

Appendix A. Proofs of theorems regarding SDE (??) and RDE (??)

Proof of Theorem ??

Let us assume the expression $\mathcal{V}(t, x(t)) \geq c_1 |x(t)|^2$. $\mathcal{V}(t_0, x_0) \leq c_2 |x_0|^2$ and the supermartingale property of $\mathcal{V}(t, x(t))$ follow from the boundedness of the Lyapunov function at t_0 .

$$\mathbb{P} \left(\sup_t |x(t, x_0)| > \epsilon_1 | \mathcal{F}_{t_0} \right) = \mathbb{P} \left(\sup_t \mathcal{V}(t, x(t)) > c_1 \epsilon_1^2 | \mathcal{F}_{t_0} \right) \leq \frac{\mathcal{V}(t_0, x_0)}{c_1 \epsilon_1^2} \leq \frac{c_2 \delta^2}{c_1 \epsilon_1^2} < \epsilon.$$

Hence theorem is proved.

Proof of Theorem ??

Define the Lyapunov function $\mathcal{V}(t, x(t)) = x^2(t)$,

$$dx^2(t) = L [x(t)]^2 dt + \sigma(t, x(t)) \frac{\partial [x(t)]^2}{\partial x} dW(t).$$

Then,

$$\mathbb{E} [x(t)]^2 - [x_0]^2 = \int_{t_0}^t \mathbb{E} L [x(s)]^2 ds. \quad (1)$$

For $c_i > 0 \forall i$, we can assume $\mathcal{V}(t, x(t)) \geq c_1 [x(t)]^2$ with $L\mathcal{V}(t, x(t)) \leq -c_3 [x(t)]^2$ and $\mathcal{V}(t_0, x_0) \leq c_2 |x_0|^2$. Then, from (1) we have:

$$\int_{t_0}^t \mathbb{E} [x(s)]^2 ds \leq \frac{1}{c_3} \mathbb{E} \mathcal{V}(t_0, x_0) \leq \frac{c_2}{c_3} |x_0|^2 < \infty.$$

In mean-square, zero solution is stable as

$$\mathbb{E} [x(t)]^2 \leq \frac{1}{c_1} \mathbb{E} [\mathcal{V}(t, x(t))] \leq \frac{1}{c_1} \mathbb{E} [\mathcal{V}(t_0, x_0)] \leq \frac{c_2}{c_1} |x_0|^2.$$

For $0 \leq t_1 \leq t_2$, and for positive constants M_1, M_2 , one get for $M > 0$,

$$\mathbb{E} [x(t_2)]^2 - [x(t_1)]^2 \leq M (t_2 - t_1),$$

$\mathbb{E} [x^2(t)]$ satisfies Lipschitz condition with $\lim_{t \rightarrow \infty} \mathbb{E} |x(t)|^2 = 0$. In that case, the zero solution of (??) is asymptotically mean-square stable.

Proof of Theorem ??

Following the argument of the previous theorem, for $\mathcal{V}(t, x(t)) \geq c_1 e^{\lambda t} \mathbb{E} [x(t)]^2$, we have

$$c_1 \mathbb{E} [x(t)]^2 \leq e^{-\lambda t} \mathbb{E} [\mathcal{V}(t, x(t))] \leq e^{-\lambda t} \mathbb{E} [\mathcal{V}(t_0, x_0)] \leq c_2 e^{-\lambda t} |x_0|^2,$$

for all positive t_0 and this completes the exponential mean-square stability. For global stability, define the Lyapunov function

$$\mathcal{V}(t, x(t)) = k \int_{t-t_0}^t e^{\epsilon s} x^2(s) ds + e^{\epsilon t} [x(t)]^2,$$

with $L\mathcal{V}(t, x(t)) \leq 0$, for $0 \leq k < 1$, $t \geq t_0$. Then

$$e^{\varepsilon t} \mathbb{E} [x(t)]^2 \leq \mathbb{E} [\mathcal{V}(t, x(t))] \leq \mathbb{E} [\mathcal{V}(t_0, x_0)] \leq e^{\varepsilon t_0} |x_0|^2 + kt_0 e^{\varepsilon t_0} |x_0|^2,$$

implies

$$\mathbb{E} [x(t)]^2 \leq |x_0|^2 (1 + kt_0) e^{-\varepsilon(t-t_0)}.$$

Choose $\varepsilon > 0$ such that

$$\begin{aligned} t_0 e^{-\varepsilon t_0} &< 1, \\ k e^{\varepsilon t_0} &< 1. \end{aligned} \tag{2}$$

