

Experimental observation of critical topology

Corresponding Author: Professor Jian-Hua Jiang

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This file contains all reviewer reports in order by version, followed by all author rebuttals in order by version.

Version 0:

Reviewer comments:

Referee #1

(Remarks to the Author)

The manuscript Experimental discovery of critical topology and its transition by Zhi-Kang Lin et al (group of Jian-Hua Jiang), provided an experimental demonstration, in phononic 1D and 2D lattices, of critical topology and gapless symmetry-protected topological (gSPT) states. They use entanglement spectrum that is defined for free fermions in condensed matter physics, and apply it to their phononic lattices. Their platform can be manipulated with greater flexibility than condensed matter materials, which enables them to visualize topological edge and corner states in the gapless bulk continuum. They also measure the mid gap states in entanglement spectrum and are able to measure edge localized entanglement wavefunctions.

The novelty of this work is mainly within the experimental observations in phononic crystals, as entanglement spectrum and its relation to topology was theoretically already studied within condensed matter physics. I find the paper interesting and potentially worthy of publication, experiments were done diligently and the paper is clearly written. However, I am missing a discussion on the following issue: The definition of the entanglement spectrum relies on the free Fermi systems where one has a well-defined Fermi level, and the properties of the system depend on where it is located (i.e., on doping in condensed matter language). Here, on the other hand we have a phononic, classical system. I understand that there is a relation between single particle states in the fermionic systems and modes in the phononic systems. However, since the results depend on the position of the Fermi level which is not present in the experimental system here, I think that this discrepancy should be elaborated in much more detail for a paper that aims publication in Nature. How do you choose the Fermi level to get the theoretical results that correspond to your phononic system? Is there some arbitrariness involved in the process of calculating the entanglement spectrum and the wavefunctions for the phononic system (it seems so), or is the Fermi level derived in a unique and consistent fashion?

A related question that needs further discussion is what is the physical interpretation of quantum entanglement in a phononic _classical_ system? Experiment was done but a theoretical interpretation of the quantum – classical connection between the theoretical concepts and the experimental system is not sufficient in my opinion.

In summary, I find the paper interesting and if the questions above are properly and thoroughly addressed I would suggest the editors to reconsider a revised version for publication.

Referee #2

(Remarks to the Author)

Dear Dr. Picardi,

The manuscript studies a phononic analogue of gapless topological states. Such states have attracted substantial attention in the theory literature on quantum phases of matter because they extend topological notions beyond the conventional setting of gapped phases, into regimes where topological properties persist at criticality despite the absence of a protecting bulk gap. The present work builds on body of work realizing classical analogues of topological band structures in photonic

and phononic platforms, where the dispersion can be engineered to emulate the single-particle physics of free-fermion topological insulators. Here, the authors focus on phase transitions between such phases and argue that distinct critical lines can themselves be topologically distinct, directly paralleling earlier theoretical work on gapless topological states. Experimentally, they use pump-probe measurements to reconstruct the analogue of the entanglement spectrum and use mid-gap states in this spectrum as a diagnostic of critical topology. They further present signatures interpreted as multicriticality separating distinct topological critical regimes.

While I find it worthwhile to have a classical engineered platform in which some of the concepts developed in the gapless-topology literature can be explored experimentally, the manuscript is at present framed in a significantly misleading way. The title speaks of "discovery," whereas the system has been deliberately engineered to realize theoretically anticipated behavior; terms such as "realization" or "observation" would be more appropriate. More importantly, the title, abstract, and introduction lean too heavily on the language of quantum criticality, gSPT states, and especially entanglement, in a way that risks overstating what has actually been achieved. This is a classical phononic simulator whose band structure mimics the single-particle physics of free-fermion models; it does not realize a genuine quantum many-body system, nor does it measure physical many-body entanglement. Although the abstract notes that the platform "emulates" free-fermion physics, this caveat is too weak relative to the strong quantum language used elsewhere. In my view, publication would require substantial rewriting of the title, abstract, and framing so that the paper is presented straightforwardly as a classical analogue realization of previously proposed gapless topological phenomena.

A related issue is the way the manuscript presents its multicriticality claim. In the Abstract, the authors write that they "uncover" that topological transitions of gSPT states lead to unavoidable multicritical points and thereby "reveal a new organizing principle" for quantum phase transitions. This wording strongly suggests a new conceptual contribution. However, my understanding is that the relation between distinct topological critical lines and multicriticality was already discussed in prior theoretical work, including papers cited by the authors themselves. If so, then the present manuscript does not reveal a new organizing principle, but rather provides an engineered experimental analogue realization or illustration of one that already existed in the theory literature. This part of the Abstract, and the corresponding discussion in the main text, should therefore be rewritten in substantially more cautious and accurate terms.

As concerns novelty, I do not think the work reaches the level expected for Nature. The relevant free-fermion models have been discussed previously in theory, including the use of entanglement-spectrum signatures as a probe of their topology. More generally, phononic analogues of free-fermion topological Hamiltonians are well established, and even the extraction of entanglement-spectrum-like observables in such platforms has prior precedent. Thus, the main advance here is the combination of these ingredients in a single engineered experiment aimed at gapless critical topology. Even the physical setting is not entirely new in spirit, since the authors are effectively exploring phase boundaries between previously studied topological phases in an analogue simulator. I therefore do not see this as a suitable paper for Nature, and likely not for Nature Physics either. That said, once the narrative is corrected and the claims are brought into line with what is actually demonstrated, I do think it could become an interesting and publishable paper in a venue such as Nature Communications or another strong specialist journal.

Some typos:

- corner stone --> cornerstone
- Fig. 1 caption: "zoom in a part" should be corrected
- "fractional winding numbers were ever proposed"
- Fig. 5 caption: "while lattice" --> whole lattice
- Page 7: "subsytem"
- Page 8: "gSPST" --> gSPT

Referee #3

(Remarks to the Author)

Experimental discovery of critical topology and its transition

This paper builds on the generalized SSH model with long-range interaction and the attainable large winding numbers. The generalized SSH model has been studied in experiments previously (see "Acoustic topological metamaterials of large winding number") and are known to support large winding number $W = 2$. The authors should clarify the difference between their paper and previous study.

My first main concern is the necessity of the entanglement spectrum to explain the critical topological phase transition. Since the number of topological edge states for the generalize SSH is determined by the winding number (see Fig. 2d), there is not doubt that the phase transition from $W = 1$ to $W = 2$ is different from the phase transition from $W = 1$ to $W = 0$. This difference lies in the existence of additional edge states in the bulk continuum during the transition from $W = 1$ to $W = 2$. The authors have used the entanglement spectrum to indicate the topological edge states. However, one can also simply measure the real space response to infer the topological edge states. This has been demonstrated previously ("Acoustic topological metamaterials of large winding number"). Therefore, the advantage of the current approach is not clear. Additionally, the current approach is very technique and can not be easily adopted by other researchers.

My second concern is the suitability of entanglement spectrum for classical waves. The entanglement spectrum originates from quantum systems. This concept is adopted for the current paper based on classical waves. However, there are some fundamental differences between classical waves and quantum waves regarding occupied wave functions. For classical

waves, whether a wave function is occupied or not is determined by the excitation conditions and there is no exclusion principle. The authors have assumed half-filling for the paper. Is this reasonable? How the results would change if filling numbers are chosen?

In Fig. 1d, the authors have illustrated three represented phases at the phase boundaries. What about the critical point phase. How does the h_x and h_y trajectory plot look like?

The dispersion relations shown for the case at the trivial phase boundary (Fig. 3d) and the case at the nontrivial phase boundary (Fig. 3f) exhibit some subtle difference. For the trivial case, two bands show nearly constant slope at and nearby the intersection point $ka = \pi$. While for the nontrivial case, the slope of the two bands shows weak variation. Such unusual slopes sometimes lead to unusual complex bands, i.e., bands with real-valued frequencies but wavenumbers can be complex-valued (see "Anomalous frozen evanescent phonons"). Can the authors provide the complex dispersion bands? These complex bands, which captures evanescent modes, might provide insight into the additional edge states.

The authors have claimed topological edge states. The robustness of the edge state is not discussed or experimentally demonstrated.

Is the matrix f, k used in Fig. 3a a two-by-two matrix? It is not clear how this matrix is obtained. In the main text, the authors have mentioned two sublattice indices α and β . Does the first index α refer to the sublattice for the pumping unit cell and the second index refer to the sublattice of the measurement unit cell?

Version 1:

Reviewer comments:

Referee #1

(Remarks to the Author)

The authors have carefully addressed all my questions regarding their manuscript, both in the revised text and in the response to the referees. I suggest publication of the manuscript.

Referee #2

(Remarks to the Author)

I thank the authors for their response and the revised manuscript. Some of my previous concerns have been addressed: in particular, the title has been changed from "discovery" to "observation," some of the gSPT language has been removed, and the authors have added a more explicit discussion of the mapping between the classical phononic experiment and the corresponding free-fermion problem. However, other important points raised in my previous report were not substantively addressed. In particular, the authors did not explicitly respond to my concern that several claimed theoretical advances were already present in prior work cited in the manuscript (see below).

I remain concerned that the manuscript still overclaims the theoretical novelty of the work. The strongest and most defensible contribution is the experimental phononic realization and ES-based diagnostic of critical topology. The paper may merit publication on this basis, provided this is made the central focus. However, the broader framing undercuts rather than strengthens the manuscript, because several statements in the abstract, introduction, and discussion present theoretical points as new discoveries when they are, in my view, already known or at least misleadingly described.

