A DYNAMICS BASED PROCEDURE FOR CLUSTERING AND CLASSIFYING SPACE DEBRIS: SUPPLEMENTARY MATERIAL

ALESSANDRA CELLETTI AND TUDOR VARTOLOMEI

1. Supplementary material

1.1. **Dynamical model.** In the study of satellite and space debris dynamics, the space surrounding Earth is divided into three main regions:

- **Low Earth Orbit (LEO):** Extends up to 2 000 km of altitude.
- **Medium Earth Orbit (MEO):** Ranges over altitudes between 2 000 and 35 786 km.
- **Geosynchronous Earth Orbit (GEO):** Includes altitudes at 35 786 km and above.

The dominant forces vary according to the region: the atmospheric drag significantly affects LEO, while MEO and GEO are influenced by Moon, Sun, and Solar radiation pressure. This study focuses on MEO and GEO, where the dynamics is governed by conservative forces. We consider a model described by the Hamiltonian function:

$$\mathcal{H} = \mathcal{H}_{Kep}(a) + \mathcal{H}_{Geo}(\mathcal{E}_O, \theta) + \mathcal{H}_{Moon}(\mathcal{E}_O, \mathcal{E}_M) + \mathcal{H}_{Sun}(\mathcal{E}_O, \mathcal{E}_S) + \mathcal{H}_{SRP}(\mathcal{E}_O, \mathcal{E}_S),$$

where the terms in the Hamiltonian are $\mathcal{H}_{Kep} = -\frac{\mathcal{G}m_E}{2a}$ - Earth's gravitational attraction with $\mathcal{G}$ the gravitational constant and m_E the mass of the Earth, $\mathcal{H}_{Geo}(\mathcal{E}_O, \theta)$ - Earth's non-spherical shape, $\mathcal{H}_{Moon}(\mathcal{E}_O, \mathcal{E}_M)$ - gravitational influence of the Moon, $\mathcal{H}_{Sun}(\mathcal{E}_O, \mathcal{E}_S)$ - gravitational influence of the Sun, $\mathcal{H}_{SRP}(\mathcal{E}_O, \mathcal{E}_S)$ - solar radiation pressure. The function arguments are the orbital elements of the debris, Moon, Sun: $\mathcal{E}_O = (a, e, i, \ell, \omega, \Omega), \mathcal{E}_M = (a_M, e_M, i_M, \ell_M, \omega_M, \Omega_M), \mathcal{E}_S = (a_S, e_S, i_S, \ell_S, \omega_S, \Omega_S)$ and θ is the sidereal time accounting for the rotation of the Earth.

The Fourier expansions of $\mathcal{H}_{Geo}(\mathcal{E}_O, \theta)$, $\mathcal{H}_{Moon}(\mathcal{E}_O, \mathcal{E}_M)$, $\mathcal{H}_{Sun}(\mathcal{E}_O, \mathcal{E}_S)$, and $\mathcal{H}_{SRP}(\mathcal{E}_O, \mathcal{E}_S)$ (see [3] and [2]) produce terms classified by the following dependence upon the angles:

- **Short-periodic terms:** Depend on fast angles (ℓ, θ) that are non-commensurable.
- **Resonant terms:** Depend on fast angles (ℓ, θ) with commensurable rates which satisfy a relation of the form $k_1\dot{\ell} + k_2\dot{\theta} = 0$ for non-zero integers k_1, k_2.
- **Semi-secular terms:** Depend on semi-fast angles (ℓ_M, ℓ_S).

- **Secular terms:** Depend on slow angles $(\omega, \Omega, \omega_{M/S}, \Omega_{M/S})$, causing long-term variations in eccentricity and inclination over centuries.

A detailed construction of the Hamiltonian function can be found in [1].

1.2. **Hamiltonian normal form.** The Hamiltonian can be expressed in terms of the Delaunay variables, which provide a convenient framework for describing the orbital motion and for applying perturbation theory. The Delaunay variables are defined as:

$$\begin{aligned}&L = \sqrt{\mathcal{G} m_E a}, \quad G = L\sqrt{1-e^2}, \quad H = G\cos i,\\ &l = \ell, \quad g = \omega, \quad h = \Omega,\end{aligned}$$

where (L, G, H) are the actions, and (l, g, h) are the conjugated angles. These variables allow the Hamiltonian to be expanded around reference values corresponding to a specific orbit.

The Hamiltonian, written in terms of these variables, can be expanded as a series:

$$\mathcal{H} = \mathcal{H}_0(L, G, H) + \sum_{\mathbf{k}} \left[\mathcal{H}_{\mathbf{k}}(L, G, H)\cos(\mathbf{k}\cdot(l, g, h)) + \mathcal{H}'_{\mathbf{k}}(L, G, H)\sin(\mathbf{k}\cdot(l, g, h))\right],$$

where:

- $\mathcal{H}_0(L, G, H)$ represents the unperturbed part of the Hamiltonian,
- $\mathcal{H}_{\mathbf{k}}(L, G, H)$ and $\mathcal{H}'_{\mathbf{k}}(L, G, H)$ are functions of the Delaunay actions, corresponding to the amplitudes of the perturbation terms,
- $\mathbf{k}\cdot(l, g, h)$ are the combinations of angles appearing in the trigonometric terms.

