

Supplementary Information: Universal quantum simulation of single-qubit nonunitary operators using duality quantum algorithm

Chao Zheng

Department of Physics, College of Science, North China University of Technology, Beijing 100144, P. R. China.

czheng@ncut.edu.cn

2.1 Three UE-terms.

In a general case, we find that $F(t)$ can be expressed by three UE-terms as

$$F(t) = e^{i\theta_0} |f_0| \sigma_0 + e^{i\theta_1} |c_1| U_1 + e^{i\theta_2} |c_2| U_2. \quad (1)$$

The three UE-parameters can be expressed as functions of f_k 's in Eq. (??), which the details are presented in the Supplementary Information.

The three UE-parameters are f_0 ,

$$c_1 = e^{i\theta_1} \sqrt{|f_3|^2 + |f_1|^2 \cos^2 \varphi_1 + |f_2|^2 \sin^2 \varphi_2} \quad (2)$$

and

$$c_2 = e^{i\theta_2} \sqrt{|f_1|^2 \sin^2 \varphi_1 + |f_2|^2 \cos^2 \varphi_2}, \quad (3)$$

where

$$\varphi_1 = \theta_1 - \theta_3 \quad \text{and} \quad \varphi_2 = \theta_2 - \theta_3. \quad (4)$$

U_1 and U_2 are two unitary operators in $SU(2)$,

$$U_1 = \begin{bmatrix} \cos \zeta_1 & e^{i\phi_1} \sin \zeta_1 \\ e^{-i\phi_1} \sin \zeta_1 & -\cos \zeta_1 \end{bmatrix} \quad \text{and} \quad U_2 = \begin{bmatrix} 0 & e^{i\phi_2} \\ -e^{-i\phi_2} & 0 \end{bmatrix}, \quad (5)$$

where ϕ_1 is decided by

$$\cos \phi_1 = \frac{|f_1| \cos \varphi_1}{\sqrt{|f_1|^2 \cos^2 \varphi_1 + |f_2|^2 \sin^2 \varphi_2}} \quad \text{and} \quad \sin \phi_1 = \frac{|f_2| \sin \varphi_2}{\sqrt{|f_1|^2 \cos^2 \varphi_1 + |f_2|^2 \sin^2 \varphi_2}}; \quad (6)$$

ϕ_2 is decided by

$$\cos \phi_2 = \frac{|f_2| \cos \varphi_2}{|c_2|} \quad \text{and} \quad \sin \phi_2 = \frac{|f_1| \sin \varphi_1}{|c_2|}; \quad (7)$$

ζ_1 is decided by

$$\cos \zeta_1 = \frac{|f_3|}{|c_1|} \quad \text{and} \quad \sin \zeta_1 = \frac{\sqrt{|f_1|^2 \cos^2 \varphi_1 + |f_2|^2 \sin^2 \varphi_2}}{|c_1|}. \quad (8)$$

Notice that f_k and θ_k ($k = 0, 1, 2, 3$) are time-dependent, the variables above are implicit functions of time t .

Four phase matching conditions for two UE-terms

As claimed in the main text, $F(t)$ can be expressed by two UE-terms as

$$F(t) = a_0 V_0 + a_1 V_1, \quad (S1)$$

as long as one of the four *phase matching* conditions are met. We now give details of the four *phase matching* conditions and the related parameters and matrices for Eq. () in the main text.

Condition I. If the two phase angles θ_0 and θ_3 in the maintext satisfy

$$(\theta_3 - \theta_0) \mod \pi = 0, \quad (S2)$$

the explicit forms of the UE-parameters are

$$a_0 = |a_0| e^{i\theta_0} = e^{i\theta_0} \sqrt{|f_0|^2 + |c_2|^2} \quad \text{and} \quad a_1 = c_1 \quad (S3)$$

referring c_1 and c_2 in the main text.

The two unitary matrices are

$$V_0 = \begin{bmatrix} \cos \zeta_0 & e^{i(\phi_2 + \theta_3 - \theta_0)} \sin \zeta_0 \\ -e^{-i(\phi_2 + \theta_3 - \theta_0)} \sin \zeta_0 & \cos \zeta_0 \end{bmatrix} \quad \text{and} \quad V_1 = U_1, \quad (S4)$$

where ζ_0 is decided by

$$\cos \zeta_0 = \frac{|f_0|}{|c_0|} \quad \text{and} \quad \sin \zeta_0 = \frac{|c_2|}{|c_0|}; \quad (S5)$$

U_1 and ϕ_2 are referred to that in the main text.

