

The effects of heterogeneity on stochastic cycles in epidemics

Supplemental Material

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S1 Linear-noise approximation

Carrying out the system-size expansion for the model with heterogeneity is tedious, but straightforward and follows the lines of¹. The final outcome is the linear-noise approximation in Eqs. (11). The variables η_i and ν_a , represent Gaussian noise, with no correlation in time, but with potential correlation between the different noise variables at equal time. These noise variables can be decomposed as

$$\begin{aligned}\eta_i &= -\sum_a u_{ia} - \sum_{ab} v_{iab} - \sum_{k \neq i} x_{ik} + \sum_{k \neq i} x_{ki} + \sum_a y_{ai} + z_i, \\ \nu_a &= \sum_i u_{ia} + \sum_{ib} v_{iba} - w_a - \sum_i y_{ai},\end{aligned}\tag{S1}$$

where, broadly speaking, each term on the right-hand side represents one possible type of event in the microscopic model. For example, u_{ia} relates to spontaneous infection of a susceptible individual of type S_i , resulting in a newly infective of type I_a . Similarly, v_{iab} represents an event in which an individual of type S_i is infected by an individual of type I_a , and the newly infected is of type I_b . The variable w_a relates to a recovery event of an individual of type I_a , death of susceptible S_i and simultaneous birth of susceptible S_k is reflected by x_{ik} ; death of an individual of type I_a and simultaneous birth of susceptible S_i is described by y_{ai} , and finally death of a recovered individual and simultaneous birth of susceptible S_i , by z_i . The signs on the right-hand-side in Eqs. (S1) reflect the fact that each of these events may either increase or reduce the number of individuals of type S_i and I_a , respectively.

Each of the noise variables on the right-hand-side of Eqs. (S1) are uncorrelated in time, and they have no cross-correlations. Within the LNA their variances are set by the corresponding reaction rates at the deterministic fixed point, i.e. we have

$$\begin{aligned}\langle u_{ia}(t)u_{ia}(t') \rangle &= \xi \chi_i q_a x_i^* \delta(t-t'), \\ \langle v_{iab}(t)v_{iab}(t') \rangle &= \beta_a \chi_i q_b x_i^* I_a^* \delta(t-t'), \\ \langle w_a(t)w_a(t') \rangle &= \rho I_a^* \delta(t-t'), \\ \langle x_{ik}(t)x_{ik}(t') \rangle &= p_k \kappa x_i^* \delta(t-t'), \\ \langle y_{ai}(t)y_{ai}(t') \rangle &= p_i \kappa I_a^* \delta(t-t'), \\ \langle z_i(t)z_i(t') \rangle &= (1 - S^* - I^*) p_i \kappa \delta(t-t').\end{aligned}\tag{S2}$$

Using the shorthand introduced in Eqs. (15), we then find

$$\begin{aligned}\langle \eta_i(t)\eta_j(t') \rangle &= -\kappa^2 \left(\frac{1}{A_i} + \frac{1}{A_j} \right) p_i p_j \delta(t-t'), \text{ for } i \neq j, \\ \langle \eta_i(t)\eta_i(t') \rangle &= 2\kappa \left(1 - \frac{\kappa p_i}{A_i} \right) p_i \delta(t-t'), \\ \langle \nu_a(t)\nu_b(t') \rangle &= 0, \text{ for } a \neq b \\ \langle \nu_a(t)\nu_a(t') \rangle &= 2C \mathfrak{X}^* q_a \delta(t-t'), \\ \langle \eta_i(t)\nu_a(t') \rangle &= -\kappa C \left(\frac{\chi_i}{A_i} + \frac{\mathfrak{X}^*}{D} \right) p_i q_a \delta(t-t'),\end{aligned}\tag{S3}$$

which are needed for the computation of the PSDs.