Using relations (2) and by the induction, we can show

$$\mathbb{E} [x(t)]^2 \leq \frac{1}{1 - k e^{\varepsilon t_0}} |x_0|^2 e^{-\varepsilon(t-t_0)}, \quad t \geq t_0. \tag{3}$$

Firstly, for $[t_0, 2t_0)$,

$$\begin{aligned} \mathbb{E} |x(t)|^2 &\leq |x_0|^2 (1 + kt_0) e^{-\varepsilon(t-t_0)} \\ &\leq |x_0|^2 (1 + k e^{\varepsilon t_0}) e^{-\varepsilon(t-t_0)} \\ &\leq \frac{1}{1 - k e^{\varepsilon t_0}} |x_0|^2 e^{-\varepsilon(t-t_0)}. \end{aligned} \tag{4}$$

For $[2t_0, 3t_0)$,

$$\mathbb{E} |x(t)|^2 \leq \frac{1}{1 - k e^{\varepsilon t_0}} |x_0|^2 e^{-\varepsilon(t-t_0)}.$$

For $[nt_0, (n+1)t_0)$, $n = 1, 2, \dots$

$$\begin{aligned} \mathbb{E} |x(t)|^2 &\leq |x_0|^2 (1 + kt_0) e^{-\varepsilon(t-t_0)} \\ &\leq |x_0|^2 e^{-\varepsilon(t-t_0)} (1 + k e^{\varepsilon t_0} + k^2 e^{2\varepsilon t_0} + \dots + k^n e^{n\varepsilon t_0}) \\ &\leq \frac{1}{1 - k e^{\varepsilon t_0}} |x_0|^2 e^{-\varepsilon(t-t_0)}. \end{aligned}$$

The zero solution of (??) is therefore globally exponentially mean-square stable since (3) holds.

Proof of Theorem ??

Consider the Lyapunov function $[x(t)]^2$ Hence,

$$\mathbb{E} [d|x(t)|^2] = \mathbb{E} [2a(\omega)f(t, \omega)x(t) + b^2(\omega)g^2(t, \omega)] dt + 2\mathbb{E} [b(\omega)g(t, x_t)x(t)] dW(t),$$

which implies for $0 \leq t_1 \leq t_2$

$$\mathbb{E} [[x(t_2)]^2 - [x(t_1)]^2] = \int_{t_1}^{t_2} \mathbb{E} [2a(\omega)f(s, \omega)x(s) + b^2(\omega)g^2(s, \omega)] ds + 2 \int_{t_1}^{t_2} \mathbb{E} [b(\omega)g(s, x_t)x(s)] dW(s).$$

For positive constants M_1, M_2 ,

$$2\mathbb{E}|a(\omega)f(t, \omega)x(t)| \leq 2\|a(\omega)\|_4\|f(t, \omega)\|_4\|x(t)\|_2 \leq M_1.$$

Similarly,

$$\mathbb{E}[b^2(\omega)g^2(t, \omega)] \leq \|b(\omega)\|_4^2\|g(t, \omega)\|_4^2 \leq M_2.$$

Therefore,

$$\mathbb{E}[|x(t_2)|^2 - |x(t_1)|^2] \leq M(t_2 - t_1),$$

for $M > 0$, $\mathbb{E}[x^2(t)]$ satisfies Lipschitz condition with $\lim_{t \rightarrow \infty} \mathbb{E}|x(t)|^2 = 0$, and this completes the proof.

Appendix B. Proofs of Propositions regarding the applications

Proof of Proposition ??

Let

$$\mathcal{V}(t, x_1, x_2, x_3) = \sum_{i=1}^3 x_i^2(t). \quad (5)$$

By Itô formula, the differential of $\mathcal{V}$ takes the form:

$$d\mathcal{V}(t, x(t)) = L\mathcal{V}(t, x(t))dt + \sum_{i=1}^3 \sigma_i x_i \frac{\partial \mathcal{V}}{\partial x_i} dW_i(t),$$

where,

$$\begin{aligned} L\mathcal{V}(t, x(t)) &= -2(\Lambda + \beta I^*)x_1^2(t) - 2\beta S^*x_1(t)x_2(t) + 2\beta I^*x_1(t)x_2(t) + 2(\beta S^* - \Lambda - \delta)x_2^2(t) \\ &\quad + 2\delta x_2(t)x_3(t) - 2(\Lambda + d)x_3^2(t) + \sum_{i=1}^3 \sigma_i^2 x_i^2(t) \\ &= (\sigma_1^2 + \beta S^* + \beta I^* - 2(\Lambda + \beta I^*))x_1^2(t) + (\sigma_2^2 + \delta + \beta S^* + \beta I^* + 2(\beta S^* - \Lambda - \delta))x_2^2(t) \\ &\quad + (\sigma_3^2 + \delta - 2(\Lambda + d))x_3^2(t). \end{aligned}$$