Condition II.

If the phase angles satisfy

$$|(\theta_0 - \theta_3) \mod 2\pi| = |(\theta_1 - \theta_2) \mod 2\pi| = \frac{\pi}{2} \quad (S6)$$

the UE-parameters and matrices become

$$a_0 = |a_0| e^{i\theta_0} = e^{i\theta_0} \sqrt{|f_0|^2 + |f_3|^2} \quad \text{and} \quad a_1 = |a_1| e^{i\theta_1} = e^{i\theta_1} \sqrt{|f_1|^2 + |f_2|^2}; \quad (S7)$$

$$V_0 = \begin{bmatrix} e^{i\zeta_0} & 0 \\ 0 & e^{-i\zeta_0} \end{bmatrix} \quad \text{and} \quad V_1 = \begin{bmatrix} 0 & e^{i\zeta_1} \\ e^{-i\zeta_1} & 0 \end{bmatrix}, \quad (S8)$$

where ζ_0 and ζ_1 are decided by

$$\cos \zeta_0 = \frac{|f_0|}{|a_0|} \quad \text{and} \quad \sin \zeta_0 = \frac{|f_3|}{|a_0|} e^{i(\theta_3 - \theta_0 - \frac{\pi}{2})} \quad (S9)$$

and

$$\cos \zeta_1 = \frac{|f_1|}{|a_1|} \quad \text{and} \quad \sin \zeta_1 = \frac{|f_2|}{|a_1|} e^{i(\theta_2 - \theta_1 - \pi/2)}, \quad (S10)$$

respectively. From Eq. (S6), $e^{i(\theta_3 - \theta_0 - \pi/2)}$ and $e^{i(\theta_2 - \theta_1 - \pi/2)}$ in Eq. (S9) and (S10) are signs.

Condition III. If the phase angles satisfy

$$(\theta_2 - \theta_0) \mod \pi = (\theta_3 - \theta_1) \mod \pi = 0, \quad (S11)$$

the two UE-parameters and terms become

$$a_0 = |a_0| e^{i\theta_0} = e^{i\theta_0} \sqrt{|f_0|^2 + |f_2|^2} \quad \text{and} \quad a_1 = |a_1| e^{i\theta_3} = e^{i\theta_3} \sqrt{|f_1|^2 + |f_3|^2}; \quad (\text{S12})$$

$$V_0 = \begin{bmatrix} \cos \zeta_0 & \sin \zeta_0 \\ -\sin \zeta_0 & \cos \zeta_0 \end{bmatrix} \quad \text{and} \quad V_1 = \begin{bmatrix} \cos \zeta_1 & \sin \zeta_1 \\ \sin \zeta_1 & -\cos \zeta_1 \end{bmatrix}, \quad (\text{S13})$$

where ζ_0 and ζ_1 are decided by

$$\cos \zeta_0 = \frac{|f_0|}{|a_0|} \quad \text{and} \quad \sin \zeta_0 = \frac{|f_2|}{|a_0|} e^{i(\theta_2 - \theta_0)} \quad (\text{S14})$$

and

$$\cos \zeta_1 = \frac{|f_3|}{|a_1|} \quad \text{and} \quad \sin \zeta_1 = \frac{|f_1|}{|a_1|} e^{i(\theta_1 - \theta_3)}, \quad (\text{S15})$$

respectively. Notice that $e^{i(\theta_2 - \theta_0)}$ and $e^{i(\theta_1 - \theta_3)}$ are a signs.