First, as I mentioned in my first report, the multicriticality claim remains overstated. In my previous report I noted that the relation between topologically distinct critical lines and multicriticality had already been discussed in prior theoretical work, including papers cited by the authors. The authors' response does not address this point; instead it reiterates the overall novelty of the experimental realization and ES measurement. The revised manuscript still says that the work "unveils" topology-driven multicriticality and a hidden hierarchical structure of topological phase diagrams. As written, this claimed conceptual novelty is at best misleading and, in my view, not correct.

Second, the discussion of previously-introduced (e.g. 'fractional') invariants is misleading, and appears to have changed in a problematic direction relative to the first version. The manuscript states that fractional winding numbers are "controversial" and that the present results "contradict" the fractional-winding-number picture of Ref. 6. I do not see this contradiction. Earlier work does not predict edge modes at every gap closing between phases with different winding numbers. Rather, the boundary-mode content depends on the critical invariant and on the nature of the criticality. In particular, the fact that the $(W=1)-(W=2)$ critical line is topological whereas the $(W=0)-(W=1)$ critical line is not, appears consistent with Refs. 3 and 6, not contradictory to them. Ref. 3 already distinguishes critical points such as (H_0+H_1) and (H_1+H_2) , assigning them different topological invariants, and shows that "only the latter" carries a localized edge mode (by relating the value of the topological invariant to the number of gapless bulk modes). More generally, Ref. 6 contains a theorem relating edge modes to topological invariants and the nature of the criticality. The authors should therefore remove the claimed contradiction unless they can identify a precise failed prediction of Ref. 6.

Third, the manuscript should be more cautious about higher-dimensional novelty. Several cited works already discuss higher-dimensional or two-dimensional forms of gapless/critical topology; examples include Refs. 4, 6, 9, and 16 in the manuscript bibliography. The authors may have introduced a new 2D spin-Chern phononic implementation and an ES-

based diagnostic for this model, but they should not imply that 2D gapless topology itself is essentially unexplored or first developed here.

Overall, I find the experimental realization interesting, but the revised manuscript still does not adequately separate what is experimentally new from what was already theoretically understood. I encourage the authors to more accurately reflect the novelty of the work: namely, the experimental realization in a classical phononic system, which should perhaps be made explicit in the title for clarity, and the implementation of entanglement-inspired probes. I would recommend publication only after the authors substantially moderate the incorrect or misleading theoretical-novelty claims.

Separately, as stated in my original report, I remain uncertain that the manuscript reaches the novelty threshold for Nature, and it may be better suited to a more specialized or topic-specific journal. However, in its favor, I agree with the authors that this is the first experimental exploration of this type of topological criticality which is potentially significant, even in a classical analogue platform, and may warrant publication if properly framed. However, in my view, the manuscript would be much stronger, and more appropriate for consideration, if it focused on this genuine experimental advance rather than relying on broader theoretical claims that are inaccurate or misleading relative to prior work.

Referee #3

(Remarks to the Author)
can be still accepted

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Response Letter for

“Experimental observation of critical topology” (Nature 2026-02-05225)

Dear Referees:

We are deeply grateful to the referees for their dedicated efforts, insightful comments, and constructive suggestions, which have been invaluable in helping us improve our manuscript. We are pleased that Referee #1 has provided a very positive assessment of our work’s innovations. While the other referees have raised important issues and concerns, we are confident that all can be thoroughly addressed, as our approach is built on a firm physical foundation and the underlying physics we uncover is both nontrivial and significant. Nevertheless, we recognize that our original presentation required significant refinement to make these key points clear and transparent. Accordingly, we have carefully and comprehensively revised the manuscript—with substantial portions rewritten—to directly address all the referees’ comments and suggestions.

We would like to first briefly address two overarching issues, with full details provided in our point-by-point responses below: (1) This work presents key innovations spanning both experimental and theoretical frontiers. Experimentally, we report the first successful observation and quantitative characterization of critical topology using the entanglement spectrum and entanglement wavefunctions—a milestone that had not been previously achieved in any system. Especially, the utility of entanglement wavefunctions as a defining signature of critical topology remains entirely unexplored in the literature prior to this work—neither theoretically nor experimentally. Furthermore, we generalize the concept of critical topology from the 1D chiral class to the 2D spin-Chern class, creating both the theoretical model and its acoustic realization in this work. Importantly, whether the entanglement spectrum can serve as a valid diagnostic for 2D critical topology was a nontrivial open question that had remained unresolved. While this remains a challenging outstanding problem for interacting systems, we provide rigorous theoretical analysis and, more importantly, convincing numerical calculations and consistent experimental results that demonstrate its validity for non-interacting 2D systems—using our spin-Chern class critical topology as an example (see the revised Supplementary Materials for full details). These advances collectively represent a significant leap beyond the existing research—bridging theory and experiments, extending critical topology from 1D to higher dimensions, introducing entanglement wavefunctions as a new diagnostic signature, and developing entirely new model systems for this emerging field. **Most importantly, our experiments**

directly demonstrate the fundamental physics and rich phenomena associated with topology at criticality, shedding new light on the deep connection between these two foundational domains of physics.

(2) Our approach is established on a firm physics ground and can be mapped exactly to quantum many-body systems. As summarized in Fig. R1, the basic logic of our approach is as follows: First, we map a non-interacting many-electron system (a free fermion system) to a non-interacting boson system when the single-particle properties are concerned (such as the topological physics studied in this work). Accompanying this mapping is the use of a virtual filling of all states below a “Fermi frequency” which mimics the filling of electrons in all states below the Fermi level. Second, a non-interacting boson system becomes a classical wave system when the boson numbers are significantly large, according to **the quantum-classical correspondence principle**. This principle leads to **exact** one-to-one correspondence between the quantum mechanical theory of phonons and the classical acoustic wave measurements, because in our experiments the number of excited acoustic phonons can easily reach to 10^{24} . By reversing these one-to-one mappings, our experiments can directly attack the cutting-edge problems in the study of critical topology in free electron quantum many-body systems, such as testing the Li-Haldane conjecture for critical topology in higher dimensions. We remark that our experimental approach can be readily generalized to interacting quantum systems where quasi-particles are still well-defined. For instance, for cold atoms in engineered optical lattices, the quasi-particle band structure can be tuned to realize various topological phases and phase transitions where the underlying physics is of similar type as what have been revealed in this work.

Finally, our detailed point-by-point responses to the comments from all three referees are presented below (referees’ comments are in *italics* and dark gray). The revised parts in both the main text and Supplementary materials are marked in [blue](#).

Sincerely,

All authors of this work

Reply to Referee #1

Comment: *The manuscript Experimental discovery of critical topology and its transition by Zhi-Kang Lin et al (group of Jian-Hua Jiang), provided an experimental demonstration, in phononic 1D and 2D lattices, of critical topology and gapless symmetry-protected topological (gSPT) states. They use entanglement spectrum that is defined for free fermions in condensed matter physics, and apply it to their phononic lattices. Their platform can be manipulated with greater flexibility than condensed matter materials, which enables them to visualize topological edge and corner states in the gapless bulk continuum. They also measure the mid gap states in entanglement spectrum and are able to measure edge-localized entanglement wavefunctions.*

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A related question that needs further discussion is what is the physical interpretation of quantum entanglement in a phononic (classical) system? Experiment was done but a theoretical interpretation of the quantum-classical connection between the theoretical concepts and the experimental system is not sufficient in my opinion.

In summary, I find the paper interesting and if the questions above are properly and thoroughly addressed I would suggest the editors to reconsider a revised version for publication.

Our reply: Thank you for in-depth review, insightful comments, and positive assessment of our innovation. Indeed, the Fermi level is a key quantity in studying the entanglement spectrum for both interacting and non-interacting electron systems. Here we mimic non-interacting electron systems with acoustic phonons in phononic crystals. In our approach, the filling of all states below the Fermi level is represented by summing up the contributions of all phonon states of which the eigen-frequency is below a “Fermi frequency”. This summation is performed when we calculate the correlation matrix in subsystem $\mathcal{A}$ which

directly gives the entanglement spectrum as its eigenvalues. In our approach, the Fermi frequency is set as the eigen-frequency of the gap-closing point. In this way, all bands below the gap-closing point contribute to the entanglement spectrum, which gives precisely the band filling needed for investigating critical topology in non-interacting electron systems. Such a mapping is based on the one-to-one correspondence between free fermion systems and free boson systems when only single-particle physics is concerned (see Fig. R1). In our approach, another step is used to go from free boson systems to classical wave systems (Fig. R1) which is realized by **the quantum-classical correspondence principle**. That is, when the number of acoustic phonons (quanta of acoustic waves) is significantly large, the behavior of the acoustic phonons coincides with the classical acoustic waves. In our experiments, the number of excited acoustic phonons can easily reach $\sim 10^{24}$. In this situation, the quantum-classical correspondence principle requires that the classical acoustic pump-probe responses $\chi_{\alpha\beta}(f, \mathbf{k})$ can be quantitatively understood via the quantum (retarded) Green's function of phonons $G_{\alpha\beta}(f, \mathbf{k})$. The underlying quantum-classical correspondence principle ensures that these two quantities are identical if they are properly normalized. In our measurements, we didn't ensure the normalization of the measured acoustic pump-probe responses, as it does not affect our final results. In fact, using the fact that the measured $\chi_{\alpha\beta}(f, \mathbf{k})$ is proportional to $G_{\alpha\beta}(f, \mathbf{k})$ is already enough to extract the properties of phonons' eigenstates for further investigation of critical topology.

