
Supplementary information

Experimental observation of critical topology

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Supplementary Materials for “Experimental observation of critical topology”, Lin, et al.

Contents

Supplementary Note 1. A brief introduction to the physics of critical topological states	2
Supplementary Note 2. Bulk entanglement spectrum as a universal fingerprint for identifying critical topology in free-fermion systems	10
Supplementary Note 3. Various critical states in 1D models	14
Supplementary Note 4. Details of the geometric parameters of experimental samples	17
Supplementary Note 5. Details of the acoustic pump-probe measurement protocol.....	19
Supplementary Note 6. Confirmation of the gapless feature via entanglement entropy scaling from experiments	23
Supplementary Note 7. Half-filling calculation and trivial contributions from lower-lying <i>s</i> -orbital phononic bands	25
Supplementary Note 8. Correlation function method for obtaining the entanglement spectrum in free-fermion systems	26
Supplementary Note 9. Robustness simulations of both edge states and the entanglement spectra.....	28
Supplementary Note 10. Entanglement wavefunctions and their scalings.....	32
Supplementary Note 11. Hamiltonian, definition of pseudo-spins, calculation of spin Chern numbers, and topological corner states of the spin Chern model	35
Reference.....	38

Supplementary Note 1. A brief introduction to the physics of critical topological states

Topological states form a cornerstone of modern condensed matter physics, extending beyond the Landau-Ginzburg-Wilson paradigm of symmetry-breaking. A notable example is symmetry-protected topological states, in which nontrivial topology is protected by global or spatial symmetries and is typically protected by a bulk spectral gap. In the past two decades, it has been widely believed that these topological properties are destroyed when the bulk gap closes. However, recent achievements have shown that many key features of topological physics [1-5], such as topological edge modes, can surprisingly emerge in the less explored realm of gapless quantum systems. Such novel phenomena can arise at continuous quantum critical points or, more generally, within gapless phases, giving rise to the notions of symmetry-enriched quantum critical points (also known as topologically nontrivial quantum critical points) and gapless symmetry-protected topological (gSPT) states, respectively. In the main text, we use the term “critical topological states” to represent the gSPT states in order to avoid possible confusion, since the latter terminology is usually denoted in interacting systems. To date, different families of gSPT states have been identified in the literature [1, 6]. The classification of these states is based on whether they contain a gapped sector and whether they are intrinsic (a term clarified below), yielding four distinct types, as summarized in Table S1. The subsequent brief explanations are mainly based on Ref. [6].

- 1) *The gSPT states*: The first example of a gSPT state, which is non-intrinsic and includes a gapped sector, was systematically investigated in Ref. [7]. Specifically, a general construction of gSPT states was introduced based on the decorated domain wall or defect picture [8]. The central idea is to decorate the G -symmetry defects of a G -symmetric conformal invariant gapless system with an H -symmetric gapped SPT phase. This decorated defect construction creates a gapped sector that acts both on G and H symmetries, resulting in a gSPT state whose

topological properties can also manifest in gapped counterparts. As a result, the topological properties of these gSPT states arise from the gapped sector and can be interpreted as a gapped SPT stacked with a conformal field theory (CFT). Typically, recent progress has shown that such gSPT states can emerge at conformally invariant critical points separating symmetry-breaking phases and gapped SPT states [3], also known as symmetry-enriched CFT or topologically nontrivial quantum critical points.

	Containing a gapped sector	No gapped sector
Non-intrinsic	Gapless SPT	Purely gapless SPT
Intrinsic	Intrinsically gapless SPT	Intrinsically purely gapless SPT

Table. S1 | Classification of gSPTs based on whether they are purely gapless (horizontal direction) or intrinsically gapless (vertical direction). The table is adapted from Ref. [6].

- 2) *The intrinsically gSPT states:* More importantly, the interplay between criticality and topology gives rise to a new class of critical topological states that have no counterpart in gapped systems, referred to as intrinsically gapless topological states [6]. This phenomenon originates from anomalous structures emerging at gapless systems. To be specific, consider a gapless system with a total symmetry group Γ fitting into the group extension $1 \rightarrow H \rightarrow \Gamma \rightarrow G \rightarrow 1$. The construction begins with a G -symmetric gapless system or the CFT characterized by a self-anomaly ω_G . An H -symmetric gapped SPT state is then stacked on top of the G -symmetry domain walls or defects. Due to the non-trivial group extension, the resulting gapped sector exhibits an (emergent) anomaly $-\omega_G$ that cancels the anomaly in the gapless sector, rendering the combined system an intrinsically gSPT state that is Γ -anomaly-free. By construction, intrinsically gSPT states also include a gapped

sector, leading to exponentially localized edge modes near the boundaries. Importantly, the topological features of intrinsically gSPT states cannot be realized in any Γ -symmetric gapped systems, thereby justifying the term “intrinsic”. This novel class of critical topological states can be realized in both noninteracting free-fermion systems and strongly interacting many-body systems, and is unique to gapless settings.

- 3) *The purely and intrinsically purely gSPT states*: From the previous discussion, we know that both gSPT and intrinsically gSPT states typically include a gapped sector, which gives rise to exponentially localized topological edge modes. In contrast, Verresen *et al.* [3, 12] first studied the critical cluster Ising model with time-reversal symmetry and demonstrated that this model lacks a gapped sector. This is evidenced by the algebraic decay of the energy splitting of the edge mode, a behavior that is forbidden in gapped systems. This result suggests the existence of a gSPT state without any gapped sector, referred to as a “purely gSPT state” [6, 10]. However, to the best of our knowledge, there are only a few known examples exhibiting such exotic phases, and constructing additional lattice models is therefore highly desirable. Furthermore, it is natural to explore the possibility of an intrinsically gSPT state that also lacks any gapped sector, a phase termed an “intrinsically purely gSPT state” [6]. However, despite ongoing efforts, explicitly constructing a lattice model that exhibits such a state remains an open challenge, making it an important direction for future research.

In interacting settings, such as quantum spin systems, gSPT states provide a natural generalization of gapped SPT states (e.g., the Haldane phase in spin-1 Heisenberg chains), but exhibit topological features that go beyond those of their gapped counterparts. Analogous to the gapped topological systems, gSPT states can also arise in free-fermion systems. In particular, gSPT states realized in non-interacting free-fermion models may be regarded as the gapless counterparts of topological insulators—representing a form of gapless band topology—which is the main focus of our

experiments. In the free-fermion setting, Ref. [12] argues that continuous transitions can host exponentially localized topological edge modes whenever the transition changes the nonzero topological index—for example, from winding number $W = 1$ to $W = 2$, or $W = 2$ to $W = 4$. In our one-dimensional (1D) experiments, the critical points separating topological phases with $W = 1$ from phases with $W = 0$ or $W = 2$ are topologically distinct: the former are topologically trivial, whereas the latter retain nontrivial topology with exponentially localized edge modes, providing a minimal example of free-fermion topological critical states. This insight provides a general guiding principle for designing tight-binding lattice models that exhibit gapless band topology in higher dimensions—an approach that is particularly well-suited to phononic platforms, where such gapless topological features can be designed and probed more directly and flexibly than in a quantum platform.

Traditionally, from the perspective of condensed matter physics, the study of topological phases has been largely limited to gapped systems, owing to the long-held belief that nontrivial topological properties are destroyed when the bulk spectral gap closes. This has fostered the impression that criticality and topology are two seemingly incompatible theoretical paradigms. From the perspective of statistical physics, it is widely believed that the universality class of a phase transition—one of the most fundamental concepts in the field—is determined solely by critical exponents, which form the standard framework for investigating phase transitions both theoretically and experimentally. Together, these long-standing beliefs have shaped our understanding of topological phases of matter and phase transitions.

However, the gapless critical topology observed in this work challenges the conventional beliefs outlined above. Specifically, within the condensed matter community, critical topological states—realizable in both free-fermion systems and strongly interacting many-body systems—represent a new class of topological states beyond those previously recognized in the literature. It overturns the long-held belief that nontrivial topology cannot persist in gapless systems, thereby opening a new

avenue for exploring novel topological phases. We note that although topological semimetals are often referred to as gapless topological states in the literature, they are fundamentally distinct from critical topological states. In particular, the nontrivial topology of a topological semimetal is not intrinsically tied to bulk gaplessness. For example, while a three-dimensional Weyl semimetal is gapless at isolated points in momentum space, any two-dimensional (2D) slice that avoids these points remains gapped. The associated Chern number of such slices underlies the emergence of Fermi arc surface states. Topological semimetals thus rely on a momentum-dependent spectral gap. By contrast, in critical topological states, topological edge states are intrinsically linked to a gapless bulk. For instance, in 1D gapless systems, there is no way to define a lower-dimensional topological invariant. Thus, critical topological states constitute a fundamentally new type of topological state, distinct from and extending beyond previously established classes, including topological insulators and topological semimetals.

On a different front, within the statistical physics community, critical topological states—also referred to as critical topological phase transitions—represent a novel class of phase transitions beyond the traditional Landau–Ginzburg–Wilson symmetry-breaking paradigm. It is widely believed that the universality class of a phase transition, in both many-body and free-fermion systems, is uniquely determined by a set of critical exponents. However, the discovery of critical topological phase transitions reveals that phase transitions with the same critical exponents can still be distinguished by whether topological edge states coexist at the gapless phase boundary. More importantly, in the existing literature, multi-critical points separating different phase boundaries are typically attributed to changes in critical exponents. By contrast, critical topological phase transitions provide a new mechanism: a topologically enforced multicritical point can arise between phase boundaries that share the *same* critical exponents but possess distinct gapless topological properties. We also emphasize that, unlike extensively studied topological phase transitions in the literature—which generally involve either topological defect-driven transitions, such as the Kosterlitz–Thouless transition, or

transitions between topologically trivial and nontrivial gapped phases, neither of which host topological edge states exactly at the transition point—critical topological phase transitions support robust topological edge states intrinsic to the gapless phase boundary itself, thereby enabling a topological classification of gapless systems. Therefore, critical topological phase transitions constitute a fundamentally new type of phase transition, extending beyond all previously recognized classes, including conventional topological phase transitions.

From the experimental perspective, although significant theoretical progress on gapless critical topology has been achieved in recent years, experimental realizations remain extremely scarce. This is because such novel phenomena are difficult to access, and their critical ground states are challenging to prepare, via nonequilibrium protocols, posing substantial obstacles for modern quantum simulation platforms, particularly in higher dimensions, where even richer critical topological physics is expected compared to 1D. Our experiments avoid these difficulties by preparing and observing critical topology within classical simulation platforms. Specifically, we start from the well-known 1D Su-Schrieffer-Hegger (SSH) model and further design a 2D lattice model to realize critical topology beyond 1D. This strategy provides more direct access to critical phase boundaries and is readily extendable to higher dimensions. Therefore, our study constitutes a first step toward the experimental observation of critical topology in the community.