Then

$$\begin{aligned} d[\mathbb{E}[\mathcal{V}(t, x(t))]] &= \mathbb{E}[L\mathcal{V}(t, x(t))]dt + \mathbb{E}\left[\sum_{i=1}^3 \sigma_i \frac{\partial \mathcal{V}}{\partial x_i} dW_i(t)\right] \\ &\leq (\sigma_1^2 + \beta S^* + \beta I^* - 2(\Lambda + \beta I^*))\mathbb{E}[x_1^2(t)]dt \\ &\quad + (\sigma_2^2 + \delta + \beta S^* + \beta I^* + 2(\beta S^* - \Lambda - \delta))\mathbb{E}[x_2^2(t)]dt \\ &\quad + (\sigma_3^2 + \delta - 2(\Lambda + d))\mathbb{E}[x_3^2(t)]dt + 2\mathbb{E}\left[\sum_{i=1}^3 \sigma_i x_i^2(t)dW_i(t)\right], \end{aligned}$$

implies

$$\begin{aligned}
\mathbb{E} [\mathcal{V}(t, x(t)) - \mathcal{V}(t_0, x_0)] &\leq (\sigma_1^2 + \beta S^* + \beta I^* - 2(\Lambda + \beta I^*)) \int_{t_0}^t \mathbb{E} [x_1^2(s)] ds \\
&\quad + (\sigma_2^2 + \delta + \beta S^* + \beta I^* + 2(\beta S^* - \Lambda - \delta)) \int_{t_0}^t \mathbb{E} [x_2^2(s)] ds \\
&\quad + (\sigma_3^2 + \delta - 2(\Lambda + d)) \int_{t_0}^t \mathbb{E} [x_3^2(s)] ds \\
&\quad + 2\mathbb{E} \left(\int_{t_0}^t \sum_{i=1}^3 \sigma_i x_i^2(s) dW_i(s) \right).
\end{aligned}$$

The stochastic integral's zero-mean property causes the final term to disappear. Additionally, all ordinary integrals $\int_{t_0}^t \mathbb{E} [x_i^2(s)]$ According to the square integrability of the stochastic process $\mathbf{x}(t)$ for $i = 1, 2, 3$, are finite. The zero solution of the linear system (??) is then mean-square stable, in accordance with Theorem ??, if these inequalities hold true

$$\begin{aligned}
\sigma_1^2 + \beta S^* + \beta I^* - 2(\Lambda + \beta I^*) &\leq 0, \\
\sigma_2^2 + \delta + \beta S^* + \beta I^* + 2(\beta S^* - \Lambda - \delta) &\leq 0, \\
\sigma_3^2 + \delta - 2(\Lambda + d) &\leq 0,
\end{aligned}$$

are satisfied, and if equation (??) holds, then

$$\begin{aligned}
|\mu(t, x) - F \cdot x| + |\sigma(t, x) - G \cdot x| &= \sqrt{(-\beta x_1 x_2)^2 + (\beta_1 x_1 x_2)^2} + \sqrt{\sum_{i=1}^3 \sigma_i^2 x_i^2 - \sigma_i^2 x_i^2} \\
&= \sqrt{2\beta^2 x_1^2 x_2^2} \\
&\leq \sqrt{2}\beta |x|^2,
\end{aligned}$$

and this is in the small neighborhood $(-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon)$ of $x = 0$. Then under conditions (??), the endemic equilibrium E^* is stochastically stable.

Proof of Proposition ??