The underlying mathematics is as follows: In the Lehmann representation, the retarded Green's function of phonons is written as

$$G_{\alpha\beta}(f, \mathbf{k}) = \sum_n \frac{\langle \alpha | u_{n,\mathbf{k}} \rangle \langle u_{n,\mathbf{k}} | \beta \rangle}{f - f_{n,\mathbf{k}} + i\gamma_{n,\mathbf{k}}}. \quad (1)$$

Here, $|u_{n,\mathbf{k}}\rangle$ is the periodic part of the Bloch wavefunction labeled by the momentum $\mathbf{k}$ and band index n . $f_{n,\mathbf{k}}$ is the phonon eigen-frequency that gives the phonon band structure. $\gamma_{n,\mathbf{k}}$ is the damping coefficient. To extract the eigenstates precisely, an SVD is performed on the response tensor to yield

$$\chi(f, \mathbf{k}) = U(f, \mathbf{k}) \Omega(f, \mathbf{k}) V^\dagger(f, \mathbf{k}), \quad (2)$$

where columns of U and V denote the left and right singular vectors, respectively. Ω is the diagonal matrix comprising singular values labeled by σ (real-valued, positive semidefinite) sorted from the largest to the smallest. Recall that the singular values of a matrix M equal the square root of the eigenvalues of the covariance matrix $MM^\dagger$. Moreover, the left singular vectors of M associated with those nonzero singular values coincide with the left eigenvectors of $MM^\dagger$. We find that

$$G(f, \mathbf{k}) G^\dagger(f, \mathbf{k}) = \sum_n \frac{|u_{n,\mathbf{k}}\rangle \langle u_{n,\mathbf{k}}|}{(f - f_{n,\mathbf{k}})^2 + \gamma_{n,\mathbf{k}}^2}. \quad (3)$$

Since for a given $\mathbf{k}$, the Bloch wavefunctions $|u_{n,\mathbf{k}}\rangle$ form an orthogonal and complete basis, the above equation is exactly an SVD form. Therefore, the left eigenvector corresponding to the largest eigenvalue is

precisely the Bloch wavefunction $\langle u_{n,\mathbf{k}} |$ when $f = f_{n,\mathbf{k}}$ (i.e., resonant excitation of the $|u_{n,\mathbf{k}}\rangle$ state). This largest eigenvalue is $d_s(f, \mathbf{k}) = [(f - f_{n,\mathbf{k}})^2 + \gamma_{n,\mathbf{k}}^2]^{-1}$ which is reached at $f = f_{n,\mathbf{k}}$. Therefore, the largest singular value of $\chi(f, \mathbf{k})$ is $\sigma_1 = \sqrt{d_s}$, of which the resonances encode the eigen-frequency $f_{n,\mathbf{k}}$ for all n and reproduce the phonon band dispersion by varying $\mathbf{k}$. Elegantly, this approach applies to situations with band degeneracies as well, generating the correct phonon band dispersion and degenerate Bloch eigenstates (associated with multiple singular values).

We emphasize that our approach to extracting the eigen-frequencies and Bloch wavefunctions are based on rigorous mathematical frame using the Lehmann representation of the (retarded) phonon Green's function and therefore are exact and unambiguous—The wavefunctions of course have a phase redundancy (gauge redundancy), but the entanglement spectrum is gauge invariant (as the correlation matrix is gauge invariant, see Eq. (1) in the main text.). Therefore, our approach, based on the exact mapping from free fermion systems to free boson systems and then to classical wave systems (see Fig. R1), gives precisely the same entanglement spectrum as in truly quantum non-interacting electron systems. This exactness sets a firm foundation for our experimental study.

In the revised manuscript, we largely rewrite the main text to make all the above points crystal clear.

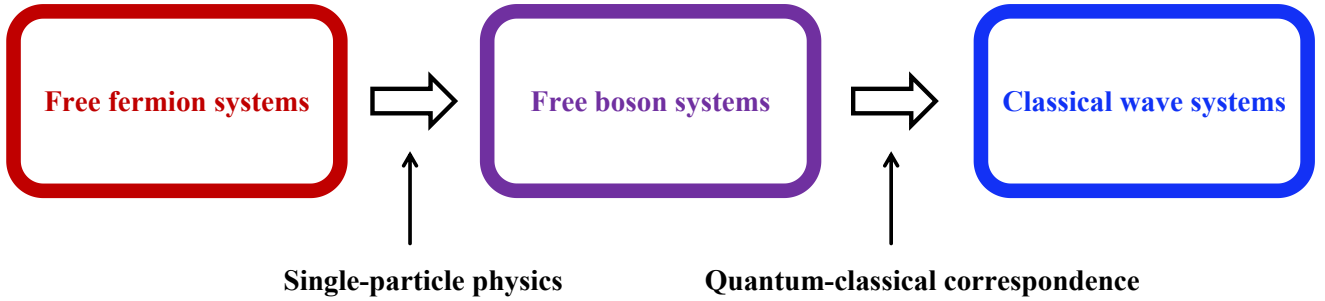


Fig. R1. Schematic of the underlying logic of our approach.

Reply to Referee #2

Comment 1: *The manuscript studies a phononic analogue of gapless topological states. Such states have attracted substantial attention in the theory literature on quantum phases of matter because they extend topological notions beyond the conventional setting of gapped phases, into regimes where topological properties persist at criticality despite the absence of a protecting bulk gap. The present work builds on body of work realizing classical analogues of topological band structures in photonic and phononic platforms, where the dispersion can be engineered to emulate the single-particle physics of free-fermion topological insulators. Here, the authors focus on phase transitions between such phases and argue that distinct critical lines can themselves be topologically distinct, directly paralleling earlier theoretical work on gapless topological states. Experimentally, they use pump-probe measurements to reconstruct the analogue of the entanglement spectrum and use mid-gap states in this spectrum as a diagnostic of critical topology. They further present signatures interpreted as multi-criticality separating distinct topological critical regimes.*

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Our reply: Thank you for your thoughtful comments and constructive suggestions, which have been invaluable in helping us strengthen our manuscript. We have carefully considered your concerns and implemented substantial revisions accordingly: we have revised the title to "Experimental observation of critical topology", completely rewritten the abstract and introduction, refined our terminology (including removing references to gSPT), and thoroughly clarified the underlying physics and quantitative logic of our approach throughout the manuscript. While we greatly appreciate your perspective, we respectfully disagree with the characterization that our work merely “puts everything known together”. As we elaborate below, our study represents a significant advance that combines novel theoretical developments with groundbreaking experimental observations in the emerging field of critical topology.

As detailed in the revised manuscript (with all changes marked in blue), our work makes key contributions spanning both experimental and theoretical frontiers. Experimentally, we report the first successful observation and quantitative characterization of critical topology using the entanglement spectrum—a milestone that had not been previously accomplished in any system. In addition, the utility of entanglement wavefunctions as a defining signature of critical topology remains entirely unexplored in the literature of critical topology prior to this work—neither theoretically nor experimentally. Furthermore, on the theoretical side, we generalize the concept of critical topology from the 1D chiral class to the 2D spin-Chern class, and we have created both the theoretical model and its corresponding acoustic realization in this work. Importantly, whether the entanglement spectrum can serve as a valid diagnostic for 2D critical topology was a nontrivial open question that had remained unresolved. While this remains a challenging outstanding problem for interacting systems, we provide rigorous theoretical analysis and, more importantly, convincing numerical calculations and consistent experimental results that demonstrate its validity for non-interacting 2D systems—using our spin-Chern class critical topology as an example (see the revised Supplementary Materials for full details). **These advances collectively represent a significant leap beyond the existing research: bridging theory and experiments, extending critical**

topology from 1D to higher dimensions, introducing entanglement wavefunctions as a new diagnostic signature, and developing entirely new model systems for this emerging field.

Beyond these specific advances, our work addresses a rapidly evolving frontier at the intersection of phase transitions and topology—two foundational paradigms that have long been viewed as incompatible: continuous critical phenomena versus discrete topological invariants, gapless critical states versus gapped topological phases, and scale-invariant critical behaviors versus localized topological boundary modes. To date, the vast majority of progress in this field has remained theoretical, with most studies limited to one-dimensional systems. Experimental demonstration of this elegant underlying physics has been highly anticipated by the community (would certainly interest a much more broad community of science as well, considering bridging the two incompatible foundational domains), yet realizing such a milestone in genuine quantum many-body systems remains exceptionally challenging. Against this backdrop, our work represents the most cutting-edge experimental advance in this field to date.

Most importantly, our experiments directly demonstrate the fundamental physics and rich phenomena associated with topology at criticality, shedding new light on the deep connection between these two foundational domains of physics: phase transitions and topology. Furthermore, the exactness of our approach—established through the rigorous one-to-one mappings illustrated in Fig. R1 and elaborated in the revised Methods section—and its ready generalizability to quantum many-body systems provide a solid foundation for future experimental investigations of critical topological phenomena. Given these significant contributions and the fundamental insights they provide into the nature of critical topology, we believe our work meets the high standards for publication in *Nature*.

We greatly appreciate your thoughtful comments and constructive suggestions, which have significantly improved the clarity and impact of our manuscript. We have carefully and comprehensively revised both the main text and Supplementary Materials to address all the issues you raised.

Comment 2: *Some typos:*

- *corner stone* → *cornerstone*
- *Fig. 1 caption: "zoom in a part" should be corrected*
- *"fractional winding numbers were ever proposed"*
- *Fig. 5 caption: "while lattice" → whole lattice*
- *Page 7: "subsytem"*
- *Page 8: "gSPST" → gSPT*

Our reply: Thank you for your meticulous review and for pointing out these typographical errors. We have carefully corrected all of them throughout the manuscript.