We also wish to emphasize that critical topology is not a trivial generalization of topology to gapless phase boundaries. Rather, the interplay between criticality and topology gives rise to qualitatively new physics beyond that of conventional topological insulators and topological semimetals. We remark that the critical topology here originates from *kinetic inversion* rather than band inversion [13]. In familiar gapped topological insulators, the underlying mechanism for realizing nontrivial topology is well understood in terms of band inversion. Consider a spatial interface between a trivial gapped phase (or vacuum) with the topological invariant $W = 0$ and a

topological insulating phase with $W = 1$ (see the yellow line in Fig. S1a). At the interface, the low-energy physics is described by a Dirac Hamiltonian with a mass term, $h_1 = -i\partial_x + m(x)$, where $m > 0$ ($m < 0$) corresponds to the trivial (topological) phase. For a spatially varying mass profile $m(x)$ interpolating between these two phases, as shown in Fig. S1b, a topological edge mode is trapped at the interface, exponentially localized with a decay length set by the inverse bulk gap, $\sim 1/|m(x \rightarrow \pm\infty)|$. This mass (or band) inversion mechanism underlies the conventional bulk–boundary correspondence in most gapped topological insulators and superconductors.

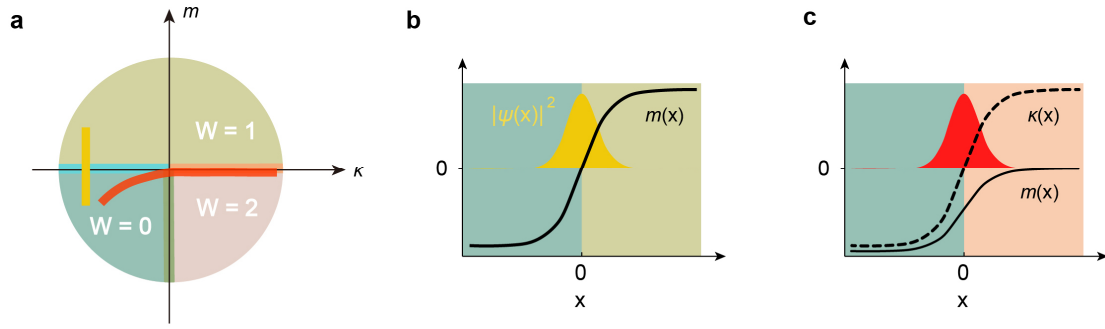


Figure S1 | Illustration of kinetic inversion for 1D topological critical states. **a**, Phase diagram in the vicinity of a multi-critical point. Any path connecting the two phases (e.g., the yellow and red lines) necessarily passes through a singular point. **b**, A spatial interface between a trivial phase and a gapped topological phase. This configuration hosts an exponentially localized zero-energy mode (yellow), arising from the sign change of the mass term. **c**, A spatial interface corresponding to the red path shown in **a**. A localized mode again emerges, now induced by a sign change of the kinetic term. Adapted from Ref. [13].

In contrast, the situation is fundamentally different for critical topological states. We again consider an interface between the vacuum and a critical topological state (the phase boundary between $W = 1$ and $W = 2$), now passing through a multicritical point that separates topologically trivial and nontrivial gapless phase boundaries, as illustrated by the red line in Fig. S1a. Near this multicritical point, the low-energy physics is instead described by [13]

$$h_2 = -\partial_x^2 - \kappa(x)\partial_x + m(x), \quad (\text{S1})$$

where $\kappa(x)$ denotes the kinetic parameter. In this case, it is not the mass term that changes sign, but rather the kinetic term, as shown in Fig. S1c. This is sufficient to guarantee an exponentially localized zero-energy solution of $h_2\psi(x) = 0$, provided that $\kappa(x)$ is asymptotically positive on one side and negative on the other. This mechanism, referred to as *kinetic inversion*, represents a new route to generating topological edge modes within the bulk continuum [13]. The resulting edge states remain stable under arbitrary deformations of h_2 (e.g., higher-order derivative terms) that do not induce bulk phase transitions, as discussed in Ref. [13]. This stability can be understood as a generalization of an index theorem to gapless systems. Notably, the localization length in this case is set by the kinetic parameter, with $\psi(x) \sim e^{-\kappa x}$ at the interface (for $m = 0$), and is thus not directly tied to any mass scale. For $x \rightarrow +\infty$, the effective Hamiltonian takes the form $h_2 \sim -\partial_x - \frac{1}{\kappa}\partial_x^2$, indicating that the localization length $\sim \frac{1}{\kappa}$ arises from an renormalization group (RG)-irrelevant perturbation relative to the leading kinetic term. This perspective parallels the gapped case, where the localization length $\sim \frac{1}{|m|}$ can likewise be viewed as the prefactor of an RG-irrelevant kinetic term about the massive-fixed point. In both gapped and gapless settings, the edge mode becomes sharply localized in the RG fixed-point limit.

Furthermore, it is found that entanglement mid-gap states can take a generalized topological index and robust signatures to characterize critical topology, even without a full spectral gap and with the breakdown of conventional topological invariants. Moreover, phase boundaries themselves undergo topological transitions through the multi-critical point, revealing a hidden hierarchical structure of the topological phase diagram: gapped topological phases as regions, gapless critical topological states as boundaries, and multi-critical points as transition points of critical topological states. Such a phase diagram unveils a mechanism of topology-driven multi-criticality and a

hidden hierarchical topological structure usually ignored before, largely enriching the understanding of phase transitions.

In summary, although the critical topological phenomena observed in our experiments share some superficial similarities with well-established gapped topological insulators, they exhibit fundamentally new physics that cannot be described within conventional topological band theory and are unique to gapless systems. This indicates that critical topology is not a trivial combination of topology, criticality, and entanglement. Instead, it reveals a rich class of physics intrinsic to gapless systems. Our work represents a first step toward this emerging direction by observing the free-fermion critical topology in 1D and 2D via entanglement-based observables.

Supplementary Note 2. Bulk entanglement spectrum as a universal fingerprint for identifying critical topology in free-fermion systems

In the main text, we focus on critical topology in non-interacting free-fermion systems, which are experimentally more accessible in classical simulators. A central question that then arises is how to identify gapless band topology in such systems. In gapped systems, integer topological invariants—such as the winding number or, more generally, quantities derived from the Berry curvatures in parameter space—are used to classify different topological insulators or semimetals protected by various symmetries. Importantly, nonzero bulk topological invariants correspond to the presence of robust edge states under open boundary conditions, as encapsulated by the well-known bulk–boundary correspondence. This principle constitutes a central ingredient of topological band theory, which has been extensively developed over the past two decades and serves as the standard paradigm for understanding gapped topological states. By contrast, at gapless phase boundaries associated with critical topology, these conventional topological invariants—based on, such as the winding number or Chern number—are no longer well defined, as they exhibit intrinsic singularities in parameter space when the gap closes. Consequently, the topological

band theory framework developed for gapped systems breaks down, necessitating a new theoretical approach to characterize critical topology in free-fermion systems.

On a different front, beyond conventional topological invariants, an exact correspondence between the bulk entanglement spectrum and the low-energy boundary spectrum under open boundary conditions has been established by Fidkowski [14] for gapped topological insulators, using the spectral-flattening construction. This result can be viewed as another manifestation of the bulk (entanglement spectrum)–boundary (energy spectrum) correspondence in the gapped topological systems, commonly referred to as the Li–Haldane correspondence [15]. However, for the critical topology discussed in our work, whether the Li-Haldane correspondence can be directly applied is a highly nontrivial issue, as the spectral flattening construction breaks down in this case.

Very recently, Ref. [16] has analytically established a generalized Li–Haldane correspondence for critical topology in free-fermion systems in *arbitrary dimensions*, using CFT methods that are inherently suited to gapless systems. A brief outline of the proof extracted from Ref. [6] is provided as follows:

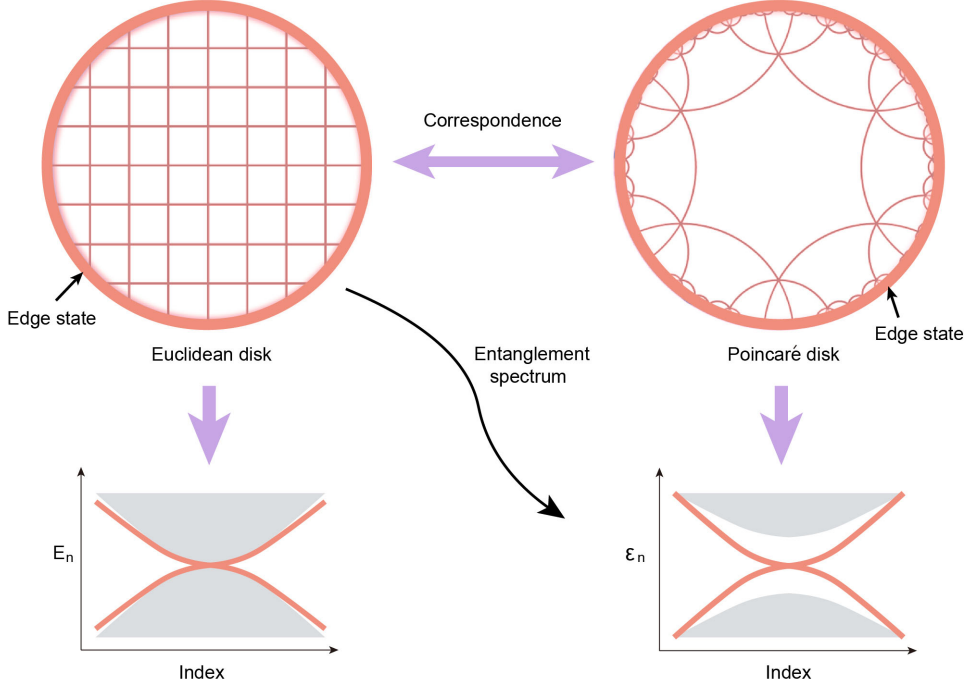


Figure S2 | Illustration of the correspondence between the energy spectrum of a hyperbolic disk and the entanglement spectrum of a flat disk. Adapted from Ref. [16].