This expansion allows us to study the perturbations systematically. Resonant terms occur when there is a commensurability between the frequencies of the angles, while secular terms result from combinations where the angular dependence vanishes, leading to long-term variations in the orbital elements.

The normal form of the Hamiltonian provides a hierarchical framework for analysing orbital dynamics. By isolating the dominant terms and eliminating short-periodic contributions through canonical transformations, one can focus on the long-term evolution of the system.

The full procedure can be found in [1], while here we limit to present the main steps to compute the proper elements.

1.3. **Proper elements.**

Definition 1. *Let $\mathcal{H}$ be a nearly-integrable Hamiltonian:*

$$\mathcal{H}(L, G, H, l, g, h) = \mathcal{Z}(L, G, H) + \varepsilon\mathcal{R}(L, G, H, l, g, h);$$

let $\mathcal{H}^{(k)}$ denote the k-th order transformed Hamiltonian:

$$\mathcal{H}^{(k)}(L^{(k)}, G^{(k)}, H^{(k)}, l^{(k)}, g^{(k)}, h^{(k)}) = \mathcal{Z}^{(k)}(L^{(k)}, G^{(k)}, H^{(k)}) + \varepsilon^{k+1}\mathcal{R}^{(k)}.$$

The variables $(L^{(k)}, G^{(k)}, H^{(k)})$ are transformed through a Lie series $L_{\chi^{(k)}}$ with generating function $\chi^{(k)}$:

$$(L^{(k-1)}, G^{(k-1)}, H^{(k-1)}) = e^{L_{\chi^{(k)}}}(L^{(k)}, G^{(k)}, H^{(k)}).$$

The proper elements are defined by

$$(L_P, G_P, H_P) = e^{L_{\chi^{(1)}}} \cdots e^{L_{\chi^{(k)}}}(L^{(k)}, G^{(k)}, H^{(k)}).$$

For space debris applications, proper elements are computed hierarchically according to the following procedure.

1. *Initial averaging*: Consider the Hamiltonian, including the effects of Earth, Moon, Sun, and SRP. Averaging over the mean anomaly l and sidereal time θ makes the semi-major axis a constant, yielding the first proper element.

2. *Extended phase space*: Since $\Omega_M(t)$ and $l_S(t)$ vary due to SRP, extend the phase space by adding conjugate variables (H_M, L_S):

$$\mathcal{H}_{ext} = \mathcal{H}_{ext}(L, G, H, H_M, L_S, l, g, h, \Omega_M, \ell_S).$$

3. *Linear coordinate shift*: Shift (G, H) to reference values (G_0, H_0), corresponding to eccentricity e_0, inclination i_0.

4. *Displacements*: Define $\eta = e - e_0$, $\iota = i - i_0$, or equivalently $(P = G - G_0, Q = H - H_0)$.

5. *Hamiltonian expansion*: Expand $\mathcal{H}_{ext}$ around $(P, Q) = (0, 0)$ to fourth order:

$$\mathcal{H}_{exp} = \mathcal{Z}_0 + \mathcal{R}_0,$$

where $\mathcal{Z}_0$ is linear in (P, Q), with frequencies (ω_P, ω_Q).

6. *Normal form*: Introduce a parameter λ and split $\mathcal{R}_0$ into the averaged part $\overline{\mathcal{R}}_0$ and the oscillatory part $\widetilde{\mathcal{R}}_0$:

$$\mathcal{H}_{exp} = \mathcal{Z}_0 + \lambda\overline{\mathcal{R}}_0 + \lambda\widetilde{\mathcal{R}}_0.$$

7. *Generate normal form*: Perform a Lie transformation with generating function $\chi^{(1)}$, leading to:

$$\mathcal{H}^{(1)} = \mathcal{Z}_0 + \lambda\overline{\mathcal{R}}_0 + \lambda^2\mathcal{R}_1,$$

and discard $\mathcal{R}_1$ to get:

$$\mathcal{H}^{(NF)} = \mathcal{Z}_0 + \lambda\overline{\mathcal{R}}_0.$$

8. *Back-transform*: Use the generating function to express the new variables as functions of the initial variables:

$$(P', Q') = (S^{\lambda}_{\chi^{(1)}})^{-1}(P, Q).$$

Proper eccentricity and inclination are computed by shifting back to Delaunay actions and then back-transforming as

$$e' = \sqrt{1 - \frac{H'^2}{L^2}}, \quad i' = \cos^{-1}\left(\frac{H'}{G'}\right) \,,$$

where L' ($L' = L$), G' and H' are the proper Delaunay actions obtained from P' and Q' as inverse transformations given in step 4.

Primary Corresponding author:

Alessandra Celletti, Department of Mathematics, University of Roma Tor Vergata, Via della Ricerca Scientifica 1, 00133 Roma (Italy), e-mail *celletti@mat.uniroma2.it*

Corresponding author:

Tudor Vartolomei, Department of Mathematics, Al. I. Cuza University, Bd. Carol I 11, 700506 Iasi (Romania), e-mail: *tudor.vartolomei@uaic.ro*

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Department of Mathematics, University of Roma Tor Vergata, Via della Ricerca Scientifica 1, 00133 Roma (Italy)

Email address: `celletti@mat.uniroma2.it`

Department of Mathematics, Al. I. Cuza University, Bd. Carol I 11, 700506 Iasi (Romania)

Email address: `tudor.vartolomei@uaic.ro`