Reply to Referee #3

Comment 1: *This paper builds on the generalized SSH model with long-range interaction and the attainable large winding numbers. The generalized SSH model has been studied in experiments previously (see “Acoustic topological metamaterials of large winding number”) and is known to support large winding number $W = 2$. The authors should clarify the difference between their paper and the previous study.*

Our reply: Thank you for bringing this important experimental work to our attention. We have added the citation for "Acoustic topological metamaterials of large winding number" (Phys. Rev. Applied 19, 054028, 2023) in our revised manuscript. As you correctly noted, that study experimentally realized a one-dimensional (1D) generalized Su–Schrieffer–Heeger (SSH) model with a well-defined integer winding number $W = 2$ via the introduction of long-range hopping terms—a system that falls squarely within the framework of conventional band topology, characterized by fully gapped bulk bands and well-defined quantized topological invariants. In that work, the authors measured conventional topological edge states residing within the bulk band gap, whose count is directly determined by the bulk topological winding number.

In contrast, our work investigates critical topological states—a distinct and rapidly emerging class of topological matter that emerges **precisely at phase transitions** between different gapped topological phases and is **inherently gapless**. In our study, long-range hopping serves not as a central focus, but merely as a technical tool to engineer distinct gap-closing phase boundaries. Critical states at different phase boundaries can exhibit distinct critical topological properties, as illustrated in Fig. 1d of the main text. Furthermore, critical topological states transcend conventional band topology: they do not support well-defined quantized topological invariants derived from standard Bloch band analysis techniques (such as winding numbers, Wilson loops, Berry phase and curvature, symmetry indicators, and band representation analysis) due to the inherent singularities and ambiguities at gap-closing points. Nevertheless, they still possess robust topological properties as captured by the entanglement spectrum and topological edge/corner states emerging within the gapless bulk continuum, **thus challenging the conventional framework of topological band theory and the standard bulk-boundary correspondence**. Moreover, these critical topological states are not restricted to 1D systems, where conformal field theory provides a well-established description. Instead, they arise generically at transitions between gapped phases with different topological numbers in any spatial dimension.

While gapless critical topology has attracted rapidly growing attention within the condensed matter physics community in recent years, it remains relatively unfamiliar to the broader scientific audience. To introduce critical topology in an accessible manner, we naturally begin with the paradigmatic 1D SSH-like model—a simple yet archetypal topological system whose gapped phases are thoroughly established in the literature. We then focus on the gap-closing states with **nontrivial critical topology** in this model,

which have not been the focus of any prior experimental investigation. This approach provides a natural bridge between the unfamiliar concept of critical topology and the well-established SSH model framework. As a step further, to generalize critical topology from 1D to higher dimensions, we create a spin-Chern model and its acoustic realization to explore and observe critical topology in 2D systems, which extends the frontier of the emerging field of critical topology.

Comment 2: *My first main concern is the necessity of the entanglement spectrum to explain the critical topological phase transition. Since the number of topological edge states for the generalized SSH is determined by the winding number (see Fig. 2d), there is no doubt that the phase transition from $W = 1$ to $W = 2$ is different from the phase transition from $W = 1$ to $W = 0$. This difference lies in the existence of additional edge states in the bulk continuum during the transition from $W = 1$ to $W = 2$. The authors have used the entanglement spectrum to indicate the topological edge states. However, one can also simply measure the real-space response to infer the topological edge states. This has been demonstrated previously (“Acoustic topological metamaterials of large winding number”). Therefore, the advantage of the current approach is not clear. Additionally, the current approach is very technical and can not be easily adopted by other researchers.*

Our reply: Thank you for raising this important question regarding the necessity of entanglement-based measurements for probing critical topological states. We acknowledge that alternative interpretive frameworks have been proposed in the literature, most notably the use of “fractional winding numbers” to describe critical topological phenomena.

However, it is important to emphasize that critical topology is far more complex than these simplified heuristic frameworks—they fail even in the well-studied SSH-like models. For example, the “fractional winding numbers” picture cannot explain why both the transitions from $W = 0$ to $W = 1$ and from $W = 0$ to $W = 2$ are topologically trivial, exhibiting no edge states, whereas the transition from $W = 1$ to $W = 2$ is topologically nontrivial. Moreover, since these transitions are inherently reversible, one cannot use the winding number of the gapped phase on only one side of the transition to explain the topological edge states at the gap-closing critical point. There is no fundamental physical justification for such an approach. These limitations are not unique to SSH-like systems; similar failures have been reported in other models (see, e.g., Commun. Phys. 8, 214 (2025)). This fundamental breakdown occurs because the underlying physics of critical topology is qualitatively distinct from that of conventional topological theory.

To date, the entanglement spectrum remains the only unambiguous, singularity-free method for characterizing and diagnosing critical topology. This diagnostic power arises from the generalization of the Li-Haldane conjecture to gapless systems using conformal field theory. Notably, the Li-Haldane conjecture for critical topology has been rigorously proven theoretically for both interacting and non-interacting 1D systems. However, its generalization to the 2D spin-Chern class—both theoretically and

experimentally—represents an original contribution of the present work.

Finally, the observation of edge states can only serve as indirect, secondary evidence for critical topology, as these edge states are not stable in the absence of chiral symmetry—a limitation that applies to phononic crystals and many other physical systems (see Supplementary Note 9; additional examples are provided). In contrast, the mid-gap modes in the entanglement spectrum are protected by the emergent antiunitary particle–hole symmetry of the entanglement Hamiltonian, regardless of the system’s chiral symmetry¹. Thus, the entanglement spectrum provides a direct, rigorous, and robust diagnostic for critical topology.

Comment 3: *My second concern is the suitability of the entanglement spectrum for classical waves. The entanglement spectrum originates from quantum systems. This concept is adopted for the current paper based on classical waves. However, there are some fundamental differences between classical waves and quantum waves regarding occupied wave functions. For classical waves, whether a wave function is occupied or not is determined by the excitation conditions and there is no exclusion principle. The authors have assumed half-filling for the paper. Is this reasonable? How the results would change if filling numbers are chosen?*

Our reply: Thank you for raising this fundamental point. Referee #1 raised similar concerns, and we have provided a comprehensive response addressing this issue in detail, which we invite you to consult (see also the revised Methods section in the main text for additional technical elaborations).

In brief, our approach is grounded in rigorous physical principles and establishes an exact mapping from our classical wave systems to quantum many-body systems. As summarized in Fig. R1, the fundamental logic proceeds in two steps: First, we map a non-interacting many-electron (free fermion) system to a non-interacting boson system when single-particle physics is concerned—including the topological physics investigated in this work. Central to this mapping is the introduction of a virtual filling of all states below a "Fermi frequency", which precisely mimics the filling of electronic states below the Fermi level. Second, a non-interacting boson system reduces to a classical wave system in the limit of large boson numbers, according to the quantum-classical correspondence principle. This principle guarantees an exact one-to-one correspondence between the quantum mechanical theory of phonons and classical acoustic wave measurements, as the number of excited acoustic phonons in our experiments readily reach to $\sim 10^{24}$. By reversing these exact one-to-one mappings, our experiments provide a direct route to addressing cutting-edge problems in the study of critical topology in free-electron quantum many-body systems, including experimental tests of the generalized Li-Haldane conjecture for critical topology in 1D and higher dimensions. Importantly, our experimental framework is readily generalizable to genuine quantum many-body systems. For example, in cold atom systems with engineered optical lattices, the quasiparticle band structure can be tuned to realize various topological phases and phase transitions whose

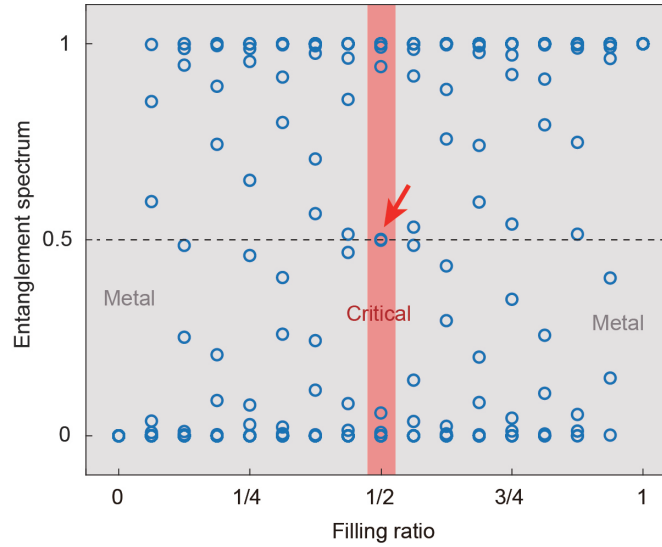


Fig. R2. Entanglement spectra at different filling ratios for the 1D tight-binding topological critical state. Only half filling with the Fermi level cutting the critical band-touching point exhibits degenerate mid-gap states (red arrow). Other fillings denoting empty states or metal states show trivial gapped spectra.

underlying physics shares fundamental similarities with that revealed in this work.

Finally, in our experiments, the filling of all states below the Fermi level is implemented by summing the contributions of all phononic states whose measured eigenfrequencies lie below the "Fermi frequency". In this work, the Fermi frequency is precisely set at the band-gap closing point. This specific choice ensures that our mapping scheme (illustrated in Fig. R1) exactly reproduces the free-fermion systems used to investigate critical topology. Crucially, this filling is purely virtual and is tailored specifically to the physical problem being studied. Shifting the filling away from the gap-closing point (i.e., away from half-filling) takes the system out of the critical topological regime. We have calculated the entanglement spectra for alternative filling factors, as presented in Fig. R2. As expected, we find that the characteristic topological signatures in the entanglement spectrum become distorted when the filling deviates from the gap-closing point.