Condition IV. If the phase angles satisfy

$$|(\theta_1 - \theta_0) \bmod 2\pi| = |(\theta_3 - \theta_2) \bmod 2\pi| = \frac{\pi}{2}, \quad (\text{S16})$$

the UE-parameters and terms become

$$a_0 = |a_0| e^{i\theta_0} = e^{i\theta_0} \sqrt{|f_0|^2 + |f_1|^2} \quad \text{and} \quad a_1 = |a_1| e^{i\theta_3} = e^{i\theta_3} \sqrt{|f_2|^2 + |f_3|^2}; \quad (\text{S17})$$

$$V_0 = \begin{bmatrix} \cos \zeta_0 & i \sin \zeta_0 \\ i \sin \zeta_0 & \cos \zeta_0 \end{bmatrix} \quad \text{and} \quad V_1 = \begin{bmatrix} \cos \zeta_1 & i \sin \zeta_1 \\ -i \sin \zeta_1 & -\cos \zeta_1 \end{bmatrix}, \quad (\text{S18})$$

where ζ_0 and ζ_1 are decided by

$$\cos \zeta_0 = \frac{|f_0|}{|a_0|} \quad \text{and} \quad \sin \zeta_0 = \frac{|f_1|}{|a_0|} e^{i(\theta_1 - \theta_0 - \pi/2)} \quad (\text{S19})$$

and

$$\cos \zeta_1 = \frac{|f_3|}{|a_1|} \quad \text{and} \quad \sin \zeta_1 = \frac{|f_2|}{|a_1|} e^{i(\theta_2 - \theta_3 - \pi/2)}, \quad (\text{S20})$$

respectively. Noticing that $e^{i(\theta_1 - \theta_0 - \pi/2)}$ and $e^{i(\theta_2 - \theta_3 - \pi/2)}$ are equal to ± 1 .

Special cases. If one UE-parameter $f_k(t)$ in the main text is equal to zero, $F(t)$ can be expressed by two UE-terms in some special cases.

(a) In the case of $f_0(t) = 0$,

$$F(t) = e^{i\theta_3} |c_1| U_1 + e^{i\theta_3} |c_2| U_2. \quad (\text{S21})$$

In fact, it is accordance with the *phase matching* condition I [see Eq. (S2)] if we set $f_0(t) = 0 \cdot e^{i\theta_3}$. From Eq. (S3) to (S5), it is not difficult to calculate the parameters and matrices in the main text that a_0 , a_1 , V_0 and V_1 are equal to c_2 , c_1 , U_2 and U_1 , respectively.

(b) In the case of $f_1(t) = 0$, if $|(\theta_0 - \theta_3) \bmod 2\pi| = \pi/2$ or $(\theta_2 - \theta_0) \bmod \pi = 0$, either *phase matching* condition II [Eq. (S6)] or III [Eq. (S11)] is met, respectively.

(c) In the case of $f_2(t) = 0$, if $|(\theta_0 - \theta_3) \bmod 2\pi| = \pi/2$ or $|(\theta_1 - \theta_0) \bmod 2\pi| = \pi/2$, either *phase matching* condition II [Eq. (S6)] or IV [Eq. (S16)] is met, respectively.

(d) In the case of $f_3(t) = 0$, it naturally satisfies *phase matching* condition I [see Eq. (S2)] because θ_3 can be set as θ_0 and $f_3(t) = 0 \cdot e^{i\theta_0}$.

Explicit expressions of UEs in the section of 'Illustration' PT-symmetric Hamiltonians.

A time-independent PT-symmetric two-level Hamiltonian has a general form of

$$H_{PT} = \begin{bmatrix} re^{i\theta} & se^{i\varphi} \\ se^{-i\varphi} & re^{-i\theta} \end{bmatrix}, \quad (\text{S22})$$

satisfying $[PT, H_{PT}] = 0$, where $P = \sigma_1$ is the parity operator and T is the time-reversal operator having the effect that $i \rightarrow -i$. The eigenvalue of H_{PT} are $\epsilon_{\pm} = r \cos \theta \pm \sqrt{s^2 - r^2 \sin^2 \theta}$, and we set the difference of them as

$$\omega = 2\sqrt{s^2 - r^2 \sin^2 \theta}. \quad (\text{S23})$$

ω is either real or imaginary depending on the system is in exact-PT or PT-broken phase, respectively. We now expand the time-evolution operator $e^{-i(t/\hbar)H_{PT}}$ as the form of Eq. (2) in the main text. The four UE-parameters are

$$f_0(t) = \left[\frac{1}{2} \left(e^{i\frac{\omega t}{2\hbar}} + e^{-i\frac{\omega t}{2\hbar}} \right) \right] e^{-i\frac{t}{\hbar} r \cos \theta}, \quad (\text{S24})$$