S2 Calculation of power spectra

We start from the result in Eqs. (14) in Sect. 3.2:

$$\begin{aligned}
\hat{S}(\omega) &= \frac{1}{i\omega + \kappa} \left[-\frac{i\omega + D}{\bar{\beta}} \hat{\mathcal{B}} + \frac{1}{\bar{\beta}} \sum_a \beta_a \hat{\nu}_a + \sum_i \hat{\eta}_i \right], \\
\hat{I}(\omega) &= \frac{1}{i\omega + D} \left[\frac{i\omega + D}{\bar{\beta}} \hat{\mathcal{B}} - \frac{1}{\bar{\beta}} \sum_a \beta_a \hat{\nu}_a + \sum_a \hat{\nu}_a \right], \\
\hat{\mathcal{X}}(\omega) &= \frac{1}{\bar{\beta}C} \left[(i\omega + E) \hat{\mathcal{B}} - \sum_a \beta_a \hat{\nu}_a \right], \\
\hat{\mathcal{B}}(\omega) &= \frac{\bar{\beta}C \sum_i \frac{\chi_i \hat{\eta}_i}{i\omega + A_i} + \sum_a \beta_a \hat{\nu}_a}{i\omega + E + \bar{\beta}C\kappa \sum_i \frac{\chi_i^2 p_i}{A_i(i\omega + A_i)}}.
\end{aligned} \tag{S4}$$

As an illustration let us now compute the power spectrum of $\hat{\mathcal{B}}$. To keep equations manageable, we define

$$\begin{aligned}
f_i(\omega) &= \bar{\beta}C \frac{\chi_i}{(i\omega + A_i)}, \\
g(\omega) &= (i\omega + E) + \bar{\beta}C\kappa \sum_i \frac{\chi_i^2 p_i}{A_i(i\omega + A_i)},
\end{aligned} \tag{S5}$$

and so we write the Fourier transform of $\tilde{\mathcal{B}}$ as

$$\hat{\mathcal{B}}(\omega) = \frac{\sum_i f_i \hat{\eta}_i + \sum_a \beta_a \hat{\nu}_a}{g}, \tag{S6}$$

where f_i , β_a , $\hat{\eta}_i$ and $\hat{\nu}_a$ are all functions of ω . We then find

$$\begin{aligned}
\mathcal{P}_{\mathcal{B}}(\omega) &= \left\langle \left(\frac{\sum_i f_i \hat{\eta}_i + \sum_a \beta_a \hat{\nu}_a}{g} \right) \left(\frac{\sum_i f_i^* \hat{\eta}_i + \sum_a \beta_a \hat{\nu}_a}{g^*} \right) \right\rangle \\
&= \frac{1}{|g|^2} \left(\sum_{i,j} f_i f_j^* \langle \hat{\eta}_i \hat{\eta}_j \rangle + \sum_{i,b} f_i \beta_b \langle \hat{\eta}_i \hat{\nu}_b \rangle + \sum_{a,j} f_j^* \beta_a \langle \hat{\eta}_j \hat{\nu}_a \rangle + \sum_{a,b} \beta_a \beta_b \langle \hat{\nu}_a \hat{\nu}_b \rangle \right) \\
&= \frac{1}{|g|^2} \left(\sum_i f_i f_i^* \langle \hat{\eta}_i \hat{\eta}_i \rangle + \sum_i \sum_{j \neq i} f_i f_j^* \langle \hat{\eta}_i \hat{\eta}_j \rangle + \sum_{i,b} (f_i + f_i^*) \beta_b \langle \hat{\eta}_i \hat{\nu}_b \rangle + \sum_a \beta_a^2 \langle \hat{\nu}_a \hat{\nu}_a \rangle \right).
\end{aligned} \tag{S7}$$

The notation $*$ denotes complex conjugation. Substituting the noise correlators from Eqs. (S3),

$$\begin{aligned}
\mathcal{P}_{\mathcal{B}}(\omega) &= \frac{1}{|g|^2} \left(2\kappa \sum_i f_i f_i^* p_i - \kappa^2 \sum_{i,j} f_i f_j^* \left(\frac{1}{A_i} + \frac{1}{A_j} \right) p_i p_j \right. \\
&\quad \left. - \bar{\beta}\kappa C \sum_i (f_i + f_i^*) \left(\frac{\chi_i}{A_i} + \frac{\phi^*}{D} \right) p_i + 2\bar{\beta}^2 C \phi^* \right),
\end{aligned} \tag{S8}$$

and, using Eq. (S6), we find

$$\mathcal{P}_{\mathcal{B}}(\omega) = \frac{2\phi^* C}{|g|^2} \left(\bar{\beta}^2 - \frac{\bar{\beta}^2 C \kappa}{D} \sum_i \frac{\chi_i p_i A_i}{\omega^2 + A_i^2} \right) - \frac{(\bar{\beta}C\kappa)^2}{|g|^2} \sum_{i,j} \frac{\chi_i p_i \chi_j p_j (A_i + A_j) (\omega^2 + A_i A_j)}{A_i A_j (\omega^2 + A_i^2) (\omega^2 + A_j^2)}, \tag{S9}$$

which is the PSD of $\mathcal{B}$, as also reported in Eq. (16) in the main text.