Comment 4: *In Fig. 1d, the authors have illustrated three represented phases at the phase boundaries. What about the critical point phase. How does the h_x and h_y trajectory plot look like?*

Our reply: The h_x - h_y trajectory plot and the entanglement spectrum at the multi-critical point for the 1D SSH model are shown in **a** and **b** in Fig. R3. The trajectory plot reveals no physical difference from the phase transition between $W = 0$ and $W = 1$: the winding loop passes through the origin, indicating an ill-defined winding number. As shown in Fig. R3b, the corresponding entanglement spectrum exhibits two nearly degenerate eigenvalues pinned close to 0.5. However, we can instead turn to the scaling behavior of the local entanglement wavefunctions (for these modes near 0.5) as a function of the subsystem length L ,

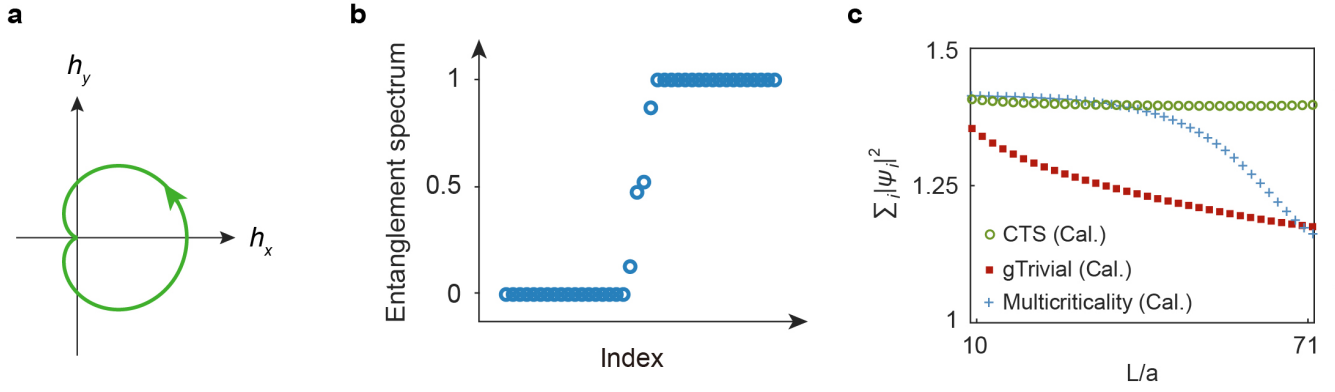


Fig. R3. h_x and h_y trajectory, entanglement spectrum, and the scaling of local entanglement wavefunctions at 1D multi-criticality.

as presented in Fig. R3c (and also in Fig. 4d of the revised main text). This scaling analysis demonstrates that the entanglement wavefunction is no longer localized at the boundary of subsystem $\mathcal{A}$, indicating the absence of well-defined critical topology at the multi-critical point. These additional results and discussions have been incorporated into the revised Supplementary Materials.

We have also emphasized this in the revised manuscript: “Moreover, we find that at multi-criticality the ES exhibits two states near 0.5, which eventually approach 0.5 in the thermodynamic limit (i.e., infinite length limit) (see tight-binding calculations in Supplementary Note 10 and Ref.²). However, there is no edge state in the bulk continuum at the multi-critical point. In this regard, the ES alone may mislead us to nontrivial critical topology at the multi-critical point, but turning to the entanglement wavefunction will resolve this issue: As shown in Fig. 4d, the multi-critical state does not support boundary localized entanglement wavefunction in the thermodynamic limit (in this limit, the entanglement wavefunction is no longer localized at the boundary of subsystem $\mathcal{A}$)—which is reasonable: as a transition state it should not have well-defined critical topology. These findings indicate that both ES and entanglement wavefunctions are indispensable characteristics of critical topology.”

Comment 5: *The dispersion relations shown for the case at the trivial phase boundary (Fig. 3d) and the case at the nontrivial phase boundary (Fig. 3f) exhibit some subtle difference. For the trivial case, two bands show nearly constant slope at and nearby the intersection point $k_a = \pi$. While for the nontrivial case, the slope of the two bands shows weak variation. Such unusual slopes sometimes lead to unusual complex bands, i.e., bands with real-valued frequencies but wavenumbers can be complex-valued (see “Anomalous frozen evanescent phonons”). Can the authors provide the complex dispersion bands? These complex bands, which captures evanescent modes, might provide insight into the additional edge states.*

Our reply: Thank you for this insightful comment from an alternative perspective. First, we would like to clarify that in fact both trivial and nontrivial phase boundaries exhibit linear dispersions around the

band-touching point. Consequently, their Dirac velocities (i.e., dispersion slopes) necessarily display comparable scaling behavior. Second, the referenced work (“Anomalous frozen evanescent phonons”) focuses exclusively on zero-frequency eigenstates, physically corresponding to static, time-frozen states, which underlie the reported anomalous evanescence. In contrast, the critical points in our system occur at finite frequencies (>4 kHz). The anomalous frozen evanescent phonon mechanism is thus irrelevant and beyond the scope of this work. On the other hand, the physical origin of the edge states in critical topology can be well understood via the Li-Haldane conjecture and the kinetic inversion mechanism³ (see details in the End Materials of this reply).

Comment 6: *The authors have claimed topological edge states. The robustness of the edge state is not discussed or experimentally demonstrated.*

Our reply: Thank you for this valuable suggestion. As we have addressed above and clarified in the revised main text, topological edge states can serve as a secondary probe of critical topology, but they lack robustness against chiral symmetry breaking. In contrast, the entanglement spectrum provides a more robust, faithful, and intrinsic diagnostic signature of critical topology. To further substantiate this point, we have added comprehensive numerical simulations in the revised Supplementary Materials (Note 9) investigating the relative stability of both edge states and entanglement spectra against random disorder, both in the presence and absence of chiral symmetry breaking.

We present the 1D simulation results in Fig. R4. Specifically, we first consider chiral-symmetric local disorders that randomly occur in the finite critical lattice (40 unit cells). To keep the chiral symmetry, the random onsite energy is forbidden. Instead, the existing hoppings can be randomly altered. Starting from the hopping parameters $t_0 = 1$, $t_1 = 2.2$, and $t_2 = 1.2$ at topological criticality, we independently set each hopping amplitude uniformly at random from the intervals $[0, 2]$, $[1, 3]$, and $[0, 2]$, respectively. The resultant eigenvalue spectrum, spatial distribution of one topological edge state, and the entanglement spectrum are displayed in Fig. R4a. As shown, the eigenvalue spectrum remains gapless and symmetric about zero energy—consistent with the preservation of chiral symmetry. Crucially, the topological features—including the localization of edge states (Fig. R4a2) and the presence of mid-gap entanglement eigenvalues at 0.5 (Fig. R4a3)—remain intact, confirming the intrinsic robustness of 1D critical topological states under chiral-symmetric disorders. For comparison, we next introduce weak, random onsite energy $E_\Delta \in [0, 0.1]$ at all lattice sites. The corresponding results are given in Fig. R4b. Without chiral symmetry, the eigenvalue spectrum loses its strict spectral symmetry about zero energy, and the edge states become delocalized. By contrast, the mid-gap entanglement eigenvalues remain tightly pinned near 0.5, demonstrating their exceptional resilience even when chiral symmetry is violated.

For 2D critical topological states, the entanglement spectrum is defined along conserved momenta k . Thus, we consider periodic perturbations to study their robustness. Specifically, the 2D spin Chern topology

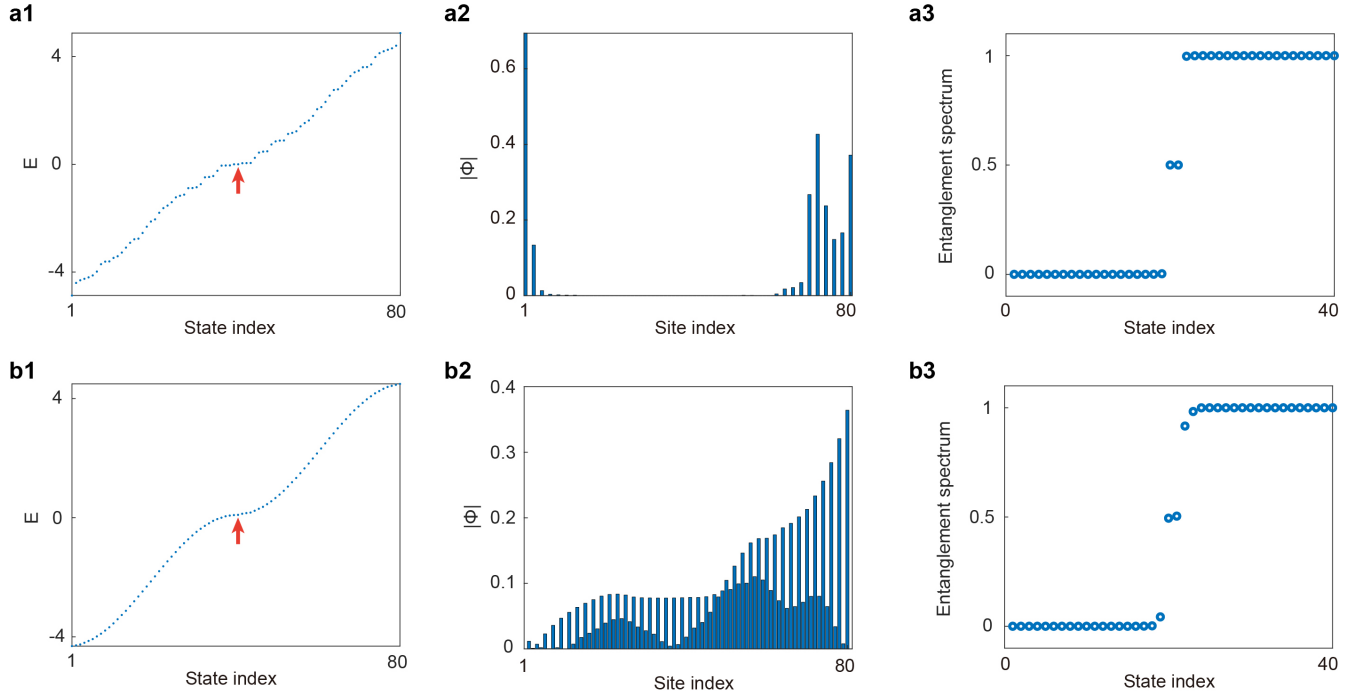


Fig. R4. Robustness simulations of 1D critical topological states. **a1-a3**, Corresponding eigenvalue spectrum, spatial distribution of one topological edge state, and entanglement spectrum, under random hoppings of $t_0 \in [0, 2]$, $t_1 \in [1, 3]$, and $t_2 \in [0, 2]$ in a finite lattice (40 unit cells), which preserve the chiral symmetry. **b1-b3**, Same as **a1-a3**, but for chiral-symmetry-breaking random onsite energy $E_\Delta \in [0, 0.1]$ at all lattice sites. Red arrows in **a1** and **b1** mark the eigenvalues of **a2** and **b2**. The hopping parameters of original critical topology are set as $t_0 = 1$, $t_1 = 2.2$, and $t_2 = 1.2$.