For $R_0 > 1$, choose

$$\mathcal{V}(t, x) = e^{\varepsilon t} \sum_{i=1}^3 x_i^2(t) + \frac{1}{R_0} \int_{t-t_0}^t e^{\varepsilon s} \sum_{i=1}^3 x_i^2(s) ds.$$

Then,

$$\begin{aligned}
\mathcal{LV}(t, x(t)) &= e^{\varepsilon t} \left[2x_1(t) (-(\Lambda + \beta I^*)x_1(t) - \beta S^*x_2(t)) + \sum_{i=1}^3 \sigma_i^2 x_i^2(t) + \left(\frac{1}{R_0} + \varepsilon\right) \sum_{i=1}^3 x_i^2(t) \right. \\
&\quad + 2x_2(t) (\beta I^*x_1(t) + (\beta S^* - \Lambda - \delta)x_2(t)) + 2x_3(t) (\delta x_2(t) - (\Lambda + d)x_3(t)) \\
&\quad \left. - \frac{1}{R_0} e^{-\varepsilon t_0} \sum_{i=1}^3 x_i^2(t - t_0) \right] \\
&\leq e^{\varepsilon t} \left[\left(\sigma_1^2 + \frac{1}{R_0} + \varepsilon - 2(\Lambda + \beta I^*) + \beta(S^* + I^*) \right) x_1^2(t) \right. \\
&\quad + \left(\sigma_2^2 + \frac{1}{R_0} + \varepsilon + \beta S^* + \beta I^* + 2(\beta S^* - \Lambda - \delta) + \delta \right) x_2^2(t) \\
&\quad \left. + \left(\sigma_3^2 + \frac{1}{R_0} + \varepsilon + \delta - 2(\Lambda + d) \right) x_3^2(t) - \frac{1}{R_0} e^{-\varepsilon t_0} \sum_{i=1}^3 x_i^2(t - t_0) \right]
\end{aligned}$$

Then,

$$\begin{aligned}
\mathbb{E} [\mathcal{LV}(t, x(t))] &\leq e^{\varepsilon t} \left[\left(\sigma_1^2 + \frac{1}{R_0} + \varepsilon - 2(\Lambda + \beta I^*) + \beta(I^* + S^*) \right) \mathbb{E} [x_1^2(t)] \right. \\
&\quad + \left(\sigma_2^2 + \frac{1}{R_0} + \varepsilon + \beta S^* + \beta I^* + 2(\beta S^* - \Lambda - \delta) + \delta \right) \mathbb{E} [x_2^2(t)] \\
&\quad \left. + \left(\sigma_3^2 + \frac{1}{R_0} + \varepsilon + \delta - 2(\Lambda + d) \right) \mathbb{E} [x_3^2(t)] - \frac{1}{R_0} e^{-\varepsilon t_0} \sum_{i=1}^3 \mathbb{E} [x_i^2(t - t_0)] \right]
\end{aligned}$$

According to Theorem. ??, $\mathbb{E} [\mathcal{LV}(t, x(t))] \leq 0$ if

$$\begin{aligned}
\sigma_1^2 + \frac{1}{R_0} + \varepsilon - (2\Lambda + \beta I^*) + \beta(I^* + S^*) &\leq 0, \\
\sigma_2^2 + \frac{1}{R_0} + \varepsilon + 3\beta S^* + \beta I^* - (2\Lambda + \delta) &\leq 0, \\
\sigma_3^2 + \frac{1}{R_0} + \varepsilon + \delta - 2(\Lambda + d) &\leq 0.
\end{aligned}$$

Choose $\varepsilon > 0$ and t_0 as the size of an interval such that

$$t_0 e^{-\varepsilon t_0} < 1, \quad \text{and} \quad \frac{1}{R_0} e^{\varepsilon t_0} < 1. \quad (6)$$

As $\mathbb{E} [\mathcal{LV}] \leq 0$, then $\mathbb{E} [\mathcal{V}(t, x(t))] \leq \mathbb{E} [\mathcal{V}(t_0, x_0)]$ then,

$$e^{\varepsilon t} \mathbb{E} \left[\sum_{i=1}^3 x_i^2(t) \right] \leq \mathbb{E} [\mathcal{V}(t, x)] \leq \mathbb{E} [\mathcal{V}(t_0, x_0)] \leq e^{\varepsilon t_0} \sum_{i=1}^3 x_{i_0}^2 + \frac{1}{R_0} t_0 e^{\varepsilon t_0} \sum_{i=1}^3 x_{i_0}^2. \quad (7)$$

we can show,

$$\mathbb{E} \left[\sum_{i=1}^3 x_i^2(t) \right] \leq \frac{1}{1 - \frac{1}{R_0} e^{\varepsilon t_0}} e^{-\varepsilon(t-t_0)} \sum_{i=1}^3 x_{i_0}^2, \quad \text{for } t \geq t_0. \quad (8)$$