$$f_1(t) = i \left[-\frac{s \cos \varphi}{i\omega} \left(e^{i\frac{\omega t}{2\hbar}} - e^{-i\frac{\omega t}{2\hbar}} \right) \right] e^{-i\frac{t}{\hbar} r \cos \theta}, \quad (\text{S25})$$

$$f_2(t) = \left[\frac{s \sin \varphi}{i\omega} \left(e^{i\frac{\omega t}{2\hbar}} - e^{-i\frac{\omega t}{2\hbar}} \right) \right] e^{-i\frac{t}{\hbar} r \cos \theta}, \quad (\text{S26})$$

and

$$f_3(t) = \left[\frac{r \sin \theta}{i\omega} \left(e^{i\frac{\omega t}{2\hbar}} - e^{-i\frac{\omega t}{2\hbar}} \right) \right] e^{-i\frac{t}{\hbar} r \cos \theta}. \quad (\text{S27})$$

Terms in the brackets are always real, while terms outside the brackets are the phases. the phase angles θ_0 and θ_3 of f_0 and f_3 satisfy *phase matching* condition I in Eq. (S2).

Anti-PT-symmetric Hamiltonians.

A Hamiltonian is anti-PT-symmetric if $\{PT, H_{APT}\} = 0$. In fact, the general form of the Hamiltonian is equal to the imaginary unit i times H_{PT} . For a time-independent two-state system,

$$H_{APT} = i \begin{bmatrix} re^{i\theta} & se^{i\varphi} \\ se^{-i\varphi} & re^{-i\theta} \end{bmatrix} = iH_{PT}. \quad (\text{S28})$$

The eigenvalues are $\epsilon_{\pm} = i \left(r \cos \theta \pm \sqrt{s^2 - r^2 \sin^2 \theta} \right)$, and we set the difference of them as

$$\lambda = 2\sqrt{r^2 \sin^2 \theta - s^2} = 2i\sqrt{s^2 - r^2 \sin^2 \theta} = i\omega, \quad (\text{S29})$$

where ω is that in Eq. (S23). The unitary expansion of the time-evolution operator $e^{-i(t/\hbar)H_{APT}}$ can be expanded as Eq. (2) in the main text, and the four UE-parameters are

$$f_0(t) = \left[\frac{1}{2} \left(e^{i\frac{\lambda t}{2\hbar}} + e^{-i\frac{\lambda t}{2\hbar}} \right) \right] e^{\frac{t}{\hbar} r \cos \theta}, \quad (\text{S30})$$

$$f_1(t) = \left[\frac{s \cos \varphi}{i\lambda} \left(e^{i\frac{\lambda t}{2\hbar}} - e^{-i\frac{\lambda t}{2\hbar}} \right) \right] e^{\frac{t}{\hbar} r \cos \theta}, \quad (\text{S31})$$

$$f_2(t) = i \left[\frac{s \sin \varphi}{i\lambda} \left(e^{i\frac{\lambda t}{2\hbar}} - e^{-i\frac{\lambda t}{2\hbar}} \right) \right] e^{\frac{t}{\hbar} r \cos \theta}, \quad (\text{S32})$$

and

$$f_3(t) = i \left[\frac{r \sin \theta}{i\lambda} \left(e^{i\frac{\lambda t}{2\hbar}} - e^{-i\frac{\lambda t}{2\hbar}} \right) \right] e^{\frac{t}{\hbar} r \cos \theta}. \quad (\text{S33})$$

Phases of the UE-parameters are outside the brackets, and they meets the *phase matching* condition II in Eq. (S6). Therefore, a time-independent anti-PT-symmetric two-level system can be simulated by two qubits.

P-pseudo-Hermitian Hamiltonians.

