

Supplementary material.

Emergent stochastic oscillations and signal detection in regular tree networks of strongly coupled excitable elements

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Models for Nodes of Ranvier

A Hodgkin-Huxley type model (HH) for a node of Ranvier contains only sodium and leak ionic currents. Thus, in eq.(2) of the main paper, the ionic current becomes $I_{\text{ion}} = I_{\text{Na}} + I_{\text{L}}$. For the sodium current we used the Hodgkin-Huxley (HH) type kinetics^{1,2}, $I_{\text{Na}} = g_{\text{Na}}m^3h(V - V_{\text{Na}})$, where $g_{\text{Na}} = 1100 \text{ mS/cm}^2$ is the maximal value of the sodium conductance and $V_{\text{Na}} = 50 \text{ mV}$ is the Na reversal potential. The gating activation and inactivation variables obey the dynamics

$$\begin{aligned}\dot{m} &= \alpha_m(V)(1 - m) - \beta_m(V)m \\ \dot{h} &= \alpha_h(V)(1 - h) - \beta_h(V)h,\end{aligned}\tag{S1}$$

with the following rate functions:

$$\begin{aligned}\alpha_m(V) &= 1.314(V + 20.4)/[1 - \exp[-(V + 20.4)/10.3]], \\ \beta_m(V) &= -0.0608(V + 25.7)/[1 - \exp[(V + 25.7)/11]], \\ \alpha_h(V) &= -0.068(V + 114)/[1 - \exp[(V + 114)/11]], \\ \beta_h(V) &= 2.52/[1 + \exp[-(V + 31.8)/13.4]].\end{aligned}\tag{S2}$$

The leak current is $I_{\text{L}} = g_{\text{L}}(V - V_{\text{L}})$ with $g_{\text{L}} = 20 \text{ mS/cm}^2$ and $V_{\text{L}} = -80 \text{ mV}$.

The Frankenhaeuser-Huxley (FH) model³ uses four ionic currents: sodium, potassium, persistent sodium and leak: $I_{\text{ion}} = I_{\text{Na}} + I_{\text{K}} + I_{\text{p}} + I_{\text{L}}$. The currents are given by

$$\begin{aligned}I_X &= P_X \frac{EF^2}{RT} \frac{[X]_0 - [X]_i e^{\frac{EF}{RT}}}{1 - e^{\frac{EF}{RT}}}, \quad X = \text{Na, K, p}, \\ I_{\text{L}} &= g_{\text{L}}(V - V_{\text{L}}),\end{aligned}\tag{S3}$$

where $E = -70 + V$; P_X are the permeabilities of sodium (Na), potassium (K) and persistent (p) ionic currents:

$$P_{\text{Na}} = \bar{P}_{\text{Na}}hm^2, \quad P_{\text{K}} = \bar{P}_{\text{K}}n^2, \quad P_{\text{p}} = \bar{P}_{\text{p}}p^2,\tag{S4}$$

where $\bar{P}_X$ are maximal values of permeabilities of corresponding channels. In Eq. (S4) $[X]_0$ and $[X]_i$ are corresponding extracellular and intracellular ionic concentrations, of Na and K ions, respectively; for the persistent current, $X = p$, we have $[X] \equiv [\text{Na}]$. The gating variables follow,

$$\begin{aligned}\dot{m} &= [\alpha_m(V)(1 - m) - \beta_m(V)m]s, \\ \dot{h} &= [\alpha_h(V)(1 - h) - \beta_h(V)h]s, \\ \dot{n} &= [\alpha_n(V)(1 - n) - \beta_n(V)n]s, \\ \dot{p} &= [\alpha_p(V)(1 - p) - \beta_p(V)p]s,\end{aligned}\tag{S5}$$

and the rate functions are:

$$\begin{aligned}\alpha_m &= 0.36(V - 22)/(1 - \exp[(22 - V)/3]), \\ \beta_m &= 0.4(13 - V)/(1 - \exp[(V - 13)/20]), \\ \alpha_h &= 0.1(10 + V)/(\exp[(V + 10)/6] - 1), \\ \beta_h &= 4.5/(1 + \exp[(45 - V)/10]), \\ \alpha_p &= 0.006(V - 40)/(1 - \exp[(40 - V)/10]), \\ \beta_p &= 0.09(25 + V)/(\exp[(V + 25)/20] - 1), \\ \alpha_n &= 0.02(V - 35)/(1 - \exp[(35 - V)/10]), \\ \beta_n &= 0.05(10 - V)/(1 - \exp[(V - 10)/10]),\end{aligned}$$

