imaginary part of the quasienergy, i.e., $\text{Im}(\varepsilon) \neq 0$.

To clarify this point, we have added the following sentence in the section “Experimental setup and quasienergy spectrum reconstruction” (Page 6, left column):

“After retrieving the Floquet operator, the quasienergies are obtained from the eigenvalues of the Floquet operator. Figure 3 compares the quasienergy spectra obtained from the tight-binding model, acoustic simulations, and experiments for the eight samples labeled in Fig. 2a...”

Referee #3 – Comment 11:

The “+” marker in Fig. 3 is hardly visible.

Response from authors:

We thank you for the helpful suggestion. To improve the visibility of the “+” markers in Fig. 3, we have highlighted them in red in the revised figure, as shown below in Fig. R11 (duplicated from Fig. 3 in the main text).

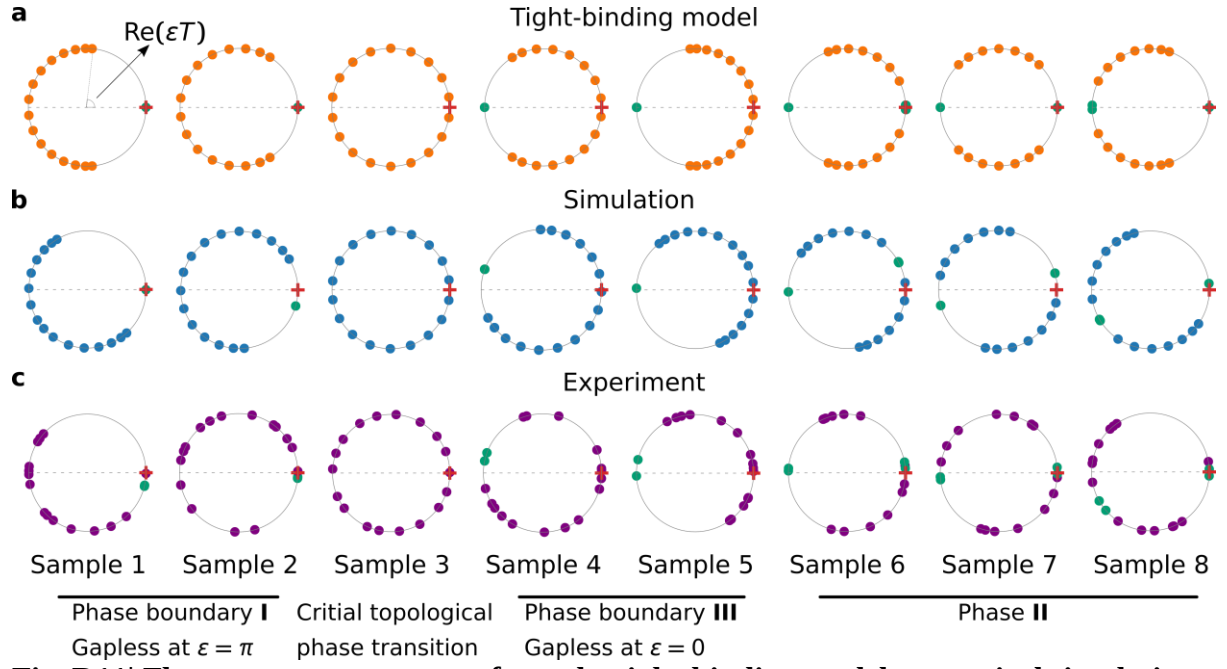


Fig. R11| Floquet operator spectra from the tight-binding model, numerical simulations, and experiments. Real parts of the quasienergies for eight representative samples from the phase diagram in Fig. 2a are plotted on a unit circle, where the angular coordinate corresponds to the real quasienergy. Orange, blue, and purple markers represent tight-binding calculations (a), acoustic simulations (b), and acoustic experiments (c), respectively. All samples host two isolated 0-quasienergy edge states (marked by “+”). Green markers indicate the locations of other edge states. All simulations and measurements are performed at a fixed acoustic frequency 9.8 kHz.