Considering a spherical subregion $A = \{\mathbf{x} \in \mathbb{R}^d \mid |\mathbf{x}| < R, t = 0\}$, on the $t = 0$ time slice in Euclidean spacetime, with the metric defined by $ds_{\mathbb{R}^{d+1}}^2 = d\tau^2 + dr^2 + r^2 d\Omega_{d-1}^2$, to construct the relation between the entanglement problem and the boundary CFT problem. Here, $d\Omega_{d-1}^2$ denotes the metric of the unit sphere, of which the boundary ∂A defines the entangling surface, regulated by a UV cutoff ϵ . $\text{Tr} \hat{\rho}_A^n$ is represented by a path integral on an n -fold branched cover, which is conformally equivalent to $S_{2\pi n R}^1 \times \mathbb{H}^d$ [17], with $\mathbb{H}^d$ being the hyperbolic space in d dimension. By introducing the coordinates (τ, u, Ω_{d-1}) , we can write the conformal mapping explicitly as

$$t = R \frac{\sinh(\frac{\tau}{R})}{\cosh u + \cosh(\frac{\tau}{R})}, \quad r = R \frac{\sinh u}{\cosh u + \cosh(\frac{\tau}{R})}, \quad (\text{S2})$$

with the angular coordinates on S^{d-1} unchanged. Based on this, the flat metric can be represented by $ds^2 = \Omega^{-2} \left(d\tau^2 + R^2 (du^2 + \sinh^2 u d\Omega_{d-1}^2) \right)$, with $\Omega = \cosh u + \cosh(\tau/R)$ being a Weyl factor. Removing this factor yields the metric on $S^1 \times \mathbb{H}^d$,

then $ds_{s^1 \times \mathbb{H}^d}^2 = d\tau^2 + R^2(du^2 + \sinh^2 u d\Omega_{d-1}^2)$ is obtained, with a cutoff $u_{max} \sim \log \frac{2R}{\epsilon}$ imposed by the entanglement surface. As a result, the entanglement Hamiltonian for region A is unitarily equivalent to the Hamiltonian

$$\hat{K}_A = 2\pi R U^\dagger \hat{H}_{\mathbb{D}} U + \Lambda, \quad (S3)$$

which is defined on $\mathbb{H}^d$ with an asymptotic boundary at $u = u_{max}$ up to an overall Casimir constant (see Fig. S2 for an illustration). Moreover, via $\rho = \tanh \frac{u}{2}$, the spatial metric $ds_{\mathbb{H}^d}^2 = R^2(du^2 + \sinh^2 u d\Omega_{d-1}^2)$ can be mapped to the standard Poincare disk, yielding

$$ds_{\mathbb{H}^d}^2 = \frac{4R^2}{(1-\rho^2)^2} (d\rho^2 + \rho^2 d\Omega_{d-1}^2). \quad (S4)$$

In 1+1D, the hyperbolic disk will degenerate into a line segment, and the CFT energy spectrum on this segment can be solved exactly, allowing to obtain an analytical expression for the entanglement spectrum. We here consider the 1+1D case to check the previous results shown in Ref. [18]. Wherein, the hyperbolic space $\mathbb{H}^1$ lacks intrinsic curvature and reduces to a flat 1D line segment. The spatial metric simplifies to $ds_{\mathbb{H}^1}^2 = R^2 du^2$, representing a flat 1D segment with an effective length determined by the entanglement cut: $L_{eff} = \int_{-u_{max}}^{u_{max}} R du \simeq 2R \log \left(\frac{2R}{\epsilon} \right)$. For a general 1 + 1D CFT defined on this strip with conformal boundary conditions, the exact energy spectrum is analytically known to be $E_n = \frac{\pi}{L_{eff}} \left(\Delta_n - \frac{c}{24} \right)$, where c is the central charge, Δ_n is the scaling dimension of the boundary primary operator. As the entanglement Hamiltonian $\hat{K}_A$ is related to the physical Hamiltonian on $\mathbb{H}^1$ evaluated over the Euclidean thermal circle of circumference $2\pi R$, the entanglement spectrum is then simply given by $\xi_n = 2\pi R E_n$. Later, substituting L_{eff} , we can directly recover the analytical result for the 1+1D entanglement spectrum:

$$\xi_n = \frac{\pi^2}{\log \left(\frac{2R}{\epsilon} \right)} \left(\Delta_n - \frac{c}{24} \right), \quad (S5)$$

confirming the results in previous literature [5, 18].

For higher-dimensional systems of interest here, although exact solutions of generic CFTs on a hyperbolic disk remain intractable and an open question, the situation simplifies considerably for CFTs in free-fermion systems. Crucially, the hyperbolic metric is Weyl-equivalent to that of the flat disk $\mathbb{D}^d$. Therefore, if the Dirac Hamiltonian on the flat disk satisfies $\mathcal{D}_{\mathbb{D}}\psi = E\psi$ and supports a zero-energy mode, the transformation properties of the Dirac operator under Weyl rescaling guarantee that the corresponding operator $\mathcal{D}_{\mathbb{H}}\tilde{\psi} = E\tilde{\psi}$ also supports a zero mode [16]. Consequently, the entanglement spectrum inherits the degeneracy associated with zero modes of the edge spectrum. This establishes a rigorous correspondence between the boundary zero modes in the entanglement spectrum and those of a $(d + 1)$ -dimensional critical topological states in free-fermion systems.

Therefore, although no standard topological band theory description applies to free-fermion critical topological states, the entanglement spectrum—according to current understanding—offers a unique diagnostic tool for identifying nontrivial topology and establishing bulk–boundary correspondence at gapless phase boundaries. We emphasize that previous uses of the entanglement spectrum as a probe of topology have been largely restricted to gapped topological insulators. More importantly, identifying nontrivial topology at gapless phase boundaries via the entanglement spectrum is not a straightforward generalization of the gapped case. Our experiment thus constitutes a first step toward observing and probing critical topology through measurements of the entanglement spectrum.

Supplementary Note 3. Various critical states in 1D models

In this section, we provide several 1D critical states that are not explored in both the main text and the literature. As illustrated in Fig. S3a, we introduce additional t_3 , t_4 , and t_5 , to obtain negative and larger winding numbers -1 , 3 , and 4 , respectively. In this regard, Figs. S3b-S3e show critical phases adjacent to gapped phases with those

negative and larger winding numbers. The specific parameters are labeled above each panel. As we can see, the critical phases which are adjacent to the trivial phase ($W = 0$), always exhibit trivial gapless topology, as implied by the bulk feature of the mid-gap states (middle panel) and the gapped entanglement spectra (right panel). Meanwhile, the critical phases between $W = 1$ and $W = 3$, or $W = 2$ and $W = 4$ denotes the critical topological states. Their mid-gap states indeed show localization around the boundary, and the entanglement spectra also support mid-gap states at 0.5 , demonstrating the correspondence between the edge spectrum and the entanglement spectrum for more critical topological states.

Furthermore, we propose a conjecture here that the number of paired mid-gap states in the entanglement spectrum for critical phases always inherits the adjacent gapped phase with the smallest absolute value of topological numbers, as supported by all numerical calculations in this study. For example, for the trivial critical phase $0 \leftrightarrow 1$ studied in the main text, and $0 \leftrightarrow 2$ and $-1 \leftrightarrow 1 \leftrightarrow 0$ provided here, the smallest absolute value of topological numbers of the adjacent gapped phases is always zero, thereby leading to a gapped entanglement spectrum. On the contrary, the critical topological state is only near the gapped topological phases with nonvanishing topological numbers, and the smallest topological number determines the entanglement spectrum, which can be evidenced by the $1 \leftrightarrow 2$, $1 \leftrightarrow 3$, and $2 \leftrightarrow 4$ cases. We remark that it is not an easy task to obtain a full phase transition diagram for a given Hamiltonian. In that way, detecting a critical topological state based on whether it is adjacent to a trivial phase is not that direct. Nevertheless, the entanglement spectrum can give a general diagnosis. Based on these observations, we here also propose a tentative criterion for free-fermion topological critical states: they are gapless phases that cannot be adiabatically deformed into a gapped phase with a trivial topological invariant under any continuous, symmetry-preserving perturbation, without inducing additional spectral degeneracies between occupied and unoccupied bands in the process. Rigorous theoretical and experimental verification, or falsification, of this definition will be a key direction for future work.

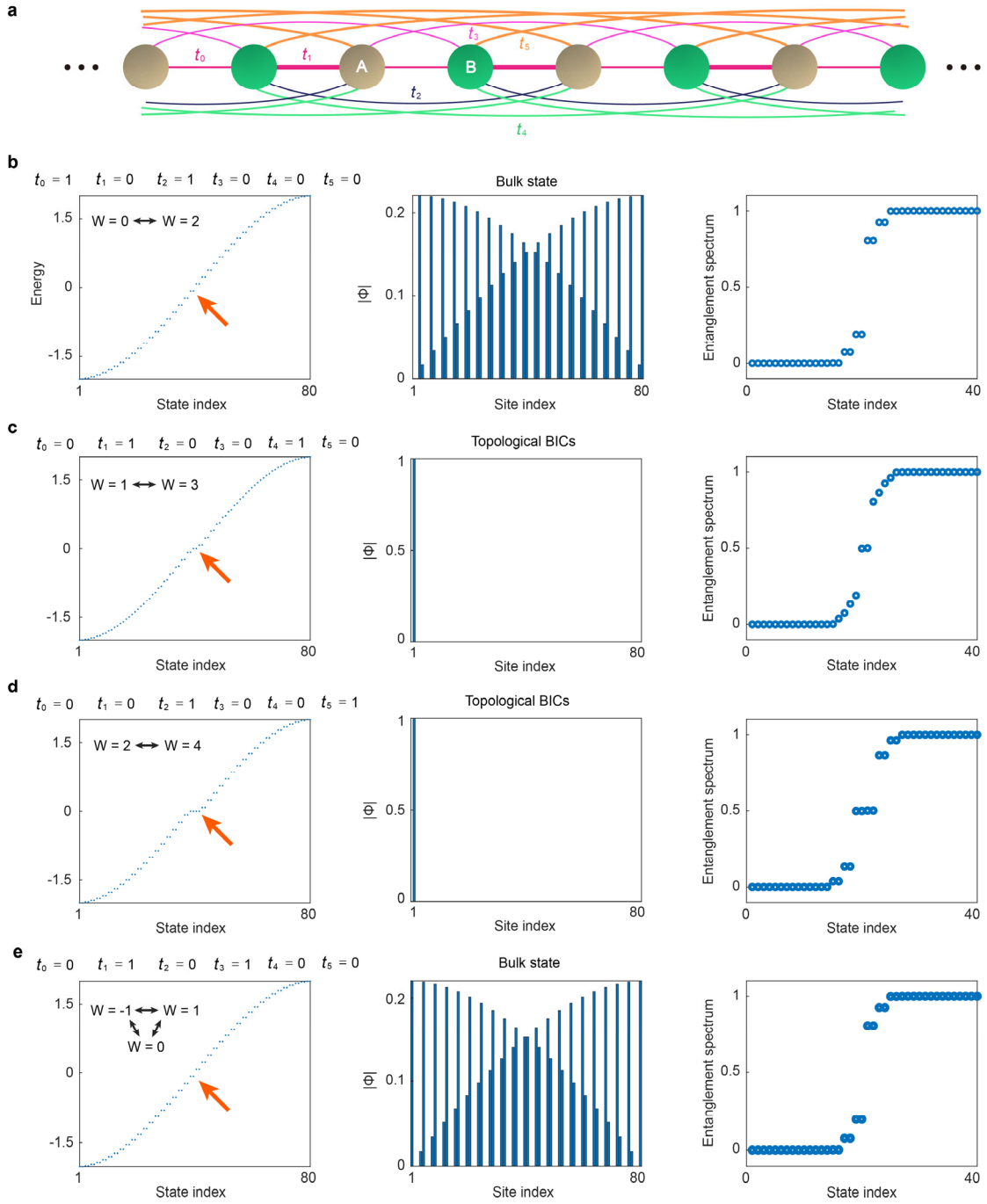


Figure S3 | Topological critical states involved with larger and negative winding numbers.

a, Illustration of the 1D SSH model with additional long-range hoppings. **b-e**, Band structures, mid-gap states, and entanglement spectra for different criticalities not explored in the main text.

Here, BICs denote bound states in the continuum.