Following the same process, we can compute the PSD for the remaining quantities, $\mathfrak{X}$, I and S . We do not report all details, but only the final results

$$\begin{aligned}
\mathcal{P}_{\mathfrak{X}}(\omega) &= \frac{1}{\bar{\beta}^2 C} \left[2\bar{\beta}^2 \mathfrak{X}^* + (\omega^2 + E^2) \left(\frac{\mathcal{P}_{\mathcal{B}}}{C} - \frac{4\bar{\beta}^2 \mathfrak{X}^*}{|g|^2} \right) \right] \\
&+ \frac{2\kappa}{\bar{\beta}|g|^2} \sum_i \frac{\chi_i^2 p_i}{A_i (\omega^2 + A_i^2)} \left[2\mathfrak{X}^* \bar{\beta}^2 (\omega^2 - EA_i) + \bar{\beta} A_i \left(\frac{A_i \mathfrak{X}^*}{D\chi_i} + 1 \right) (\omega^2 + E^2) \right] \\
&+ \frac{2\bar{\beta} C \kappa^2}{|g|^2} \sum_{i,j} \frac{\chi_i^2 p_i \chi_j^2 p_j [E(\omega^2 + A_i A_j) + \omega^2 (A_j - A_i)]}{A_i A_j (\omega^2 + A_i^2) (\omega^2 + A_j^2)} \left(\frac{A_i \mathfrak{X}^*}{D\chi_i} + 1 \right), \\
\mathcal{P}_I(\omega) &= \frac{\mathcal{P}_{\mathcal{B}}}{\bar{\beta}^2} + \frac{4C\mathfrak{X}^* (\bar{\beta}^2 - \bar{\beta}^2) (DE + \omega^2)}{|g|^2 \bar{\beta}^2 (\omega^2 + D^2)} + \frac{4\kappa C^2 \mathfrak{X}^* (\bar{\beta}^2 - \bar{\beta}^2)}{(|g|^2) (\omega^2 + D^2) (\bar{\beta})} \sum_i \frac{\chi_i^2 p_i (DA_i - \omega^2)}{A_i (\omega^2 + A_i^2)}, \\
\mathcal{P}_S(\omega) &= \frac{1}{\omega^2 + \kappa^2} \left[\left(\frac{\omega^2 + D^2}{\bar{\beta}^2} \right) \mathcal{P}_{\mathcal{B}} + 2\kappa \left(1 - \kappa \sum_i \frac{p_i}{A_i} \right) - C\kappa \left(\frac{\mathfrak{X}^*}{D} + \sum_i \frac{\chi_i p_i}{A_i} \right) \right] \\
&+ \frac{2C}{|g|^2 (\omega^2 + \kappa^2)} \left[-\frac{2\mathfrak{X}^* \bar{\beta}^2}{\bar{\beta}^2} + \kappa \left(\frac{\mathfrak{X}^*}{D} + \sum_i \frac{\chi_i p_i}{A_i} \right) \right] \left[DE + \omega^2 - \frac{|g|^2}{2} + \bar{\beta} C \kappa \sum_j \frac{\chi_j^2 p_j (DA_j - \omega^2)}{A_j (\omega^2 + A_j^2)} \right] \\
&+ \frac{2C\kappa}{|g|^2 (\omega^2 + \kappa^2)} \sum_i \frac{\chi_i p_i}{A_i (\omega^2 + A_i^2)} \left[C \left(\frac{A_i \mathfrak{X}^*}{D} + \chi_i \right) - 2A_i + \kappa \left(1 + A_i \sum_k \frac{p_k}{A_k} \right) \right] \\
&\left\{ A_i DE + \omega^2 (A_i + E - D) + \bar{\beta} C \kappa \sum_j \frac{\chi_j^2 p_j [D(A_i A_j + \omega^2) + \omega^2 (A_j - A_i)]}{A_j (\omega^2 + A_j^2)} \right\}. \quad (\text{S10})
\end{aligned}$$