A P -pseudo-Hermitian Hamiltonian, say H_{PPH} , satisfies $PH_{PPH}^\dagger P = H_{PPH}$. For a time-independent two-level system,

$$H_{PPH} = \begin{bmatrix} re^{i\theta} & s \\ u & re^{-i\theta} \end{bmatrix}, \quad (\text{S34})$$

of which the eigenvalues are $\varepsilon_{\pm} = r \cos \theta \pm \sqrt{us - r^2 \sin^2 \theta}$. The difference of the two eigenvalues is

$$\omega = 2\sqrt{us - r^2 \sin^2 \theta}. \quad (\text{S35})$$

The time-evolution operator $e^{-i(t/\hbar)H_{PPH}}$ can be expanded as the form of Eq. (2) in the main text with the four UE-parameters as follow

$$f_0(t) = \left[\frac{1}{2} \left(e^{i\frac{\omega t}{2\hbar}} + e^{-i\frac{\omega t}{2\hbar}} \right) e^{\frac{t}{\hbar} r \cos \theta} \right] e^{-i\frac{t}{\hbar} r \cos \theta}, \quad (\text{S36})$$

$$f_1(t) = i \left[-\frac{s+u}{2i\omega} \left(e^{i\frac{\omega t}{2\hbar}} - e^{-i\frac{\omega t}{2\hbar}} \right) e^{\frac{t}{\hbar} r \cos \theta} \right] e^{-i\frac{t}{\hbar} r \cos \theta}, \quad (\text{S37})$$

$$f_2(t) = i \left[-\frac{s-u}{2i\omega} \left(e^{i\frac{\omega t}{2\hbar}} - e^{-i\frac{\omega t}{2\hbar}} \right) e^{\frac{t}{\hbar} r \cos \theta} \right] e^{-i\frac{t}{\hbar} r \cos \theta}, \quad (\text{S38})$$

and

$$f_3(t) = \left[\frac{r \sin \theta}{i\omega} \left(e^{i\frac{\omega t}{2\hbar}} - e^{-i\frac{\omega t}{2\hbar}} \right) e^{\frac{t}{\hbar} r \cos \theta} \right] e^{-i\frac{t}{\hbar} r \cos \theta}. \quad (\text{S39})$$

Comparing with *phase matching* conditions (see Sec. 2), the phases of the four UE-parameters meet both the condition I and IV in Eq. (S2) and (S16).

Anti-P-pseudo-Hermitian Hamiltonians

Timed by the imaginary unit i , H_{PPH} becomes an anti- P -pseudo-Hermitian (anti-PPH) Hamiltonian, which satisfies $PH_{APPH}^\dagger P = -H_{APPH}$. A time-independent two-dimensional one is

$$H_{APPH} = i \begin{bmatrix} re^{i\theta} & s \\ u & re^{-i\theta} \end{bmatrix} = iH_{PPH}. \quad (\text{S40})$$

The eigenvalues are $\varepsilon_{\pm} = i \left(r \cos \theta \pm \sqrt{us - r^2 \sin^2 \theta} \right)$, and the difference of them is

$$\lambda = 2\sqrt{r^2 \sin^2 \theta - us} = 2i\sqrt{us - r^2 \sin^2 \theta} = i\omega, \quad (\text{S41})$$

where ω is that in Eq. (S35).

(i) In an arbitrary phase, the time-evolution operator $e^{-i(t/\hbar)H_{APPH}}$ has the form of Eq. (2) in the main text, where the four UE-parameters are equal to

$$f_0(t) = \left[\frac{1}{2} \left(e^{i\frac{\lambda t}{2\hbar}} + e^{-i\frac{\lambda t}{2\hbar}} \right) e^{\frac{t}{\hbar} r \cos \theta} \right], \quad (\text{S42})$$

$$f_1(t) = \left[\frac{s+u}{2i\lambda} \left(e^{i\frac{\lambda t}{2\hbar}} - e^{-i\frac{\lambda t}{2\hbar}} \right) e^{\frac{t}{\hbar} r \cos \theta} \right], \quad (\text{S43})$$

$$f_2(t) = \left[\frac{s-u}{2i\lambda} \left(e^{i\frac{\lambda t}{2\hbar}} - e^{-i\frac{\lambda t}{2\hbar}} \right) e^{\frac{t}{\hbar} r \cos \theta} \right], \quad (\text{S44})$$

and

$$f_3(t) = i \left[\frac{r \sin \theta}{i\lambda} \left(e^{i\frac{\lambda t}{2\hbar}} - e^{-i\frac{\lambda t}{2\hbar}} \right) e^{\frac{t}{\hbar} r \cos \theta} \right]. \quad (\text{S45})$$

Noticing that λ is either real or imagine, the terms in the brackets are real. The phases of the four UE-parameters are 0, 0, 0 and $\pi/2$, so none of the *phase matching* conditions are met. Therefore, three qubits are necessary to simulate this system in a general case by either the six- or eight-dimension protocols.