Supplementary Note 4. Details of the geometric parameters of experimental samples

The airborne phononic crystals are taken as the platform to explore the critical topological states. Here, the building block of the phononic crystal is the cylinder-shaped resonator cavity, whose boundary can be regarded as the hard wall for acoustic waves. For each cavity, we only consider the lowest p_z mode of the sound pressure, which takes the role of the onsite potential in tight-binding models. The connecting tubes denote the hopping terms, whose radii are proportional to the coupling strength. The unit cell structures for the experimental SSH and the spin Chern lattices are schematically illustrated in Fig. S4, where the geometric dimensions are given in detail. The phononic crystal samples are manufactured by 3D-printing using the photosensitive resin, which has an additional thickness of 2mm. The resin can be regarded as the hard-wall boundaries for airborne sound waves, as it has a huge acoustic impedance mismatch with the air background, at least in the frequency regime we are interested. In experiments, the phononic SSH lattice for realizing critical topological states consists of 40 unit cells in total, while the 2D bilayer phononic lattice includes 20×20 unit cells in a rhombus shape. To locate both the gapless trivial (gTrivial) states and critical topological states, we fine-tuned the geometric parameters to search for phase transition criticalities between different topological phases, which share the same critical feature with tight-binding models. It is worth noting that the chiral symmetry is always slightly broken in phononic systems, which leaves the topological states derived from the frequency of band degeneracy and hybridize a bit with bulk states at criticality, as shown in Fig. S5. However, the topological signature exhibited in the entanglement spectrum is more robust, as there exists an emergent particle-hole symmetry in the entanglement spectrum to protect the robustness of mid-gap states [19].

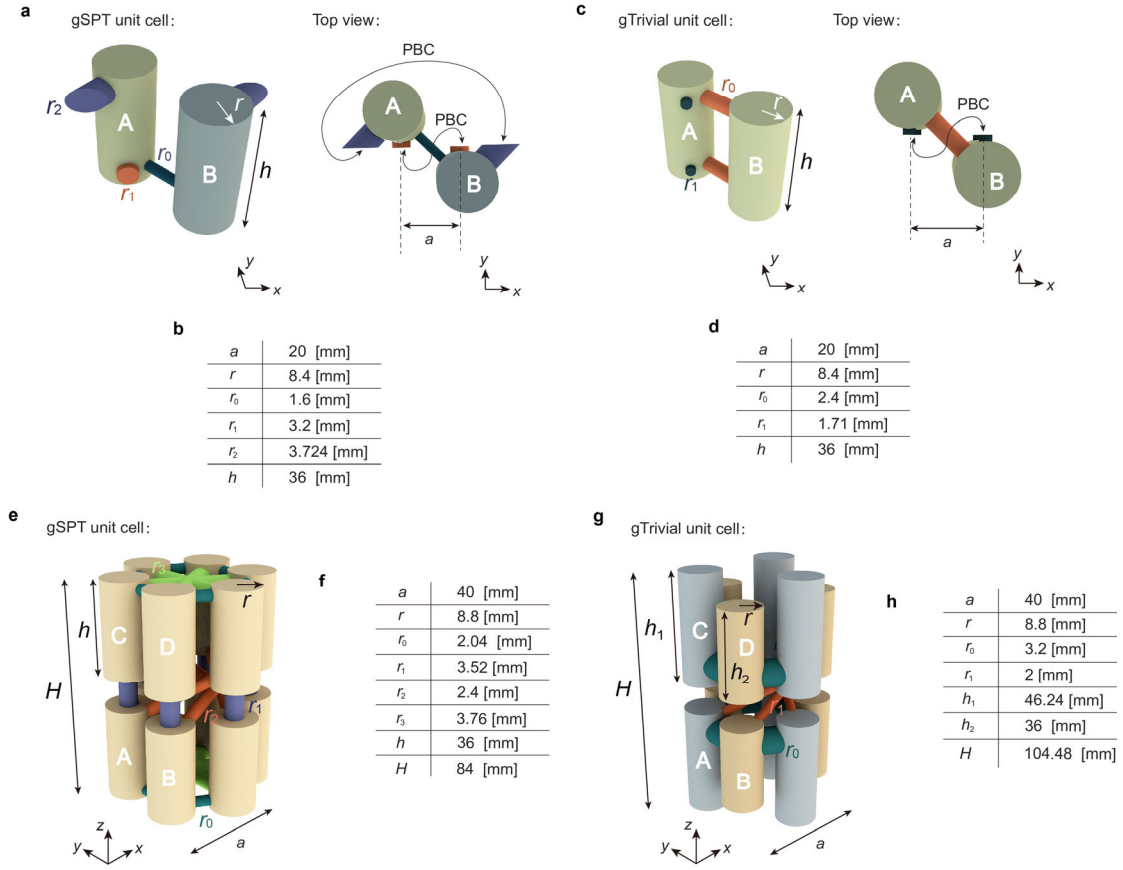


Figure S4 | Detailed geometric dimensions of air-structure unit cells for the gTrivial and critical topological states for both 1D and 2D phononic lattices. Here, PBC denotes the periodic boundary condition.

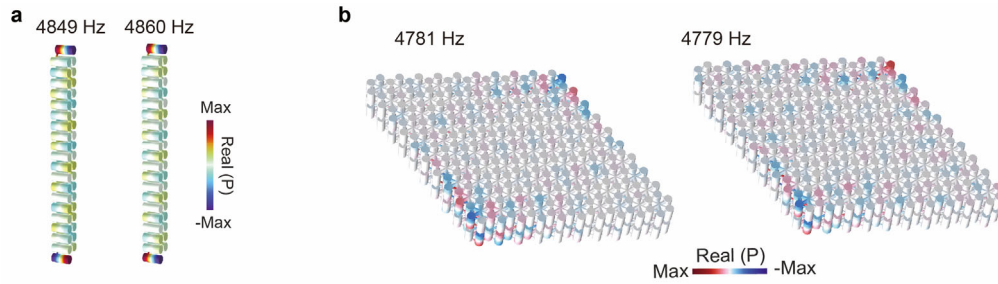


Figure S5 | Simulation of boundary states for 1D (a) and 2D (b) phononic critical topological phases.

Supplementary Note 5. Details of the acoustic pump-probe measurement protocol

As clarified in the main text, the basic logic of our experimental approach is shown in Fig. S6: First, we map a non-interacting electron system (a free-fermion system) to a non-interacting boson system when only single-particle properties are concerned (such as the topological physics studied in this work). Accompanying this mapping is the use of a virtual filling of all states below a “Fermi frequency” which mimics filling of electrons for all states below the Fermi level. Second, a non-interacting boson system becomes a classical wave system when the boson numbers are significantly large, according to the quantum-classical correspondence principle. This principle leads to exact one-to-one correspondence between the quantum mechanical theory of phonons and the classical acoustic wave measurements, because in our experiments the number of excited acoustic phonons can easily reach to 10^{24} . Therefore, by reversing these one-to-one mappings, our classical phononic system can directly mimic the critical topological states in free-fermion systems.

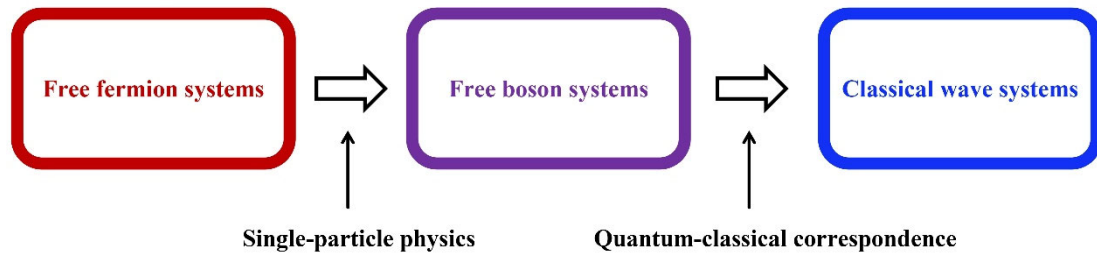


Figure S6 | Schematic of the underlying logic of our experiments. From left to right: First, map free fermion systems to free boson systems when only single-particle physics is concerned. Then, map from free boson systems to classical wave systems based on the quantum-classical correspondence principle.

The measurement protocol is then elaborated in this section. The experimental results throughout this work are all derived from the pump-probe response measurement. In the experimental setup, a moving-coil loudspeaker with a diameter

of 5mm functions as the acoustic monopole source. The loudspeaker is positioned within a cavity near the center (edge) of the phononic crystal via an access port to excite the bulk (edge) responses. The loudspeaker is driven by a vector network analyzer (Keysight E5061B), which supplies the excitation signal. The stimulated frequency range covers the concerned phononic band dispersions. The source signal is amplified by an amplifier to enhance the signal-to-noise ratio. Meanwhile, the acoustic pressure field responses—both in amplitude and phase—inside all other acoustic cavities are captured using a sub-wavelength microphone (MC401, from BSWA TECH) with a 6mm diameter. The microphone is also linked to the network analyzer, enabling synchronous recording of the detected sound responses throughout the measurement procedure. Specifically, the measured acoustic field is denoted by the response tensor $\chi_{\alpha\beta}(p, q, f)$. Here, the acoustic source is situated at (α, q) and the microphone is placed at (β, p) . α and β represent different acoustic cavity sublattices within the unit cell labeled by p or q . The monopole source is applied with a harmonic time dependence of $e^{i2\pi ft}$. In which case, χ is in direct proportion to the frequency-domain single-particle retarded Green's function G , which is the solution of

$$(\nabla^2 + k_c^2)G_{\alpha\beta}(p, q, f) = \delta(p - q). \quad (\text{S6})$$

Here, $k_c = 2\pi f/c$, f and c are the sound frequency and sound speed in air, respectively. In the Lehmann spectral representation and based on the quantum-classical correspondence principle, the Green's function can be expressed in the following form,

$$G_{\alpha\beta}(p, q, f) = \sum_n \frac{\langle q\alpha|\phi_n\rangle\langle\phi_n|p\beta\rangle}{f - f_n + i\gamma_n}, \quad (\text{S7})$$

where $|\phi_n\rangle$ is the real-space acoustic eigenfunction of the whole phononic crystal corresponding to the n th eigen-frequency f_n . γ_n is a frequency-damping coefficient. Strictly, the frequency eigenvalue of the acoustic equation is quadratic ($\sim f^2$), we here simplify it to be linear ($\sim f$), without changing any physical consequence including topology.