The power spectra of fluctuations for the individual subgroups of infectives and susceptibles are found as

$$\begin{aligned}
\mathcal{P}_{x_i}(\omega) &= \frac{1}{\omega^2 + A_i^2} \left[\left(\frac{\kappa \chi_i p_i}{A_i} \right)^2 \mathcal{P}_{\mathcal{B}} + 2\kappa \left(1 - \frac{\kappa p_i}{A_i} \right) p_i \right] \\
&+ \frac{2\bar{\beta} C \kappa^2 \chi_i p_i^2}{|g|^2 A_i (\omega^2 + A_i^2)} \left[\frac{\mathfrak{X}^*}{D} + \frac{\chi_i (\omega^2 - A_i^2)}{A_i (\omega^2 + A_i^2)} + \frac{\kappa}{A_i} \sum_j \frac{\chi_j p_j (A_i + A_j)}{\omega^2 + A_j^2} \right] \left(E + \sum_k \frac{\bar{\beta} C \kappa \chi_k^2 p_k}{\omega^2 + A_k^2} \right) \\
&+ \frac{2\bar{\beta} C \kappa^2 \chi_i p_i^2 \omega^2}{|g|^2 A_i (\omega^2 + A_i^2)} \left[\frac{2\chi_i}{\omega^2 + A_i^2} - \frac{\kappa}{A_i} \sum_j \frac{\chi_j p_j (A_i + A_j)}{A_j (\omega^2 + A_j^2)} \right] \left(1 - \sum_k \frac{\bar{\beta} C \kappa \chi_k^2 p_k}{A_k (\omega^2 + A_k^2)} \right), \quad (\text{S11})
\end{aligned}$$

$$\mathcal{P}_{y_a}(\omega) = q_a^2 \mathcal{P}_I + \frac{2C\mathfrak{X}^* q_a (1 - q_a)}{\omega^2 + D^2}. \quad (\text{S12})$$

S3 Phase Lag

In order to explore the the phase lag we use the so-called complex coherence function, $\mathcal{CCF}_{ij}$, between subgroups i and j , defined as

$$\mathcal{CCF}_{ij}(\omega) = \frac{\langle \hat{x}_i \hat{x}_j^* \rangle}{\sqrt{\langle \hat{x}_i \hat{x}_i^* \rangle \langle \hat{x}_j \hat{x}_j^* \rangle}} = \frac{\mathcal{P}_{x_i x_j}}{\sqrt{\mathcal{P}_{x_i} \mathcal{P}_{x_j}}}, \quad (\text{S13})$$

where $\hat{x}_i$ and $\mathcal{P}$ are functions of ω .

For $i \neq j$ this is in general a complex-valued function (of ω). The argument of $\mathcal{CCF}_{ij}$, given by

$$\mathfrak{L}_{x_i x_j}(\omega) = \tan^{-1} \frac{\text{Im } \mathcal{CCF}_{ij}(\omega)}{\text{Re } \mathcal{CCF}_{ij}(\omega)} = \tan^{-1} \frac{\text{Im } \mathcal{P}_{x_i x_j}(\omega)}{\text{Re } \mathcal{P}_{x_i x_j}(\omega)}, \quad (\text{S14})$$

is known as the phase spectrum; it describes the phase-lag between the time series $x_i(t)$ and $x_j(t)$ ².

The cross spectra of the population in the susceptible classes normalized with respect to the total population ($x_i = n_i/N$) is given by

$$\mathcal{P}_{x_i x_j}(\omega) = \langle \hat{x}_i \hat{x}_j^* \rangle = \left\langle \left(\frac{-\chi_i x_i^* \hat{\mathcal{B}} + \hat{\eta}_i}{i\omega + A_i} \right) \left(\frac{-\chi_j x_j^* \hat{\mathcal{B}}^* + \hat{\eta}_j}{-i\omega + A_j} \right) \right\rangle. \quad (\text{S15})$$

This can be written as

$$\mathcal{P}_{x_i x_j}(\omega) = \frac{(\omega^2 + A_i A_j) \mathcal{W}_{ij} - \omega (A_i - A_j) \mathcal{U}_{ij}}{(\omega^2 + A_i^2) (\omega^2 + A_j^2)} + i \frac{(\omega^2 + A_i A_j) \mathcal{U}_{ij} + \omega (A_i - A_j) \mathcal{W}_{ij}}{(\omega^2 + A_i^2) (\omega^2 + A_j^2)}, \quad (\text{S16})$$