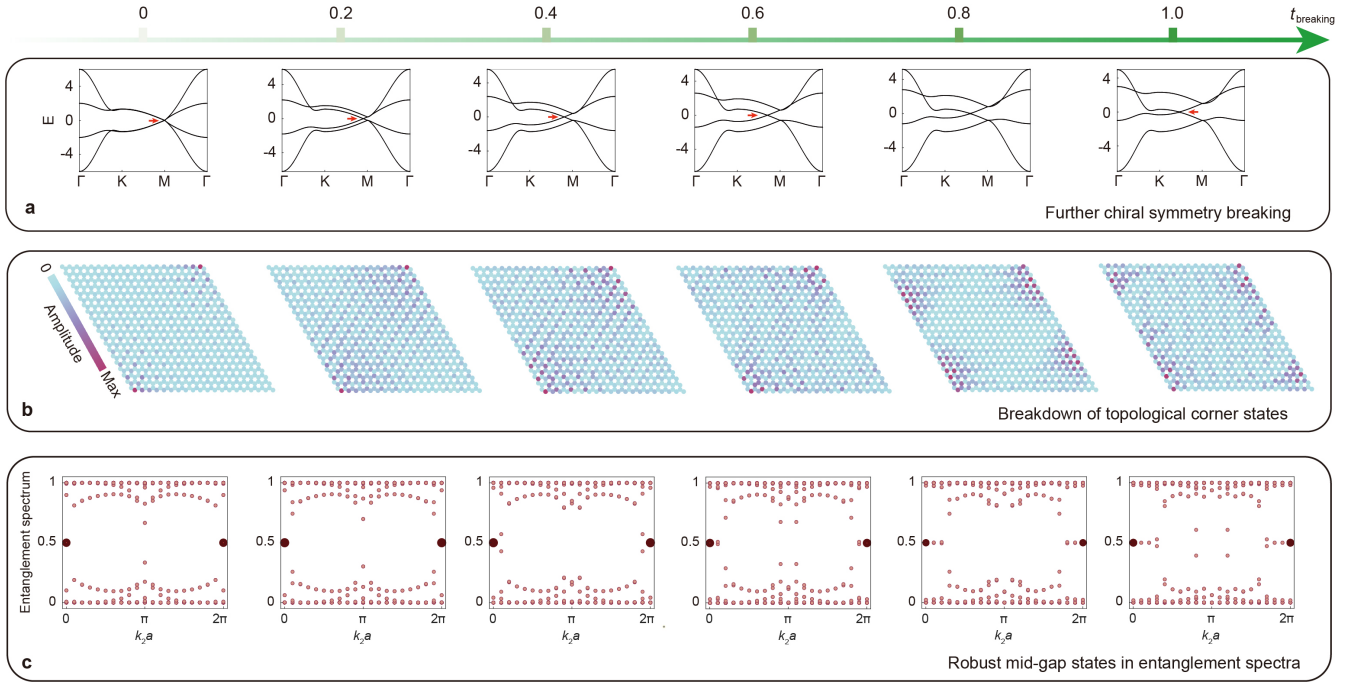


Fig. R5. Robustness simulations of 2D critical topological states. **a**, Bulk band structures along high symmetry lines under the progressively increasing chiral-symmetry-breaking perturbed hopping t_{breaking} . **b**, Corresponding spatial distributions of topological corner states with increasing delocalization. **c**, Corresponding entanglement spectra.

is irrelevant to the chiral symmetry, we introduce chiral-symmetry-breaking terms $h_{AD} = h_{BC} = t_{\text{breaking}}$ into the original Bloch Hamiltonian (see Note 11 in the revised Supplementary Materials). As illustrated in Fig. R5a, as the strength of t_{breaking} increases from 0 to 1, the band-touching points (marked by red arrows) shift away from M points toward the KM lines. Concurrently, the gapless band structure becomes more asymmetrical about zero energy. Moreover, chiral-symmetry breaking induces hybridization between the topological corner states and bulk states, leading to their delocalization, as evidenced in Fig. R5b. However, the mid-gap states in the entanglement spectra (see Fig. R5c) remain robust against stronger perturbations, showing their stability over topological boundary states as signatures of 2D critical topology.

Comment 7: *Is the matrix $\chi(f, k)$ used in Fig. 3a a two-by-two matrix? It is not clear how this matrix is obtained. In the main text, the authors have mentioned two sublattice indices α and β . Does the first index α refer to the sublattice for the pumping unit cell and the second index refer to the sublattice of the measurement unit cell?*

Our reply: You are correct. For our 1D critical topology model, the momentum-resolved response function is represented by a 2×2 matrix $\chi_{\alpha\beta}(f, \mathbf{k})$, where α and β correspond to the sublattices used for acoustic pumping and measurement, respectively. This response tensor is computed by Fourier transforming the

directly measured real-space acoustic response data, as explicitly detailed in the revised Methods section and Supplementary Note 5.

References

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2. Zhong, W.-H., Lin, H.-Q. & Yu, X.-J. Quantum entanglement of fermionic symmetry-enriched quantum critical points in one dimension. *Phys. Rev. B* **112**, 075129 (2025).
3. Verresen, R. Topology and edge states survive quantum criticality between topological insulators. *arXiv preprint arXiv:2003.05453* (2020).
4. Casini, H., Huerta, M. & Myers, R. C. Towards a derivation of holographic entanglement entropy. *J. High Energy Phys.* **2011**, 36 (2011).
5. Yu, X.-J., Yang, S., Lin, H.-Q. & Jian, S.-K. Universal entanglement spectrum in one-dimensional gapless symmetry protected topological states. *Phys. Rev. Lett.* **133**, 026601 (2024).
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End Materials

1. Generalized Li-Haldane correspondence. Here we provide an outline of the proof of the Li-Haldane correspondence for critical topological states in higher dimensions. More systematic analysis can be found in our Supplementary Materials.

A critical topological state can be understood as a conformal field theory (CFT) wherein symmetries and anomalies preclude the existence of trivial boundary conditions at any physical boundary or interface. Similarly, evaluating the reduced density matrix requires introducing an ultraviolet (UV) cutoff, which is inevitably subjected to the exact same symmetry and anomaly constraints. This profound parallel strongly suggests that the Li-Haldane correspondence may remain valid even for critical systems governed by CFTs.

To establish a concrete connection between the entanglement problem and the boundary CFT problem, we consider a spherical subregion $A = \{\mathbf{x} \in \mathbb{R}^d \mid |\mathbf{x}| < R, t = 0\}$, on the $t = 0$ time slice in Euclidean spacetime with metric $ds_{\mathbb{R}^{d+1}}^2 = d\tau^2 + dr^2 + r^2 d\Omega_{d-1}^2$, where $d\Omega_{d-1}^2$ is the metric of unit sphere, whose boundary ∂A defines the entangling surface, regulated by a UV cutoff ϵ . Within the replica formalism, $\text{Tr} \hat{\rho}_A^n$ is represented by a path integral on an n -fold branched cover, which is conformally equivalent to $S_{2\pi n R}^1 \times \mathbb{H}^{d+1}$, where $\mathbb{H}^d$ is hyperbolic space in d dimension. Introducing coordinates (τ, u, Ω_{d-1}) , the conformal mapping