The entanglement spectra in the main text are derived from the response functions which contain the information of eigen-frequencies and eigen-wavefunctions. However, the direct measurement of the real-space response tensor indicates that both the acoustic loudspeaker and the microphone have to traverse all acoustic cavities, leading to time-consuming experimental procedures. Nevertheless, given the sufficiently large experimental sample size, the influence of open boundaries on bulk responses becomes negligible, which enables us to regard the system as effectively subject to periodic boundary conditions. Then we can only measure the momentum-space response tensor $\chi_{\alpha\beta}(f, \mathbf{k}) \propto \sum_{R_p} e^{-i\mathbf{k}(R_p - R_q)} G_{\alpha\beta}(p, q, f)$, the Fourier transformation of the real-space response tensor. Here, R_p and R_q are the coordinates of the phononic unit cells labeled by p and q . To obtain this, the acoustic excitations for bulk responses are conducted only at different cavities within one central unit cell. The final response tensors are 2×2 and 4×4 matrices for the 1D and 2D phononic samples, respectively. Furthermore, based on the linear response theory, the momentum-space response tensor is also proportional to the momentum-space Green function,

$$\chi_{\alpha\beta}(f, \mathbf{k}) \propto G_{\alpha\beta}(f, \mathbf{k}) = \sum_n \frac{\langle \alpha | u_{n,\mathbf{k}} \rangle \langle u_{n,\mathbf{k}} | \beta \rangle}{f - f_{n,\mathbf{k}} + i\gamma_{n,\mathbf{k}}}. \quad (\text{S8})$$

Here, $|u_{n,\mathbf{k}}\rangle$ denotes the periodic part of the phononic Bloch wavefunction corresponding to the eigen-frequency $f_{n,\mathbf{k}}$ labeled by the momentum $\mathbf{k}$ and band index n . Eq. S8 implies that the response function includes both the phononic eigen-frequencies and wavefunctions. To extract the eigen-information more precisely, we apply the singular value decomposition (SVD) to the momentum-space response tensor, which yields

$$\chi(f, \mathbf{k}) = U(f, \mathbf{k}) \Omega(f, \mathbf{k}) V^\dagger(f, \mathbf{k}), \quad (\text{S9})$$

where columns of U and V denote the left and right singular vectors, respectively. Ω is the diagonal matrix with non-negative real singular values sorted from the largest to the smallest. For instance, Ω in the 2D phononic case has the form of

$$\begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \sigma_3 & \\ & & & \sigma_4 \end{bmatrix} \text{ with } \sigma_1 \leq \sigma_2 \leq \sigma_3 \leq \sigma_4, \quad (\text{S10})$$

for all values of f and $\mathbf{k}$. Skillfully, the eigen-information is hidden in U and V . Recall that the largest singular values of a matrix M equal the square roots of eigenvalues of the covariance matrix $MM^\dagger$. Moreover, the corresponding left singular vectors are precisely the eigenvectors of $MM^\dagger$ associated with those largest eigenvalues. Here, the matrix M represents the green function G , and we have

$$G(f, \mathbf{k})G^\dagger(f, \mathbf{k}) = \Sigma_n \frac{|u_{n,\mathbf{k}}\rangle\langle u_{n,\mathbf{k}}|}{(f - f_{n,\mathbf{k}})^2 + \gamma_{n,\mathbf{k}}^2}, \quad (\text{S11})$$

whose eigenvectors are nothing but the Bloch wavefunctions $|u_{n,\mathbf{k}}\rangle$, corresponding to the eigenvalues $\frac{1}{(f - f_{n,\mathbf{k}})^2 + \gamma_{n,\mathbf{k}}^2}$. Therefore, the largest singular values in Ω are proportional to $\frac{1}{\sqrt{(f - f_{n,\mathbf{k}})^2 + \gamma_{n,\mathbf{k}}^2}}$, which would exhibit sharp resonance peaks around

the eigen-frequencies $f_{n,\mathbf{k}}$, so that they can reproduce the phononic band dispersions and locate the occupied states as well as band-touching points from response measurements. Furthermore, the critical band degeneracy can be confirmed by the number of the largest singular values. For example, the four singular values in Eq. S10 are nearly the same at the critical four-fold band degeneracy at the M point in the 2D phononic topological critical states, which can be hinted by the localized distribution of σ_4 . More importantly, at these resonant frequencies $f_{n,\mathbf{k}}$, the first column of U corresponds precisely to the Bloch state $|u_{n,\mathbf{k}}\rangle$. In the 2D phononic case with two bands occupied, the second column of U at $f_{1,\mathbf{k}}$ just corresponds to $|u_{2,\mathbf{k}}\rangle$, as it mostly contains contributions from the nearest band except the first one. We then discretize the Bloch wavevectors into 40 non-equivalent points in the first Brillouin zone (BZ) for the 1D phononic lattice and a 20×20 mesh points in the rhombic BZ for the 2D case. With all Bloch states obtained, the momentum-space correlation matrix can be constructed as

$$C_{\alpha\beta}(\mathbf{k}) = \Sigma_n \langle \alpha | u_{n,\mathbf{k}} \rangle \langle u_{n,\mathbf{k}} | \beta \rangle, \quad (\text{S12})$$

where the summation runs over the first and the first two bands for the 1D and 2D cases, respectively. By further performing a Fourier transformation, the real-space correlation function is obtained as

$$C_{q\alpha,p\beta}^A = \frac{1}{\mathcal{N}} \sum_{\mathbf{k}} e^{i\mathbf{k}(R_p - R_q)} C_{\alpha\beta}(\mathbf{k}). \quad (\text{S13})$$

Here, the summation is performed over the first BZ for the 1D case. In 2D, the summation is only for one momentum direction. $\mathcal{N}$ denotes the number of momentum points upon the Fourier transformation. R_p and R_q are the coordinates of the phononic unit cells labeled by p and q , distributed in the subsystem A . The eigenvalue spectrum of C^A directly yields the experimentally measured entanglement spectra, and the corresponding eigenvectors give the entanglement wavefunctions.

Supplementary Note 6. Confirmation of the gapless feature via entanglement entropy scaling from experiments

In the main text, the gapless or critical feature of phononic topological critical states and gTrivial states is validated by the resonance of higher-order singular values extracted from experiments. In practice, the gapless feature can also be certified by the entanglement entropy scaling behavior, as elaborated below. The entanglement entropy can be directly obtained based on the entanglement spectrum $\{\varepsilon_n\}$ via [20]

$$S = -\text{Tr} \rho_A \log \rho_A = -\sum_n [\varepsilon_n \log \varepsilon_n + (1 - \varepsilon_n) \log(1 - \varepsilon_n)]. \quad (\text{S14})$$

Therefore, we can also derive the entanglement entropy from the experiments. For 1D gapless free-fermion systems with conformal critical points, the entanglement entropy is represented as [21]

$$S_A \cong \frac{c}{6} \log L, \quad (\text{S15})$$

where $c = c_L + c_R = 2$ is the total central charge of the underlying CFT, $c_L(c_R)$ is the chiral central charge of the left (right)-moving gapless mode. Eq. S15 implies that the entanglement entropy exhibits a linear scaling to $\log L$, with a coefficient $1/3$, for both topological critical states and gTrivial states. Figures S7a and S7b give

the entanglement entropy scaling derived from experimental data for 1D topological critical states and gTrivial states, both of which indeed display the nearly linearity versus L . The red lines are fitted using ‘polyfit’ in Matlab, giving the slope closer to $1/3$.

The scaling behavior of the entanglement entropy for 2D free-fermion gapless systems is more complicated, which is related to the Widom conjecture [21]. Nevertheless, when the Fermi surface reduces to conformal critical points, which just corresponds to our case, the entanglement entropy scaling can be described by [22]

$$S_A \sim L, \quad (\text{S16})$$

exhibiting a pure area law. The experiment results also confirm it, as shown in Figs. S7c and S7d.

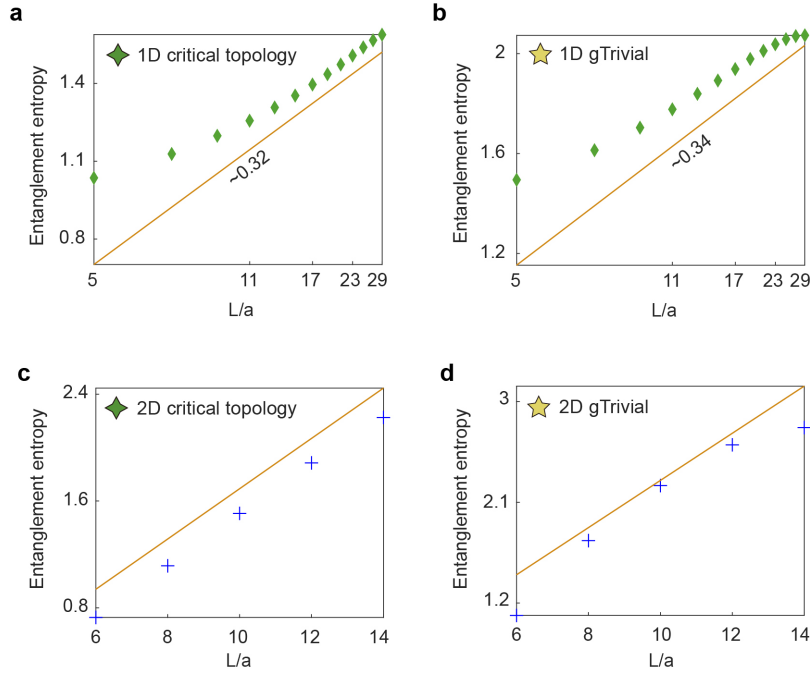


Figure S7 | Entanglement entropy scalings versus the length of subsystem, for the 1D topological critical states, 1D gTrivial states, 2D topological critical states, and 2D gTrivial states, from experimental data. The green dots represent the experimental results. The red lines are fitted, using ‘polyfit’ in Matlab.

Supplementary Note 7. Half-filling calculation and trivial contributions from lower-lying s -orbital phononic bands

Half-filling is typically implemented by selecting half of the eigenmodes of the supercell with either the lowest- or highest-energy (or frequency). For the 1D critical case with a 2-fold degenerate band-touching point, the half-filling is well defined for supercell eigenstates for both open and periodic boundary conditions. While for the 2D tight-binding spin Chern model studied here, due to the spin-mixing issue, the 4-fold degeneracy of band-touching points obstructs the well-defined half-filled occupied states in numerical calculation with periodic boundary conditions for the supercell. Nevertheless, a sufficiently small perturbation, which reduces the degenerate points to be 2-fold and meanwhile keeps both the protecting symmetry and topology, is considered for calculating the entanglement spectra of the 2D critical states. One can see Supplementary Note 9 for more parameter details. We remark that the gapless critical states with 2-fold degenerate band-touching points are also sufficient to demonstrate the critical topology in 2D.

For the topological gapless phononic bands, we focus on those arising from the p_z -orbital sound modes of each cavity. Here, as shown for the 1D case, the lower-energy s -orbital bands (marked in blue in Fig. S8a) lie far below the p -orbital bands of interest (marked in green in Fig. S8a), and do not contribute to the topological signatures observed in the entanglement spectrum. To verify this conclusion, we explicitly include the two s -orbital phononic bands together with the lowest p_z -orbital band beneath the band-touching point as occupied states in the entanglement spectrum calculation (Fig. S8b). As shown, the mid-gap entanglement states remain robust, confirming the topological irrelevance of the s -orbital bands. Other methods for topological invariants (e.g., the Wilson loop approach) confirm the same conclusion as well.