where we introduced the notation

$$\begin{aligned} \mathcal{U}_{ij}(\omega) &= \chi_j x_j^* \text{Im} \langle \hat{\eta}_i \hat{\mathcal{B}} \rangle - \chi_i x_i^* \text{Im} \langle \hat{\eta}_j \hat{\mathcal{B}} \rangle, \\ \mathcal{W}_{ij}(\omega) &= (\chi_i \chi_j x_i^* x_j^*) \mathcal{P}_{\mathcal{B}} + \langle \hat{\eta}_i \hat{\eta}_j \rangle - \chi_j x_j^* \text{Re} \langle \hat{\eta}_i \hat{\mathcal{B}} \rangle - \chi_i x_i^* \text{Re} \langle \hat{\eta}_j \hat{\mathcal{B}} \rangle. \end{aligned} \quad (\text{S17})$$

From these we obtain the phase lag as

$$\mathfrak{L}_{x_i x_j}(\omega) = \tan^{-1} \frac{\omega (A_i - A_j) \mathcal{W}_{ij} + (\omega^2 + A_i A_j) \mathcal{U}_{ij}}{(\omega^2 + A_i A_j) \mathcal{W}_{ij} - \omega (A_i - A_j) \mathcal{U}_{ij}}, \quad (\text{S18})$$

which yields the theoretical lines in Fig. 9a.

To explore the phase lag between the susceptible subgroups when normalized by the total susceptible population ($x'_i = n_i/NS$), we first need to compute the cross-spectra of the renormalized signals $\mathcal{P}_{x'_i x'_j}(\omega)$. As in Section 3.1, we start from the ansatz

$$\frac{n_i}{NS} = x'_i + \frac{1}{\sqrt{N}} \tilde{x}'_i. \quad (\text{S19})$$

We then have

$$\frac{n_i}{NS} = \frac{n_i/N}{S} = \frac{x_i + \frac{1}{\sqrt{N}} \tilde{x}_i}{S + \frac{1}{\sqrt{N}} \tilde{S}} \equiv x'_i + \frac{1}{\sqrt{N}} \tilde{x}'_i, \quad (\text{S20})$$

and so (after expanding in $1/\sqrt{N}$)

$$\tilde{x}'_i = \frac{S^* \tilde{x}_i - x_i^* \tilde{S}}{(S^*)^2}. \quad (\text{S21})$$

In Fourier space this turns into

$$\hat{x}'_i = \frac{S^* \hat{x}_i - x_i^* \hat{S}}{(S^*)^2}. \quad (\text{S22})$$

For the cross spectra we then find

$$\mathcal{P}_{x'_i x'_j}(\omega) = \langle \hat{x}'_i \hat{x}'_j{}^* \rangle = \left\langle \frac{(S^* \hat{x}_i - x_i^* \hat{S})(S^* \hat{x}_j^* - x_j^* \hat{S}^*)}{(S^*)^4} \right\rangle, \quad (\text{S23})$$

which can be rewritten as

$$\mathcal{P}_{x'_i x'_j}(\omega) = \frac{1}{(S^*)^3} \left(S^* \text{Re}[\mathcal{P}_{x_i x_j}] + \frac{\kappa^2 p_i p_j}{S^* A_i A_j} \mathcal{P}_S - \frac{Y_{ijR} + Y_{jiR}}{\bar{\beta}(\omega^2 + \kappa^2)} \right) + \frac{i}{(S^*)^3} \left(S^* \text{Im}[\mathcal{P}_{x_i x_j}] - \frac{Y_{ijI} - Y_{jiI}}{\bar{\beta}(\omega^2 + \kappa^2)} \right). \quad (\text{S24})$$

We have introduced the notation $Y_{ijR} = \text{Re}[Y_{ij}]$ and $Y_{ijI} = \text{Im}[Y_{ij}]$ with