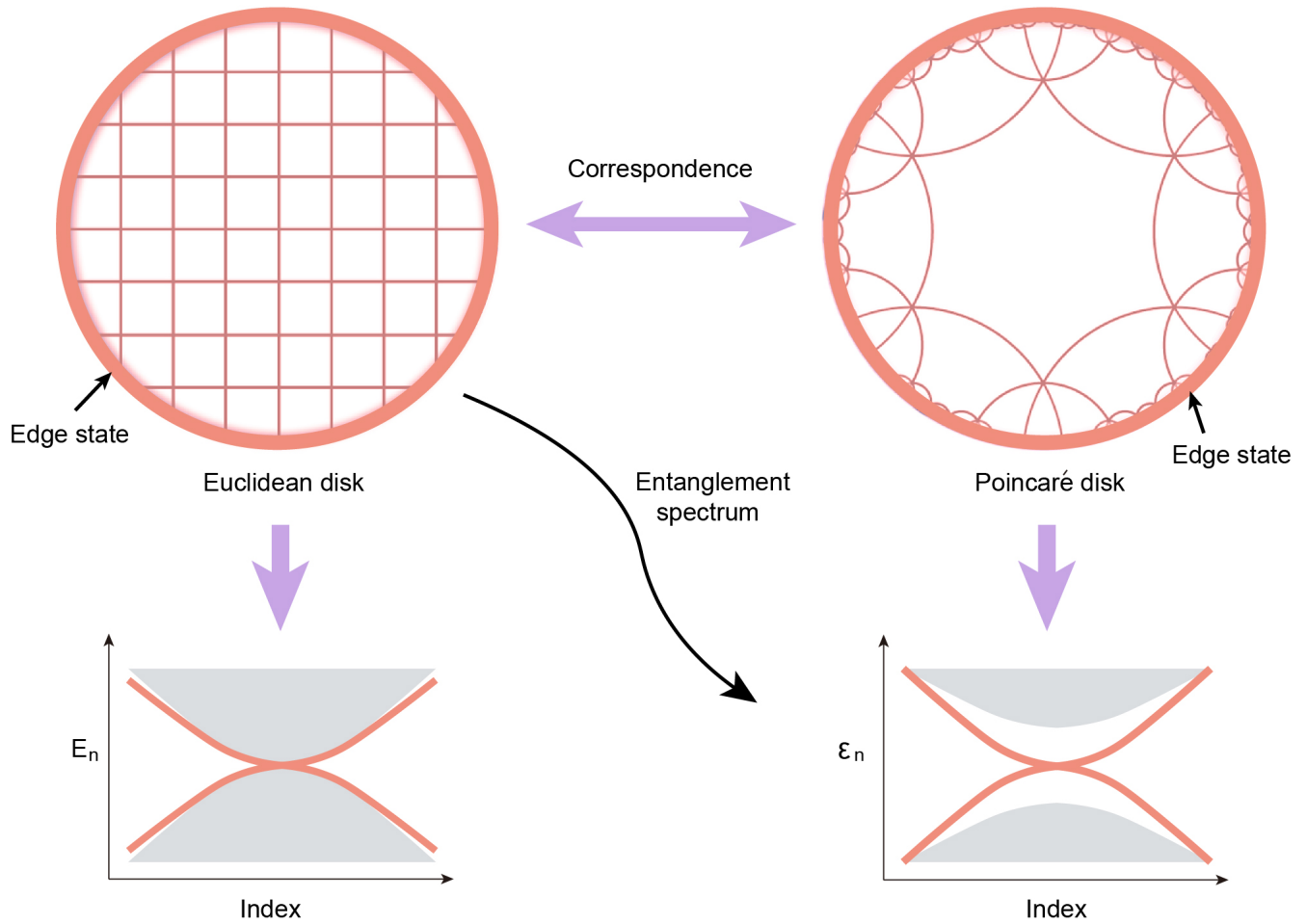


Fig. R6. This figure illustrates the correspondence between the energy spectrum of a hyperbolic disk and the entanglement spectrum of a flat disk. Due to the suppression of bulk contributions in the hyperbolic geometry, the low-energy modes of the continuous bulk spectrum are no longer preserved. Since the underlying spatial metrics differ only by a Weyl transformation, the exact zero modes originating from the boundary spectrum remain robustly preserved.

can be written explicitly as

$$t = R \frac{\sinh(\tau/R)}{\cosh u + \cosh(\tau/R)}, \quad r = R \frac{\sinh u}{\cosh u + \cosh(\tau/R)}, \quad (4)$$

while leaving the angular coordinates on S^{d-1} unchanged. Under this transformation, the flat metric becomes $ds^2 = \Omega^{-2} (d\tau^2 + R^2 (du^2 + \sinh^2 u d\Omega_{d-1}^2))$, where $\Omega = \cosh u + \cosh(\tau/R)$ is a Weyl factor. Removing this factor yields the metric on $S^1 \times \mathbb{H}^d$, we get $ds_{S^1 \times \mathbb{H}^d}^2 = d\tau^2 + R^2 (du^2 + \sinh^2 u d\Omega_{d-1}^2)$, with a cutoff $u_{\max} \sim \log \frac{2R}{\epsilon}$ imposed by the entanglement surface. Consequently, the entanglement Hamiltonian for region A is unitarily equivalent to a Hamiltonian defined on $\mathbb{H}^d$ with an asymptotic boundary at $u = u_{\max}$ up to an overall Casimir constant (see Fig. R6 for an illustration)

$$\hat{K}_A = 2\pi R U^\dagger \hat{H}_{\mathbb{D}} U + \Lambda. \quad (5)$$

Furthermore, the spatial metric $ds_{\mathbb{H}^d}^2 = R^2 (du^2 + \sinh^2 u d\Omega_{d-1}^2)$ can be mapped to the standard Poincaré disk via $\rho = \tanh \frac{u}{2}$, yielding

$$ds_{\mathbb{H}^d}^2 = \frac{4R^2}{(1-\rho^2)^2} (d\rho^2 + \rho^2 d\Omega_{d-1}^2). \quad (6)$$

In 1+1D, the hyperbolic disk degenerates into a line segment. The CFT energy spectrum on this segment can be exactly solved, which allows us to obtain an analytical expression for the entanglement spectrum in 1+1D. As a simple application, we consider the 1+1D case to see if we can recover the previous results in Ref⁵. In this case, the hyperbolic space $\mathbb{H}^1$ lacks intrinsic curvature and trivially reduces to a flat 1D line segment. The spatial metric simplifies to $ds_{\mathbb{H}^1}^2 = R^2 du^2$, representing a flat 1D segment with an effective length determined by the entanglement cut: $L_{\text{eff}} = \int_{-u_{\max}}^{u_{\max}} R du \simeq 2R \log(2R/\epsilon)$. For a general 1+1D CFT defined on such a finite strip with conformal boundary conditions, the exact energy spectrum is analytically known to be $E_n = \frac{\pi}{L_{\text{eff}}} (\Delta_n - \frac{c}{24})$, where c is the central charge and Δ_n are the scaling dimensions of the boundary primary operators. Since the entanglement Hamiltonian $\hat{K}_A$ is related to the physical Hamiltonian on $\mathbb{H}^1$ evaluated over the Euclidean thermal circle of circumference $2\pi R$, the entanglement spectrum ξ_n is simply given by $\xi_n = 2\pi R E_n$. Substituting L_{eff} , we elegantly recover the analytical result for the 1+1D entanglement spectrum :

$$\xi_n = \frac{\pi^2}{\log(2R/\epsilon)} \left(\Delta_n - \frac{c}{24} \right). \quad (7)$$

This result is consistent with previous literature^{5,6}.

For higher-dimensional cases, although finding exact solutions for general CFTs on a hyperbolic disk remains intractable, the situation significantly simplifies for Dirac free fermions when many-body CFTs admit a single particle picture. The hyperbolic metric is Weyl-equivalent to that of the flat disk $\mathbb{D}^d$. Therefore, if the Dirac Hamiltonian on the flat disk satisfies $\hat{D}_{\mathbb{D}} \psi = E \psi$ and admits a zero-energy mode, the transformation properties of the Dirac operator under Weyl rescaling guarantee that the corresponding operator $\hat{D}_{\mathbb{H}^d} \tilde{\psi} = E \tilde{\psi}$ on $\mathbb{H}^d$ also supports a zero mode (see revised Supplementary Materials for

details.) As a result, the entanglement spectrum inherits the degeneracy arising from the zero modes of the edge spectrum. This rigorously establishes a correspondence between the boundary zero modes in the entanglement spectrum and those of a $(d+1)$ -dimensional critical topological state in free fermion systems.

2. Kinetic inversion mechanism. We compare the well-established band inversion mechanism in gapped topological insulators and the kinetic inversion mechanism in critical topological states, as illustrated in Fig. R7. Consider a spatial interface between a trivial gapped phase (or vacuum) with the topological invariant $W = 0$ and a topological insulating phase with $W = 1$ (see Fig. R7a). At the interface, the low-energy physics is described by a Dirac Hamiltonian with a mass term $h_1 = -i\partial_x + m$, where $m > 0$ ($m < 0$) corresponds to the trivial (topological) phase. In this case, a topological edge mode is trapped at the interface. In contrast, we again consider an interface between the vacuum and a critical topological state (the phase boundary between $W = 1$ and $W = 2$), as illustrated in Fig. R7b. At the interface, the low-energy physics is instead³ described by $h_2 = -i\partial_x^2 - \kappa(x)\partial_x + m(x)$, where $\kappa(x)$ denotes the kinetic parameter. In this case, it is not the mass term that changes sign, but rather the kinetic term (see Fig. R7b). This is sufficient to guarantee an exponentially localized zero-energy solution of $h_2\psi(x) = 0$, provided that $\kappa(x)$ is asymptotically positive on one side and negative on the other. This mechanism, referred to as kinetic inversion, represents a new route to generating topological edge modes within the bulk continuum for critical topology. Notably, the localization length in this case is set by the kinetic parameter, with $\psi(x) \sim e^{-\kappa x}$ at the interface (for $m = 0$), and is therefore not directly tied to any mass scale.