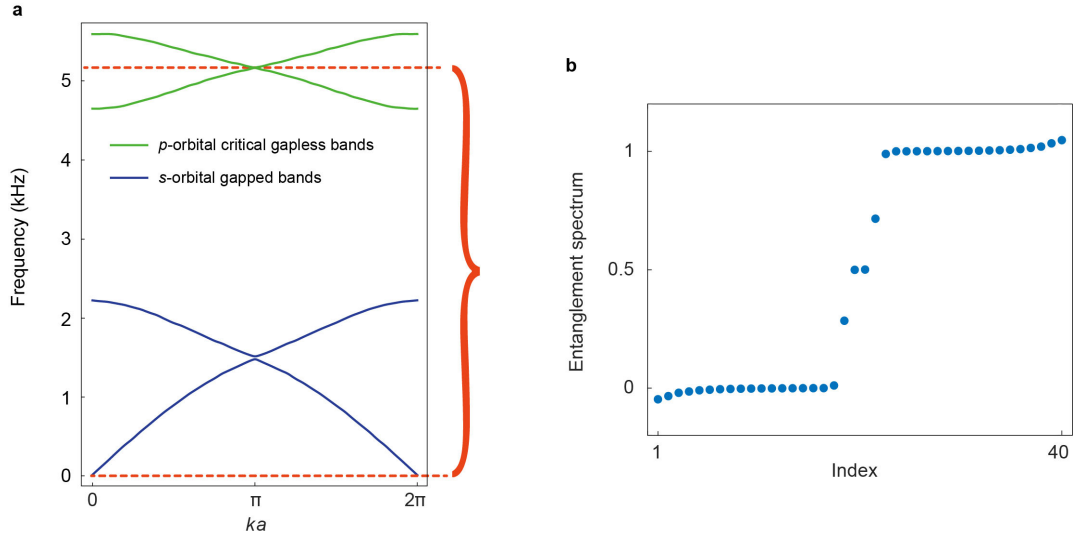


Figure S8 | Phononic band structure for 1D critical topology and entanglement spectrum calculated from lower three bands.

Supplementary Note 8. Correlation function method for obtaining the entanglement spectrum in free-fermion systems

The entanglement spectrum is defined as the eigenvalues of the entanglement Hamiltonian matrix H^E , which, in the free-fermion systems, can be obtained more conveniently by computing the eigenvalues of the system's correlation matrix [23]. Here, we give a brief derivation of this connection. The entanglement Hamiltonian matrix H^E is defined via

$$\rho_A = \frac{1}{Z} e^{-H^E}, \quad (\text{S17})$$

where $\rho_A = \text{Tr}_{i \in B}(\rho)$ is the reduced density matrix by tracing out all degrees of freedom in the subsystem B . Here, the whole system is partitioned into A and its complement B . $\rho = |G\rangle\langle G|$ is the density matrix of the whole system with $|G\rangle = \prod_{\alpha \in occ.} \psi_{\alpha}^{\dagger} |0\rangle$ denoting the ground state. $\psi_{\alpha}^{\dagger}$ (ψ_{α}) denotes the fermionic creation (annihilation) operator at energy level α . *occ.* denotes the occupied states. $|0\rangle$ is empty state. we have $|\alpha\rangle = \psi_{\alpha}^{\dagger} |0\rangle$, $\psi_{\alpha} |0\rangle = 0$, and $\{\psi_{\alpha}, \psi_{\beta}^{\dagger}\} = \delta_{\alpha, \beta}$. Specifically,

for a free-fermion system, the entanglement Hamiltonian has the following quadratic form:

$$H^E = \sum_{i,j \in A} \hat{c}_i^\dagger h_{i,j}^E \hat{c}_j. \quad (\text{S18})$$

Here, $Z = \text{Tr}(e^{-H^E})$ is the normalization coefficient and $h_{i,j}^E$ is the entanglement Hamiltonian matrix element. $\hat{c}_i^\dagger(\hat{c}_j)$ is the fermionic creation (annihilation) operator at i (j) and satisfies the anti-commutation relationship $\{\hat{c}_i, \hat{c}_j^\dagger\} = \delta_{i,j}$. By diagonalizing h^E , we have $\phi_n(i)$ as the eigenfunction with eigenvalue ξ_n . Then the fermionic operator can be expressed by $\hat{c}_i = \sum_n \phi_n(i) \hat{a}_n$ with $\hat{a}_n$ denoting the creation operator at the level n , which leaves ρ_A to be a diagonalized form:

$$\rho_A = \frac{1}{Z} e^{-\sum_n \xi_n \hat{a}_n^\dagger \hat{a}_n}. \quad (\text{S19})$$

Here, the set $\{\xi_n\}$ forms nothing but the single-particle entanglement spectrum, which is directly related to the eigenvalues of the two-point correlation functions, as elaborated in the following.

For a free-fermion system, the two-point correlation matrix in A is defined by

$$C_{i,j}^A = \langle G | \hat{c}_i^\dagger \hat{c}_j | G \rangle, \quad (\text{S20})$$

where i and j label the degrees of freedom in A. By utilizing the relation between $\hat{c}_i^\dagger$ and $\psi_\alpha^\dagger$,

$$\hat{c}_i^\dagger = \sum_\alpha \langle \alpha | i \rangle \psi_\alpha^\dagger = \sum_\alpha \psi_\alpha^*(i) \psi_\alpha^\dagger, \quad (\text{S21})$$

we have

$$C_{i,j}^A = \sum_{\alpha \in \text{occ.}} C_\alpha^{\text{single}}(i, j). \quad (\text{S22})$$

Here, $C_\alpha^{\text{single}}(i, j) = \psi_\alpha^*(i) \psi_\alpha(j)$ is the single-particle correlation matrix. The correlation matrix $C_{i,j}^A$ can also be expressed as $C_{i,j}^A = \text{Tr}_A(\rho_A \hat{c}_i^\dagger \hat{c}_j)$. Based on Eq. S19, $C_{i,j}^A$ can reduce itself to

$$C_{i,j}^A = \sum_n \phi_n^*(i) \phi_n(j) \frac{1}{e^{\xi_{n+1}}}. \quad (\text{S23})$$

Therefore, we have $\sum_j C_{i,j}^A \phi_n(j) = \frac{1}{e^{\xi_{n+1}}} \phi_n(i)$, which means that $\phi_n(i)$ denotes the eigenfunction of the correlation matrix $C_{i,j}^A$ with eigenvalue $\frac{1}{e^{\xi_{n+1}}}$, where ξ_n exactly forms the single-particle entanglement spectrum, as shown in Eq. S19. In our paper, we utilize Eq. S22 to calculate and probe the entanglement spectrum $\varepsilon_n = \frac{1}{e^{\xi_{n+1}}}$ from both simulation and experiments. ε_n ranges from 0 to 1 based on its definition. More importantly, the mid-gap states at $\varepsilon_n = 0.5$ corresponds to the zero modes of the originally entanglement spectrum defined from the entanglement Hamiltonian, and the corresponding eigenvectors denote the entanglement wavefunctions.

All theoretical calculations of entanglement spectra as well as entanglement wavefunctions in this work—for both 1D and 2D models—are based on the correlation matrix method elaborated above. And these calculation results confirm that a generalized Li–Haldane correspondence for critical topology can hold in both 1D and higher dimension for non-interacting fermionic systems.

Supplementary Note 9. Robustness simulations of both edge states and the entanglement spectra

Topological edge states can be a secondary probe of critical topology, as they may not be robust against local disorders and chiral symmetry breaking. In contrast, the entanglement spectrum provides a more robust, faithful, and intrinsic signature of critical topology. To demonstrate this, we here give tight-binding robustness simulations of both edge states and the entanglement spectra for 1D and 2D critical topology, against certain perturbations.

We first consider chiral-symmetric local disorders that randomly occur in the finite 1D critical lattice (with 40 unit-cells in total). To keep the chiral symmetry, the random onsite energy is forbidden. Instead, the existing hoppings can be randomly altered. Starting from the hopping parameters $t_0 = 1$, $t_1 = 2.2$, and $t_2 = 1.2$ at

topological criticality, we independently set each hopping amplitude uniformly at random from the intervals $[0, 2]$, $[1, 3]$, and $[0, 2]$, respectively. The resultant eigenvalue spectrum, spatial distribution of one topological edge state, and the entanglement spectrum are displayed in Fig. S9a. As shown, the eigenvalue spectrum remains gapless and symmetric about zero energy—consistent with the preservation of chiral symmetry. Crucially, the topological features—including the localization of edge states (Fig. S9a2) and the presence of mid-gap entanglement eigenvalues at 0.5 (Fig. S9a3)—remain intact, confirming the intrinsic robustness of 1D critical topological states under chiral-symmetric disorders. For comparison, we next introduce weak, random onsite energy $E_\Delta \in [0, 0.1]$ at all lattice sites. The corresponding results are given in Fig. S9b. Without chiral symmetry, the eigenvalue spectrum loses its strict spectral symmetry about zero energy, and the edge states become delocalized. By contrast, the mid-gap entanglement eigenvalues remain tightly pinned near 0.5, demonstrating their exceptional resilience even when chiral symmetry is violated.

For 2D critical topological states, the entanglement spectrum is defined along conserved momenta k . Thus, we consider periodic perturbations to study their robustness. Specifically, the definition of 2D bulk spin Chern topology is irrelevant to the chiral symmetry, hence we introduce chiral-symmetry-breaking terms $h_{AD} = h_{BC} = t_{\text{breaking}}$ into the original Bloch Hamiltonian (see Eq. S24 in the following) to test the robustness. As illustrated in Fig. S10a, as the strength of t_{breaking} increases from 0 to 1, the fourfold-degenerate band-touching points (marked by red arrows) shift away from M points toward the KM lines, and split into double degenerate Dirac points. Concurrently, the gapless band structure becomes more asymmetrical about zero energy. Moreover, chiral symmetry breaking induces hybridization between the topological corner states and bulk states, leading to the delocalization of corner states, as evidenced in Fig. S10b. However, the mid-gap states in the entanglement spectra (see Fig. S10c) remain robust against stronger and stronger perturbations, showing their further stability as signatures of 2D critical

topology over topological boundary states. Note that a sufficiently small perturbation, such as $t_{\text{breaking}} = 10^{-12}$, or $t_2 = -0.5 + 10^{-12}$, is considered for calculating the entanglement spectra in Fig. 5h in the main text, which does not destroy any topology as well as the gapless feature of the system, and meanwhile results in well-defined half-filled occupied states in semimetals with multiple-degenerate (> 2) band-touching points.

We further remark that chiral symmetry is an ideal intrinsic symmetry that can only be approximated in real experimental settings, including the phononic crystals. Although the chiral symmetry is slightly broken due to the inevitable intra-sublattice couplings in our acoustic designs, the topological edge and corner states indeed partly hybridize with bulk states, but remain rather localized, as simulated and experimentally probed here. Nevertheless, turning to the mid-gap entanglement spectra avoids these concerns and thus is more direct, robust, and universal.