$$\begin{aligned} Y_{ij}(\omega) = & \frac{(\omega^2 + i\omega(\kappa - A_j) + \kappa A_j) \kappa p_i}{(\omega^2 + A_j^2) A_i} \left\{ \frac{\kappa \chi_j p_j}{A_j} \left(D \mathcal{P}_B - \sum_a \beta_a \text{Re} \langle \hat{\nu}_a \hat{\mathcal{B}} \rangle - \bar{\beta} \sum_k \text{Re} \langle \hat{\eta}_k \hat{\mathcal{B}} \rangle \right) \right. \\ & \left. \omega \text{Im} \langle \hat{\eta}_j \hat{\mathcal{B}} \rangle - D \text{Re} \langle \hat{\eta}_j \hat{\mathcal{B}} \rangle - \kappa \bar{\beta} p_j \left[C \left(\frac{\chi_j}{A_j} + \frac{\mathfrak{X}^*}{D} \right) - 2 + \frac{\kappa}{A_j} + \kappa \sum_k \left(\frac{p_k}{A_k} \right) \right] \right\} \\ & + i \frac{[\omega^2 + i\omega(\kappa - A_j) + \kappa A_j] \kappa p_i}{(\omega^2 + A_j^2) A_i} \left[\frac{\kappa \chi_j p_j}{A_j} \left(\omega \mathcal{P}_B + \sum_a \beta_a \text{Im} \langle \hat{\nu}_a \hat{\mathcal{B}} \rangle + \bar{\beta} \sum_k \text{Im} \langle \hat{\eta}_k \hat{\mathcal{B}} \rangle \right) \right. \\ & \left. \left. - \omega \text{Re} \langle \hat{\eta}_j \hat{\mathcal{B}} \rangle - D \text{Im} \langle \hat{\eta}_j \hat{\mathcal{B}} \rangle \right] \right\}. \quad (\text{S25}) \end{aligned}$$

From these, we can find the phase-lag as

$$\mathfrak{L}_{x'_i x'_j}(\omega) = \tan^{-1} \frac{S^* \text{Im}[\mathcal{P}_{x_i x_j}] - \frac{Y_{ijI} - Y_{jiI}}{\bar{\beta}(\omega^2 + \kappa^2)}}{S^* \text{Re}[\mathcal{P}_{x_i x_j}] + \frac{\kappa^2 p_i p_j}{S^* A_i A_j} \mathcal{P}_S - \frac{Y_{ijR} + Y_{jiR}}{\bar{\beta}(\omega^2 + \kappa^2)}}. \quad (\text{S26})$$

This expression was used to obtain the analytical predictions shown in Fig. 9b.

S4 Table of Symbols

		Meaning	Defined in Eqs.
Symbols	N	Total population size.	
	K	Number of susceptible subgroups.	
	M	Number of infective subgroups.	
	S_i	Susceptible subgroup i .	
	I_a	Infective subgroup a .	
	n_i	Number of individuals of type S_i .	
	m_a	Number of individuals in class I_a .	
	p_i	Probability of being assigned a susceptibility χ_i at birth.	
	q_a	Probability of being assigned an infectiousness β_a upon infection.	
	χ_i	Susceptibility of subgroup i .	
	β_a	Infectiousness of subgroup a .	
	ρ	Recovery rate.	
	κ	Death/birth rate.	
	ξ	Spontaneous infection rate.	
	$\bar{\chi}$	Mean susceptibility at birth.	(2)
	$\bar{\mathfrak{X}}$	Aggregate susceptibility of the population.	(2)
	$\bar{\beta}$	Mean infectiousness upon infection.	(3)
	$\mathcal{B}$	Total ‘infective power’ in the population.	(3)
	x_i	Fraction of susceptible individuals in subgroup i in the limit of infinite system size.	
	y_a	Fraction of infected individuals in subgroup a in the limit of infinite system size.	
	S	Total density of susceptible individuals in the population.	(5)
	I	Total density of infective individuals in the population.	(5)
	$\frac{\mathfrak{X}_n}{\chi^n}$	Nth moment of the aggregate susceptibility.	
	$\frac{\chi^n}{\chi^n}$	Nth moment of susceptibilities at birth.	
	A_i, C, D, E	Notation introduced to simplify equations.	(15)
	$\mathcal{P}_z$	Power spectral density of z .	(16)
	g	Notation introduced to simplify equations.	(17)
	$\frac{\omega_d}{\beta^2}$	Dominant cycle frequency.	(18)
	$\mathbb{S}$	Sharpness of the PSD.	(19)
	$\mathcal{L}_{z_1 z_2}$	Phase lag between signals z_1 and z_2 .	(20)
			(21)
Accents	$\bar{\chi}$	Expected value at birth or upon infection.	(2), (3)
	$\dot{x}$	Deterministic evolution (time derivative).	(4), (6), (7)
	x^*	Deterministic fixed point.	(9), (10)
	$\tilde{x}$	Stochastic fluctuations about the deterministic fixed point.	(11), (12)
	$\hat{x}$	Fourier transform with respect to time.	(13), (14)
	x^*	Complex conjugate.	

References

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