Firstly, for the interval $[t_0, 2t_0)$, using (6), then (??) implies

$$\begin{aligned} \mathbb{E} \left[\sum_{i=1}^3 x_i^2(t) \right] &\leq \left(1 + \frac{1}{R_0} t_0\right) e^{-\varepsilon(t-t_0)} \sum_{i=1}^3 x_{i_0}^2 \\ &\leq \left(1 + \frac{1}{R_0} e^{\varepsilon t_0}\right) e^{-\varepsilon(t-t_0)} \sum_{i=1}^3 x_{i_0}^2 \leq \frac{1}{1 - \frac{1}{R_0} e^{\varepsilon t_0}} e^{-\varepsilon(t-t_0)} \sum_{i=1}^3 x_{i_0}^2. \end{aligned} \quad (9)$$

For the interval $[2t_0, 3t_0)$, using (6), (9)

$$\begin{aligned} \mathbb{E} \left[\sum_{i=1}^3 x_i^2(t) \right] &\leq \left(1 + \frac{1}{R_0} t_0\right) e^{-\varepsilon(t-t_0)} \sum_{i=1}^3 x_{i_0}^2 \\ &\leq e^{-\varepsilon(t-t_0)} \sum_{i=1}^3 x_{i_0}^2 + \frac{1}{R_0} e^{\varepsilon t_0} e^{-\varepsilon(t-t_0)} \left(1 + \frac{1}{R_0} e^{\varepsilon t_0}\right) e^{-\varepsilon(t_0-t_0)} \sum_{i=1}^3 x_{i_0}^2 \\ &= e^{-\varepsilon(t-t_0)} \sum_{i=1}^3 x_{i_0}^2 + \left(R_0 e^{\varepsilon t_0} + \frac{1}{R_0} e^{2\varepsilon t_0}\right) e^{-\varepsilon(t-t_0)} \sum_{i=1}^3 x_{i_0}^2 \\ &= \left(1 + \frac{1}{R_0} e^{\varepsilon t_0} + \frac{1}{R_0} e^{2\varepsilon t_0}\right) e^{-\varepsilon(t-t_0)} \sum_{i=1}^3 x_{i_0}^2 \leq \frac{1}{1 - \frac{1}{R_0} e^{\varepsilon t_0}} e^{-\varepsilon(t-t_0)} \sum_{i=1}^3 x_{i_0}^2. \end{aligned}$$

For $[nt_0, (n+1)t_0)$, $n \in \mathbb{Z}$

$$\begin{aligned} \mathbb{E} \left[\sum_{i=1}^3 x_i^2(t) \right] &\leq \left(1 + \frac{1}{R_0} t_0\right) e^{-\varepsilon(t-t_0)} \sum_{i=1}^3 x_{i_0}^2 \\ &\leq \left(1 + \frac{1}{R_0} e^{\varepsilon t_0} + \frac{1}{R_0} e^{2\varepsilon t_0} + \dots + \frac{1}{R_0} e^{n\varepsilon t_0}\right) e^{-\varepsilon(t-t_0)} \sum_{i=1}^3 x_{i_0}^2 \\ &\leq \frac{1}{1 - \frac{1}{R_0} e^{\varepsilon t_0}} e^{-\varepsilon(t-t_0)} \sum_{i=1}^3 x_{i_0}^2. \end{aligned}$$

So under conditions (??) and inequality (8) holds. zero solution of (??) is globally exponentially m.s stable. Consequently, the endemic point of (??) is stochastically globally exponentially stable.

Proof of Proposition ??

Assuming $\mathcal{V}(t, x(t)) = x^2(t)e^{-\lambda\sigma t}$, $\lambda > 0$ is sufficiently large, one gets

$$\begin{aligned} \mathcal{L}\mathcal{V}(t, x(t)) &= \mathcal{V}_t(t, x(t)) + \mathcal{V}_x(t, x(t))(rx(t)) + \frac{1}{2!}\mathcal{V}_{xx}(t, x(t))\sigma^2 \sin^2(x(t)) \\ &= -\lambda\sigma x^2(t)e^{-\lambda\sigma t} + 2rx^2(t)e^{-\lambda\sigma t} + \sigma^2 \sin^2(x(t))e^{-\lambda\sigma t} \\ &= (2r - \lambda\sigma) e^{-\lambda\sigma t} x^2(t) + \sigma^2 \sin^2(x(t))e^{-\lambda\sigma t}. \end{aligned}$$