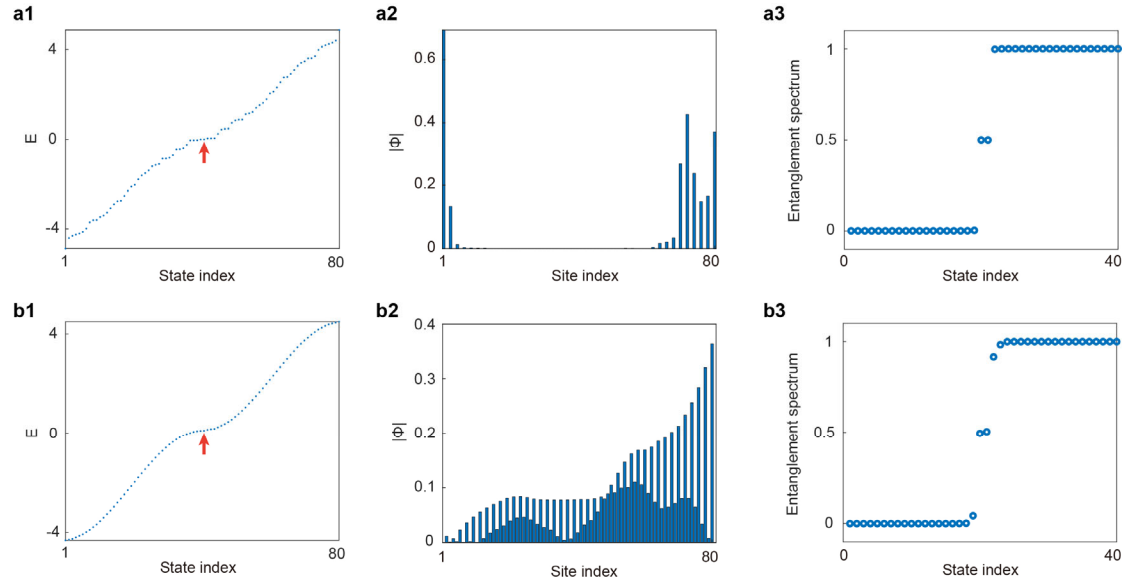


Figure S9 | Robustness simulations of 1D critical topological states. **a1-a3**, Corresponding eigenvalue spectrum, spatial distribution of one topological edge state, and entanglement spectrum, under random hoppings of $t_0 \in [0, 2]$, $t_1 \in [1, 3]$, and $t_2 \in [0, 2]$ in a finite lattice (40 unit cells) preserving the chiral symmetry. **b1-b3**, Same as **a1-a3**, but for chiral-symmetry-breaking random onsite energy $E_\Delta \in [0, 0.1]$ at all lattice sites. Red arrows in **a1** and **b1** mark the eigenvalues of **a2** and **b2**. The hopping parameters of original critical topology are set as $t_0 = 1$, $t_1 = 2.2$, and $t_2 = 1.2$.

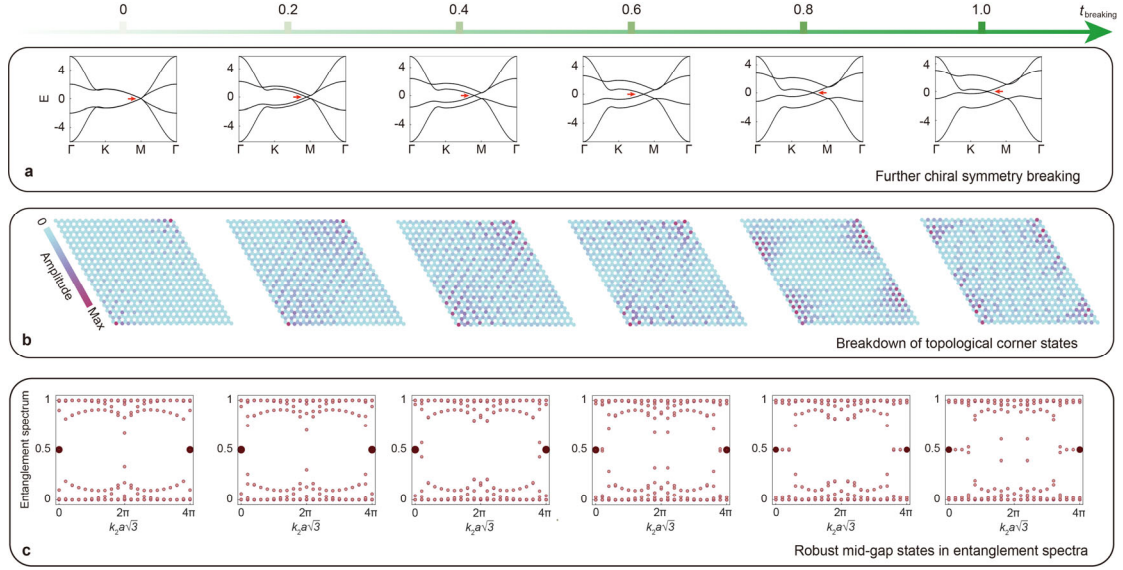


Figure S10 | Robustness simulations of 2D critical topological states. **a**, Bulk band structures along high symmetry lines under the progressively increasing chiral-symmetry-breaking perturbed hopping t_{breaking} . **b**, Corresponding spatial distributions of topological corner states, showing increasing delocalization. **c**, Corresponding entanglement spectra, remaining gapless.

Supplementary Note 10. Entanglement wavefunctions and their scalings

The entanglement wavefunction can provide a second evidence for detecting topological critical states and exclude the trivial critical phase which may show in-gap states near 0.5 in the entanglement spectrum. The special 1D multi-criticality at $0 \leftrightarrow 1 \leftrightarrow 2$, as marked by the red dot in Fig. S11a, just denotes such trivial case. Figure S11c shows the wavefunction of the mid-gap state labeled by the red arrow in Fig. S11b. It seems like an edge state. However, when the wavefunction is calculated in a larger lattice, the wavefunction exhibits a scaling-invariant feature, as shown in Fig. S11e. The scaling-invariance of the wavefunction implies its bulk origin. In contrast, the entanglement spectra exhibit in-gap states closer to 0.5 in a larger lattice, as shown in Figs. S11d and S11f. To further check its gapless topology, we need to turn to the

entanglement wavefunctions. As shown in Fig. S11g, although the wavefunction of the mid-gap entanglement state shows localization for the multi-criticality, the local density of states (LDOS) summed over the first four sites near the left boundary in the subsystem A exhibits an L -dependence, where L denotes the length of the system A. Such dependence implies that the entanglement wavefunction is not a pure edge state, which agrees with its trivial gapless topology. For comparison, we also give the entanglement wavefunction and its LDOS scaling for the topological critical states in Fig. S11h. As we can see, the LDOS scaling shows invariance versus L , demonstrating its pure edge state feature.

For 2D experiments, we here obtain the same phenomena about the entanglement wavefunction and its LDOS scaling for the topological critical states. As shown in Fig. S12, the wavefunction of one entanglement mid-gap state at $k = 0$ shows edge-state-like localization, and the LDOS summed over the first four rows is nearly invariant versus L , which further confirms the nontrivial gapless topology for the 2D phononic critical states from experiments. Alike to 1D, the multicriticality also exhibits a quadratic degeneracy, as shown in Fig. S12c, implying the Lifshitz transition between different critical phases. By contrast, the entanglement spectrum at multicriticality shows no mid-gap states near 0.5, so we do not need to double-check its trivial gapless topology via entanglement wavefunctions and their scaling with the subsystem's length.

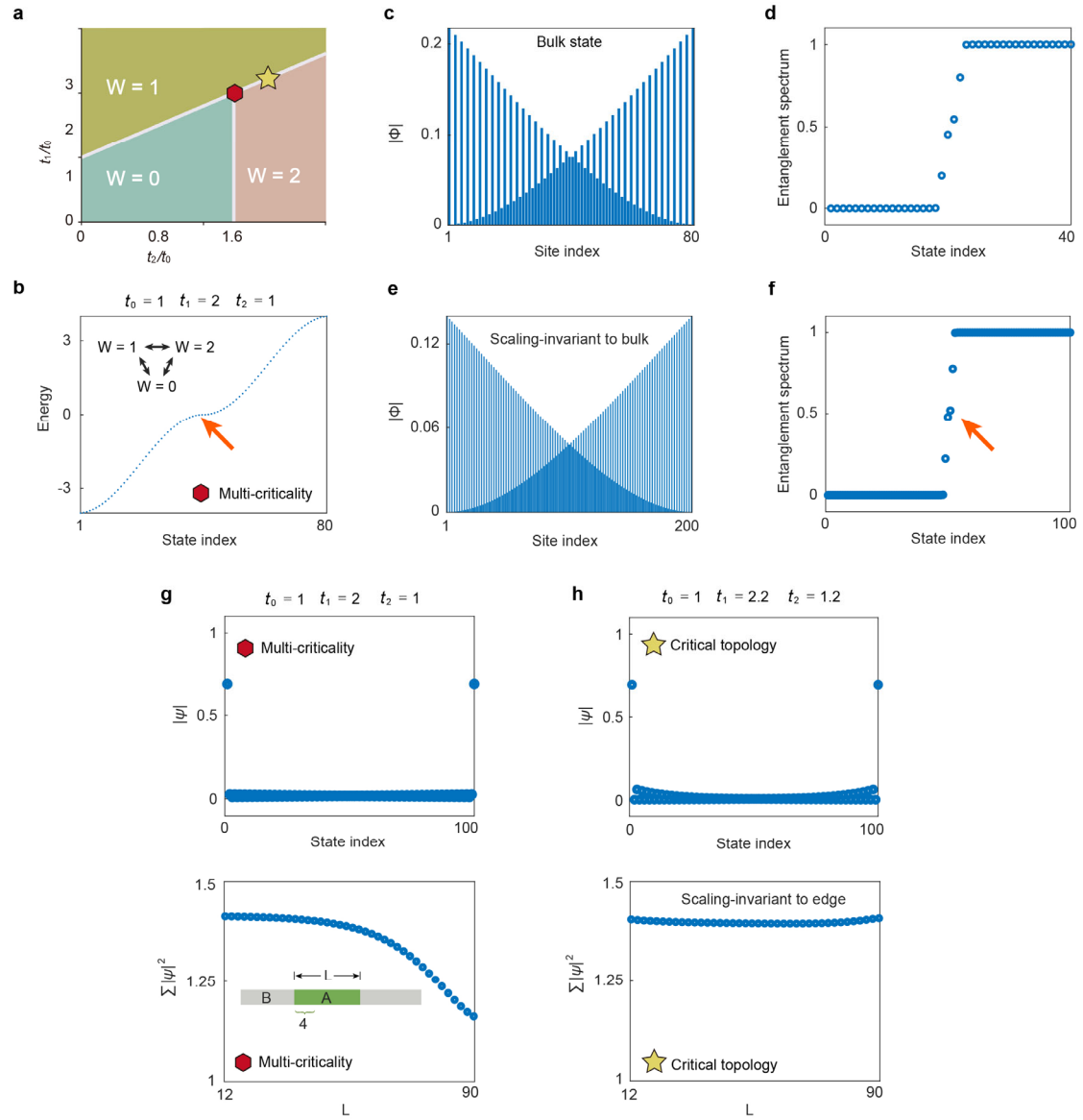


Figure S11 | Entanglement spectra and entanglement wavefunctions at multi-criticality.

a, Phase diagram of the 1D SSH model. **b**, Gapless band structure at multi-criticality. **c** and **d**, wavefunction of one mid-gap state, and the entanglement spectrum calculated in a lattice with 80 sites. **e** and **f**, Same as **c** and **d**, but in a larger lattice with 200 sites. **g** and **h**, Entanglement wavefunctions and their LDOS scaling versus L for multi-critical and topological critical states, respectively.