where s is a scaling factor. The parameters for the model were taken from the original Frankenhaeuser-Huxley paper³ with three modifications to reduce the frequency of periodic spiking when a sufficiently-high constant current is injected: (i) the rate equations for the gating variables included a scale factor $s = 0.3$; (ii) the maximal permeability of potassium channels is reduced to $\bar{P}_{\text{K}} = 3.6 \times 10^{-4} \text{ cm/sec}$; and (iii) the leak conductance was reduced to $g_{\text{L}} = 6 \text{ mS/cm}^2$. Other parameters are the same as in the original FH model: $[\text{Na}]_0 = 114.5 \text{ mM}$, $[\text{Na}]_i = 13.74 \text{ mM}$, $[\text{K}]_0 = 2.5 \text{ mM}$, $[\text{K}]_i = 120 \text{ mM}$; $\bar{P}_{\text{Na}} = 8 \times 10^{-3} \text{ cm/sec}$; $\bar{P}_{\text{p}} = 54 \times 10^{-5} \text{ cm/sec}$; $V_{\text{L}} = 0.026 \text{ mV}$. Constants R and F are the universal gas constant and the Faraday constant, and $T = 293.15 \text{ K}$.

Inverse of Tridiagonal Toeplitz Matrix

In order to calculate the effective current I_{mod} eq. (20) and noise intensity D_{mod} eq. (21), we need to evaluate the inverse matrix $\mathbf{B}^{-1}$ of $\mathbf{B}$. To this end, we apply results from Ref.⁴ in which the components of the inverse of a tridiagonal Toeplitz matrix are given in terms of two sequences $\{v_i\}$ and $\{u_i\}$ with $i = 1, 2, \dots, G$. If we apply their results to the matrix $\mathbf{B}$, eq. (17), we obtain for its (i, j) th component

$$\mathbf{B}_{ij}^{-1} = u_{\text{Min}(i,j)} v_{\text{Max}(i,j)} \begin{cases} \frac{d^i}{d^j} & \text{if } i < j \\ 1 & \text{else} \end{cases}. \quad (\text{S6})$$

The sequences $\{u_i\}$ and $\{v_i\}$ are only defined up to a multiplicative constant and it is convenient to set $u_1 = 1$. Then the sequences can be calculated from

$$v_1 = \frac{1}{d_1}, \quad v_k = -\frac{d}{d_k} v_{k-1}, \quad k = 2, 3, \dots, G$$

with

$$d_G = -(d+1), \quad d_i = -(d+1) - \frac{d}{d_{i+1}}, \quad i = 1, \dots, G-2, G-1$$

and

$$u_G = \frac{1}{w_G v_G}, \quad u_k = -\frac{d}{w_k} u_{k+1}, \quad k = 1, \dots, G-2, G-1$$

with

$$w_1 = -(n_1 + 1), \quad w_i = -(d+1) - \frac{d}{w_{i-1}}, \quad i = 2, 3, \dots, G.$$

Since we are interested in the dynamics of the first order node ($g = 0$) when only peripherals are subject to noisy currents, we only need to evaluate a single component, $(1, G)$, of $\mathbf{B}^{-1}$. For this component we find from eq. (S6), $\mathbf{B}_{1,G}^{-1} = v_G$. Applying the definition of v_k , eq. (S7), multiple times, we obtain

$$v_G = (-1)^{G-1} \frac{d^{G-1}}{\prod_{k=1}^G d_k}. \quad (\text{S7})$$

Next, we evaluate the product for which one can show by using mathematical induction that

$$\prod_{k=1}^j d_k = (-1)^j \left(\sum_{k=0}^j d^k - \frac{d}{d_{j+1}} \sum_{l=0}^{j-1} d^l \right), \quad j < G. \quad (\text{S8})$$

Finally, multiplication by d_G and applying eq. (S7) yields

$$\prod_{k=1}^G d_k = (-1)^G \sum_{l=0}^G d^l = (-1)^G N. \quad (\text{S9})$$

This yields for the $(1, G)$ component of $\mathbf{B}^{-1}$

$$\mathbf{B}_{1G}^{-1} = -\frac{d^{G-1}}{N}. \quad (\text{S10})$$

Using this in eqs. (19) yields the effective current and noise intensity, respectively.

References

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