(ii) In some special cases (see Sec. 2.5), the system can be simulated by two qubits. For example, when the system is in the anti-PT-symmetric phase, i.e. $s = u$, one of the UE-parameters $f_2(t)$ of the time-evolution operator $e^{-i(t/\hbar)H_{APPH}}$ becomes zero. As discussed in the first section above, it meets both the *phase matching* condition II [Eq. (S6)] and IV [Eq. (S16)]. So the system can be simulated by two qubits in this anti-PT-symmetric phase.

Single-qubit measurements: σ_3 -measurement.

As a simple case, we illustrate how to simulate a single-qubit σ_3 -measurement. Assuming $|0\rangle$ and $|1\rangle$ are the two eigen states of σ_3 , the effect of this measurement is to apply either of the two nonunitary matrices

$$M_0 = |0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \frac{1}{2}(\sigma_0 + \sigma_3) \quad \text{or} \quad M_1 = |1\rangle\langle 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{2}(\sigma_0 - \sigma_3) \quad (\text{S46})$$

with probabilities ${}_e\langle 0|\psi\rangle_e$ or ${}_e\langle 1|\psi\rangle_e$, respectively. We now apply a $U = H_2, C_{0-\sigma_0}, C_{1-\sigma_3}$ and another H_2 as quantum circuit in Fig.(6) in the main text. Now, the system evolves to a state

$$|0\rangle_a M_0 |\psi\rangle_e + |1\rangle_a M_1 |\psi\rangle_e. \quad (\text{S47})$$

Finally, a measurement is performed on the ancillary qubit. If an output $|k\rangle_a$ is obtained, the work qubit will evolve to $|k\rangle_e$ with a probability of ${}_e\langle k|\psi\rangle_e$ ($k = 0, 1$). Although the ancillary qubit is annihilated, the work qubit is kept for further use.

The Abrams–Lloyd’s gate

The explicit forms of the UE-terms and parameters of $N = \sum_{k=0}^4 n_k N_k$ are unitary matrices

$$N_0 = \begin{bmatrix} \frac{1}{\sqrt{2}\sqrt{2-\sqrt{2}}} \sigma_0 & -\frac{\sqrt{2}-1}{\sqrt{2}\sqrt{2-\sqrt{2}}} \sigma_0 \\ \frac{\sqrt{2}-1}{\sqrt{2}\sqrt{2-\sqrt{2}}} \sigma_0 & \frac{1}{\sqrt{2}\sqrt{2-\sqrt{2}}} \sigma_0 \end{bmatrix}, \quad (\text{S48})$$

$$N_1 = \begin{bmatrix} -\frac{1}{2}(i\sigma_2) & \frac{1}{2}\sigma_0 \\ -\frac{1}{2}(i\sigma_2) & -\frac{1}{2}\sigma_0 \end{bmatrix}, \quad (\text{S49})$$

$$N_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}}(\sigma_3 + \sigma_1) & 0 \\ 0 & \sigma_0 \end{bmatrix}, \quad (\text{S50})$$

$$N_3 = \begin{bmatrix} 0 & \sigma_0 \\ -\frac{1}{\sqrt{2}}(\sigma_3 + \sigma_1) & 0 \end{bmatrix} \quad (\text{S51})$$

and

$$N_4 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \sigma_0 & \frac{1}{\sqrt{2}} \sigma_0 \\ \frac{1}{\sqrt{2}} \sigma_0 & \frac{1}{\sqrt{2}} \sigma_0 \end{bmatrix}; \quad (\text{S52})$$

$n_0 = \sqrt{2-\sqrt{2}}, n_1 = 1, n_2 = n_3 = 1/\sqrt{2}, n_4 = (\sqrt{2}-1)/\sqrt{2}$ are the UE-parameters.

The normalizing factor in the main text is

$$f_N = \sqrt{\sum_{k=0}^4 |n_k|^2} = \sqrt{\frac{11}{2} - 2\sqrt{2}} \quad (\text{S53})$$