For sufficiently big λ , the last term vanishes. Hence, if

$$2r - \lambda\sigma < 0, \quad \sigma > 0, \quad (10)$$

then, $\mathcal{L}\mathcal{V} \leq 0$ and the trivial solution is stochastically stable according to Theorem. ?.?. As $\sin^2(x) \leq x^2$, then

$$\mathbb{E}\mathcal{L}\mathcal{V}(t, x(t)) \leq (\sigma^2 + 2r - \lambda\sigma) e^{-\lambda\sigma t} \mathbb{E}[x^2(t)],$$

Hence, if $2r - \lambda\sigma + \sigma^2 < 0$, then equation (??) is globally mean-square exponentially stable according to Theorem. ?.?. For the Lyapunov function $\mathcal{V}(t, x(t)) = x^2(t)$, we get

$$\mathbb{E}\mathcal{L}\mathcal{V}(t, x(t)) \leq (\sigma^2 + 2r) \mathbb{E}[x^2(t)].$$

According to Theorem. ??, the mean square stability condition is $\sigma^2 + 2r < 0$.

Proof of Proposition ??

Consider the random Lyapunov function $\mathcal{V}(t, S(t, \omega)) = S^2(t, \omega)e^{-\lambda\sigma t}$ with $\sigma, \lambda > 0$. Then,

$$\begin{aligned} \mathcal{L}\mathcal{V}(t, S(t, \omega)) &= -\lambda\sigma S^2(t, \omega)e^{-\lambda\sigma t} - 2A(\omega)S^2(t, \omega)e^{-\lambda\sigma t} + 2\sigma C(\omega)S^2(t, \omega)e^{-\lambda\sigma t} \\ &\leq -\lambda\sigma S^2(t, \omega)e^{-\lambda\sigma t} + S^2(t, \omega)(1 + A^2(\omega))e^{-\lambda\sigma t} + 2\sigma C(\omega)S^2(t, \omega)e^{-\lambda\sigma t}. \end{aligned} \quad (11)$$

$$\begin{aligned} \mathbb{E}\mathcal{L}\mathcal{V}(t, S(t, \omega)) &\leq -\lambda\sigma \mathbb{E}[S^2(t, \omega)] e^{-\lambda\sigma t} + \left(1 + \mathbb{E}[A^4(\omega)]\right)^{\frac{1}{4}} \mathbb{E}[S^2(t, \omega)] e^{-\lambda\sigma t} \\ &\quad + 2\sigma \mathbb{E}[C^2(\omega)]^{\frac{1}{2}} \mathbb{E}[S^2(t, \omega)] e^{-\lambda\sigma t} \\ &= \left[1 - \lambda\sigma + \mathbb{E}[A^4(\omega)]^{\frac{1}{4}} + 2\sigma \mathbb{E}[C^2(\omega)]^{\frac{1}{2}}\right] \mathbb{E}[S^2(t, \omega)] e^{-\lambda\sigma t}. \end{aligned}$$

Then, the mean-square exponential stability condition of (??) is $\|A(\omega)\|_4 + 2\sigma\|C(\omega)\|_2 < \lambda\sigma - 1$, according to Theorem. ??.

Wy introducing a random Lyapunov function $\mathcal{V}(t, S(t, \omega)) = S^2(t, \omega)$, one gets

$$\begin{aligned} \mathbb{E}\mathcal{L}\mathcal{V}(t, S(t, \omega)) &= -2\mathbb{E}[A(\omega)S^2(t, \omega)] + 2\sigma\mathbb{E}[C(\omega)S^2(t, \omega)] \\ &\leq \left[-2\mathbb{E}[A^2(\omega)]^{\frac{1}{2}} + 2\sigma\mathbb{E}[C^2(\omega)]^{\frac{1}{2}}\right] \mathbb{E}[S^2(t, \omega)], \end{aligned}$$

if only $[-\mathbb{E}[A^2(\omega)]\frac{1}{2} + \sigma \mathbb{E}[C^2(\omega)]\frac{1}{2} < 0$. Therefore, according to Theorem. ??, the model (??) is mean-square stable if $\sigma\|C(\omega)\|_2 < \|A(\omega)\|_2$ for $\sigma > 0$. For stochastic stability, the last term vanishes for sufficiently large λ in (11) which implies $L\mathcal{V}(t, x(t)) \leq 0$. Hence, according to Theorem. ??, equation (??) is stochastically stable if only $A(\omega)$ is a non-negative random variable.