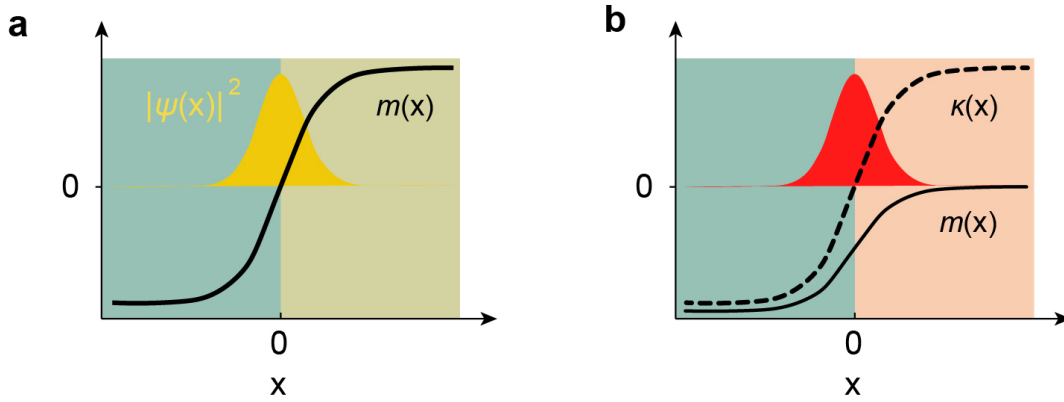


Fig. R7. Illustration of band inversion for topological gapped states (a) and kinetic inversion (b) for topological critical states. See Ref.³ for details.

Response Letter for “Experimental observation of critical topology” (Nature 2026-02-05225A)

Referee #2 (Remarks to the Author):

I thank the authors for their response and the revised manuscript. Some of my previous concerns have been addressed: in particular, the title has been changed from “discovery” to “observation,” some of the gSPT language has been removed, and the authors have added a more explicit discussion of the mapping between the classical phononic experiment and the corresponding free-fermion problem. However, other important points raised in my previous report were not substantively addressed. In particular, the authors did not explicitly respond to my concern that several claimed theoretical advances were already present in prior work cited in the manuscript (see below).

I remain concerned that the manuscript still overclaims the theoretical novelty of the work. The strongest and most defensible contribution is the experimental phononic realization and ES-based diagnostic of critical topology. The paper may merit publication on this basis, provided this is made the central focus. However, the broader framing undercuts rather than strengthens the manuscript, because several statements in the abstract, introduction, and discussion present theoretical points as new discoveries when they are, in my view, already known or at least misleadingly described.

First, as I mentioned in my first report, the multicriticality claim remains overstated. In my previous report I noted that the relation between topologically distinct critical lines and multicriticality had already been discussed in prior theoretical work, including papers cited by the authors. The authors’ response does not address this point; instead, it reiterates the overall novelty of the experimental realization and ES measurement. The revised manuscript still says that the work “unveils” topology-driven multicriticality and a hidden hierarchical structure of topological phase diagrams. As written, this claimed conceptual novelty is at best misleading and, in my view, not correct.

Second, the discussion of previously-introduced (e.g. ‘fractional’) invariants is misleading, and appears to have changed in a problematic direction relative to the first version. The manuscript states that fractional winding numbers are “controversial” and that the present results “contradict” the fractional-winding-number picture of Ref. 6. I do not see this contradiction. Earlier work does not predict edge modes at every

*gap closing between phases with different winding numbers. Rather, the boundary-mode content depends on the critical invariant and on the nature of the criticality. In particular, the fact that the $(W=1)$ - $(W=2)$ critical line is topological whereas the $(W=0)$ - $(W=1)$ critical line is not, appears consistent with Refs. 3 and 6, not contradictory to them. Ref. 3 already distinguishes critical points such as (H_0+H_1) and (H_1+H_2) , assigning them different topological invariants, and shows that *only the latter* carries a localized edge mode (by relating the value of the topological invariant to the number of gapless bulk modes). More generally, Ref. 6 contains a theorem relating edge modes to topological invariants and the nature of the criticality. The authors should therefore remove the claimed contradiction unless they can identify a precise failed prediction of Ref. 6.*

Third, the manuscript should be more cautious about higher-dimensional novelty. Several cited works already discuss higher-dimensional or two-dimensional forms of gapless/critical topology; examples include Refs. 4, 6, 9, and 16 in the manuscript bibliography. The authors may have introduced a new 2D spin-Chern phononic implementation and an ES-based diagnostic for this model, but they should not imply that 2D gapless topology itself is essentially unexplored or first developed here. Overall, I find the experimental realization interesting, but the revised manuscript still does not adequately separate what is experimentally new from what was already theoretically understood. I encourage the authors to more accurately reflect the novelty of the work: namely, the experimental realization in a classical phononic system, which should perhaps be made explicit in the title for clarity, and the implementation of entanglement-inspired probes. I would recommend publication only after the authors substantially moderate the incorrect or misleading theoretical-novelty claims.

Separately, as stated in my original report, I remain uncertain that the manuscript reaches the novelty threshold for Nature, and it may be better suited to a more specialized or topic-specific journal. However, in its favor, I agree with the authors that this is the first experimental exploration of this type of topological criticality which is potentially significant, even in a classical analogue platform, and may warrant publication if properly framed. However, in my view, the manuscript would be much stronger, and more appropriate for consideration, if it focused on this genuine experimental advance rather than relying on broader theoretical claims that are inaccurate or misleading relative to prior work.

Our response: Thank you for your carefully reviewing our revised manuscript and for recognizing the experimental novelty and significance of our work as *the first experimental realization of critical topology in 1D and 2D systems and its direct observation through entanglement-based measurements*. While the underlying theoretical framework has been developed in recent years, its experimental realization and verification in both 1D and 2D have remained absent. Bridging this gap between theory and experiment is the central motivation and the main contribution of the present

work. Meanwhile, the ES-based measurement approach developed in this work gets to results equivalent in the quantum limit and can be extensible to truly quantum systems. We also appreciate the Referee's constructive suggestions regarding the presentation of the theoretical context and the framing of the manuscript.

We agree that the theoretical framework of critical topology, together with its associated multicritical phenomena, have been established through a series of recent theoretical work. As the Referee pointed out, and as discussed in Refs. [4], [6], [9], and [16] (acknowledged in the revised manuscript), important theoretical advances have already been made in understanding critical topology in both one and higher dimensions. We therefore do not intend to claim the theoretical discovery of critical topology or its multicritical structure in the present work. Actually, our original intention of the usage of the word “unveil” is to emphasize that critical topology and its multicritical structure can be unveiled by entanglement-based analysis of using both ES and entanglement wavefunctions, which, to the best of our knowledge, has not yet been achieved before. However, our statement could be misleading. We removed these statements according to your suggestion.

In this light, we substantially revised the manuscript to distinguish more clearly between previously established theoretical developments and the new experimental advances in this work. Particularly, we have removed or rephrased statements that may be interpreted as claiming theoretical discovery of critical topology, topology-driven multicriticality, or higher-dimensional critical topology. For instance, in the revised Abstract, we have removed the word “unveil”, and the corresponding part has been changed to *“Through entanglement-based analysis, we illustrate that transitions among phase boundaries with distinct critical topology give rise to multi-critical points in the phase diagram, evidencing topology-driven multi-criticality and a hierarchical structure of topological phases. Our work demonstrates a fundamental connection between topology and criticality, paving an experimental route toward exploring topological physics at phase transitions.”* In the revised Results part, we have changed the word “uncover” to be “confirm”, the corresponding modification reads: *“The above observations also confirm a hierarchical structure in topological phase diagrams... , as predicted in previous theories [6, 4, 16, 20, 8].”* ---We added references to theoretical works to give due credits to them.

Regarding the issue of fractional invariants, the Referee is correct---we thank the Referee for this important clarification. We agree that our previous wording was a little misleading. Our intention was not to claim that the theoretical framework of Refs. [3] and [6] is incorrect, but rather to emphasize a conceptual difference between various approaches in describing critical topology. We also agree that the topological invariant defined from the zero-pole structure of the characteristic polynomial gives a rigorous definition of the one-dimensional topological critical systems, as demonstrated in Ref. [3]. By contrast, the conventional integer-valued topological invariants constructed from Berry-curvature-type quantities generally become ill-defined at bulk gap-closing points because in this framework the Bloch vector necessarily encounters a singularity. Overall, the extension of quantized topological invariants for critical topology beyond

the 1D chiral symmetry class—for example, for the two-dimensional spin-Chern model created and studied in this manuscript—remains an important theoretical question that warrants further investigation.

The point we intended to emphasize is that the entanglement-spectrum-based approach adopted in the present work provides a complementary and experimentally accessible diagnostic that is directly applicable to critical systems without requiring the construction of a critical topological invariant. Moreover, this approach has **direct** connections to edge modes, and can naturally extend to **arbitrary spatial dimensions** and **beyond the chiral symmetry class** through the generalized Li–Haldane conjecture, making it particularly suitable for our experimental realization (even for truly quantum systems). Following the Referee's suggestion, we have revised the relevant discussion and removed the statement that our results "contradict" the fractional-winding-number picture of Ref. [6].

Regarding the novelty of higher-dimensional systems, we have removed all statements that may mislead readers to feel that 2D gapless topology itself is essentially unexplored or first developed here. In fact, as we have stated in the previous version of the manuscript, *“The limited existing theories on 2D free-fermion CTS have focused on the phase transition between distinct Chern insulators or superconductors [6, 27, 42]”*, which are difficult to realize and investigate in experiments. The Referee is correct that our innovation is that we instead create a time-reversal symmetric 2D spin Chern model that can be realized and studied in our phononic systems and is investigated by our ES-based experimental approach.

Accordingly, we have revised the Abstract, Introduction, and Discussion parts to consistently reflect the experimental novelty of our work instead of any potentially misleading statements regarding theoretical progresses which is not our intension. We believe that these changes more accurately represent the scope and novelty of this work and fully address the Referee's concerns regarding the distinction between the existing theory and our experimental innovations. At last, we are deeply grateful to the Referee's remarks and suggestions which make all these clear and realizable.

PS: Changes in the manuscript are labeled in blue.