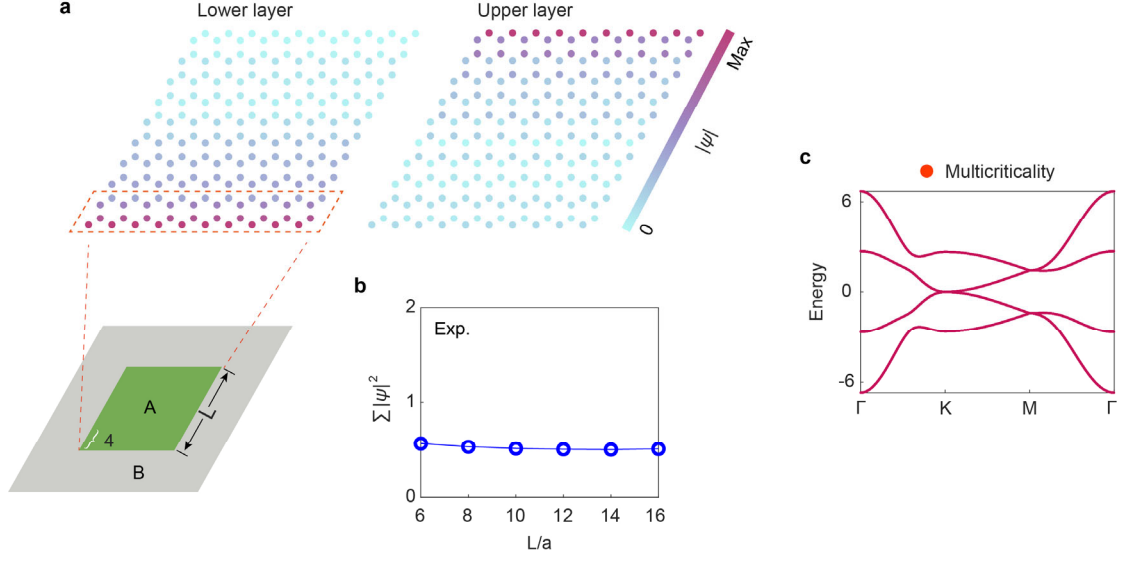


Figure S12 | Entanglement wavefunctions for 2D topological critical states. **a**, Entanglement wavefunctions of one mid-gap state at $k = 0$, showing edge-state-like localization from experimental data. **b**, Invariant scaling of local density of states of two mid-gap entanglement wavefunctions, versus the length of the subsystem A. The summation is performed over the first four rows of the subsystem, as outlined by the dashed box in **a**. **c**, Band structure along the high symmetry line at the multicriticality, showing a quadratic dispersion at the K point.

Supplementary Note 11. Hamiltonian, definition of pseudo-spins, calculation of spin Chern numbers, and topological corner states of the spin Chern model

The pseudo-spin Chern model illustrated in Fig. 5a in the main text or in Fig. S13a is derived from a bilayer honeycomb lattice with additional intra- and inter-layer hoppings, where A and B (C and D) denote the two non-equivalent sites in the unit cell for the lower (higher) monolayer. The Bloch Hamiltonian in the bases of A , B , C , and D is written as

$$H(\mathbf{k}) = \begin{pmatrix} h_{intra} & h_{inter} \\ h_{inter}^\dagger & h_{intra} \end{pmatrix}, \quad (\text{S24})$$

where, $h_{intra} = \begin{pmatrix} \Delta & h_{AB} \\ h_{AB}^* & -\Delta \end{pmatrix}$ and $h_{inter} = \begin{pmatrix} h_{AC} & 0 \\ 0 & h_{BD} \end{pmatrix}$ characterize the intra- and inter-layer parts, respectively, and

$$\begin{aligned} h_{AB} &= t_0 + 2t_0 \cos\left(\frac{k_x}{2}\right) e^{-i\frac{\sqrt{3}}{2}k_y} + 2t_3 \cos(k_x) + t_3 e^{-i\sqrt{3}k_y}, \\ h_{AC} &= h_{BD}^* = t_1 + 2t_2 \cos\left(\frac{\sqrt{3}k_y}{2}\right) e^{-i\frac{1}{2}k_x} + t_2 e^{ik_x}. \end{aligned} \quad (\text{S25})$$

Although it is a spinless model, we can take the linear combination of sublattices within different layers as the spin-up and spin-down space, i.e.,

$$AC_{\pm} = \frac{1}{\sqrt{2}}(A \pm iC), \quad BD_{\pm} = \frac{1}{\sqrt{2}}(B \pm iD). \quad (\text{S26})$$

Consequently, we can perform a basis transformation from (A, B, C, D) to (AC_-, BD_-, AC_+, BD_+) via the transformation matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \otimes \sigma_0, \quad (\text{S27})$$

with σ_0 being the identity matrix. Then the layer-mixed spin operator is

$$S_z = \sigma_z \otimes \sigma_0 \quad (\text{S28})$$

in the basis of (AC_-, BD_-, AC_+, BD_+) with $\sigma_{x,y,z}$ denoting Pauli matrices. Then, the pseudo-spin operator in the original basis of (A, B, C, D) is

$$S_y = U^\dagger S_z U = \sigma_y \otimes \sigma_0. \quad (\text{S29})$$

Based on this, the original occupied two bands can be projected to pseudo-spin space, as described by

$$H_{spin}(\mathbf{k}) = \begin{pmatrix} \langle \phi_{1,\mathbf{k}} | S_y | \phi_{1,\mathbf{k}} \rangle & \langle \phi_{1,\mathbf{k}} | S_y | \phi_{2,\mathbf{k}} \rangle \\ \langle \phi_{2,\mathbf{k}} | S_y | \phi_{1,\mathbf{k}} \rangle & \langle \phi_{2,\mathbf{k}} | S_y | \phi_{2,\mathbf{k}} \rangle \end{pmatrix}. \quad (\text{S30})$$

By diagonalization, we obtain

$$H_{spin}(\mathbf{k}) \begin{bmatrix} \alpha_{\uparrow\downarrow}(\mathbf{k}) \\ \beta_{\uparrow\downarrow}(\mathbf{k}) \end{bmatrix} = \epsilon_{\uparrow\downarrow} \begin{bmatrix} \alpha_{\uparrow\downarrow}(\mathbf{k}) \\ \beta_{\uparrow\downarrow}(\mathbf{k}) \end{bmatrix}, \quad (\text{S31})$$

which can divide two occupied bands into two groups with positive or negative expectation values of the pseudo-spin, i.e., $\epsilon_\uparrow$ and $\epsilon_\downarrow$. In our case, with always an open spin gap, namely $\text{Min}[\epsilon_\uparrow(\mathbf{k}) - \epsilon_\downarrow(\mathbf{k})] > 0$ in the whole Brillouin zone, the two occupied bands can then be projected into two spin sectors

$$\phi_{\uparrow\downarrow} = \alpha_{\uparrow\downarrow}\phi_1 + \beta_{\uparrow\downarrow}\phi_2. \quad (\text{S32})$$

Based on the spin-resolved wavefunctions, one can calculate the spin-resolved Chern number

$$C_{\uparrow\downarrow} = \frac{1}{2\pi} \int dk_x dk_y \text{Tr}[\Omega_{\uparrow\downarrow}(\mathbf{k})], \quad (\text{S33})$$

where $\Omega_{\uparrow\downarrow}(\mathbf{k})$ are the Berry curvatures for the spin-resolved occupied bands. The final spin Chern number is then obtained by

$$C_{spin} = \frac{1}{2}(C_\uparrow - C_\downarrow). \quad (\text{S34})$$

By tuning the parameters, the topological phase diagram exhibiting different spin Chern numbers is obtained, as shown in Fig. 5b in the main text and Fig. S13. We further give the band structures at different gapped and critical phases in Fig. S13. As the pseudo-spins are not conserved, the gapped phases with $C_{spin} = 1$ and $C_{spin} = 2$ labeled in Fig. S13 support 2 and 4 topological corner states, respectively, instead of helical edge states, as shown in the insets. When $\Delta = 0$, the spin Chern insulator reduces to a C_2T -symmetric fragile insulator (T denotes time-reversal symmetry), which hosts Z number of corner states localized at each obtuse-angle corner, protected by Z -classified Wilson loop-observable Euler or spin Chern numbers [24]. For the labeled critical phase transitions, as expected, only the criticality between $C_{spin} = 1$ and $C_{spin} = 2$ supports the topological corner states within the bulk continuum. Notably, when the C_2T -symmetry is broken, namely $\Delta \neq 0$, one needs to further calculate the spin-resolved entanglement spectrum to verdict the results shown in the main text, especially at the phase transition lines between $C_s = 1$ and $C_s = 2$, which leaves to the further work.

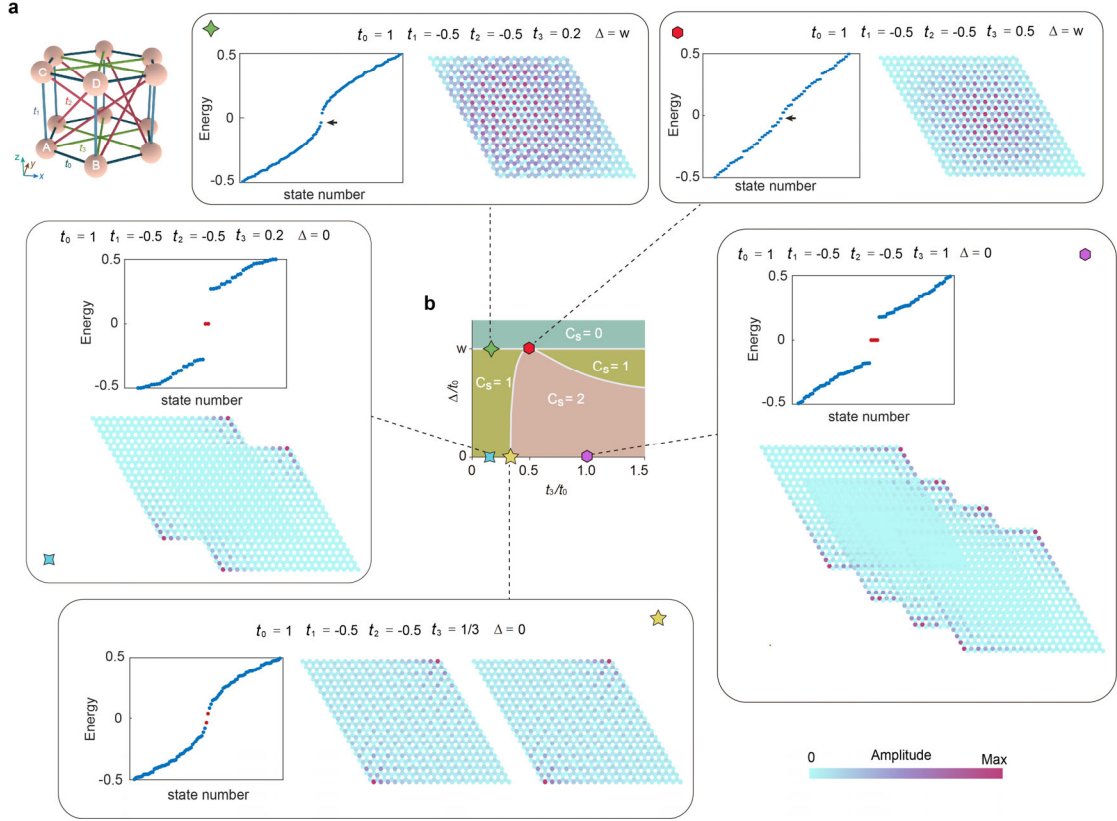


Figure S13 | Eigenvalue spectra and wavefunctions of 2D spin Chern model at different parameters. a, Illustration of the unit cell for the 2D spin Chern model. **b**, Phase transition diagram. The insets show band structures of a finite supercell and wavefunctions (only one layer) of mid-gap states at different gapped and critical phases. Here, $w \approx 0.661437$.

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