

Supplementary section

Appendix A. Details of the coefficients that appear in the Eqs. (42) – (45)

The coefficients that appear in the Eqs. (42) – (45) are reported as follows:

$$m = \frac{a_c^2 I_4 - I_6}{I_4}, \quad n = \frac{I_4 I_7}{I_8}, \quad s = \frac{(a_c^2 I_4 - I_6) (a_c^2 I_2 - I_1)}{a_c^2 I_3 R},$$

$$I_1 = \int_{\mathcal{Z}=0}^{\mathcal{Z}=1} W_1''(\mathcal{Z}) W_1(\mathcal{Z}) d\mathcal{Z}, \quad I_2 = \int_{\mathcal{Z}=0}^{\mathcal{Z}=1} W_1^2(\mathcal{Z}) d\mathcal{Z},$$

$$I_3 = \int_{\mathcal{Z}=0}^{\mathcal{Z}=1} W_1(\mathcal{Z}) \theta_1(\mathcal{Z}) d\mathcal{Z}, \quad I_4 = \int_{\mathcal{Z}=0}^{\mathcal{Z}=1} \theta_1^2(\mathcal{Z}) d\mathcal{Z},$$

$$I_6 = \int_{\mathcal{Z}=0}^{\mathcal{Z}=1} \theta_1(\mathcal{Z}) \theta_1''(\mathcal{Z}) d\mathcal{Z}, \quad I_7 = \int_{\mathcal{Z}=0}^{\mathcal{Z}=1} H^2(\mathcal{Z}) d\mathcal{Z},$$

$$I_8 = \int_{\mathcal{Z}=0}^{\mathcal{Z}=1} W_1'(\mathcal{Z}) \theta_1(\mathcal{Z}) H(\mathcal{Z}) d\mathcal{Z}, \quad I_9 = \int_{\mathcal{Z}=0}^{\mathcal{Z}=1} H(\mathcal{Z}) H''(\mathcal{Z}) d\mathcal{Z},$$

Appendix B. Estimation of an approximate value of R using the STGM

Multiplying Eqs. (28) and (29) by W and θ respectively and integrating the resulting equations with respect to $\mathcal{Z}$ in $[0, 1]$, yields

$$\langle W W'' \rangle - \pi^2 a^2 \langle W^2 \rangle + R \pi^2 a^2 \langle W \theta \rangle = 0, \quad (65)$$

$$\langle \theta \theta'' \rangle + \langle W \theta \rangle - \pi^2 a^2 \langle \theta^2 \rangle = 0, \quad (66)$$

where $\langle \dots \rangle = \int_{\mathcal{Z}=0}^{\mathcal{Z}=1} (\dots) d\mathcal{Z}$ and primes denote the $\mathcal{Z}$ -derivative. We now consider $W = \mathcal{A} W_1$, $\theta = \mathcal{B} \theta_1$, in which $\mathcal{A}$ and $\mathcal{B}$ are the constants and depending upon the considered boundary condition, W_1 and θ_1 are the corresponding trial functions which satisfies the boundary condition (30) – (31) but need not satisfy the differential equation (28) – (29).

Substituting W and θ into Eqs. (65) – (66) and solving for R gives

$$R = \frac{[\langle W_1 W_1'' \rangle - \pi^2 a^2 \langle W_1^2 \rangle] [\langle \theta \theta'' \rangle - \pi^2 a^2 \langle \theta_1^2 \rangle]}{\pi^2 a^2 \langle W_1 \theta_1 \rangle^2}. \quad (67)$$

The trial functions W_1 and θ_1 are taken in the form

$$W_1 = \mathcal{Z}^2 - \mathcal{Z}, \quad (68)$$

$$\theta_1 = \mathcal{Z}^2 - \left(\frac{2 + 2\mathcal{K} + Bi}{1 + \mathcal{K} + Bi} \right) \mathcal{Z} + \frac{\mathcal{K}}{1 + \mathcal{K} + Bi}. \quad (69)$$

By substituting the above considered trial functions into Eq. (67) and completing the integration, the value of R_c and a_c are obtained.

Appendix C: Comparison of values of a_c and R_c obtained from STGM and MSE

$\mathcal{K}$	$(R_c)_{\text{STGM}}$	a_c	$(R_c)_{\text{MSE}}$	a_c	Percentage relative error in STGM $\left(\frac{(R_c)_{\text{STGM}} - (R_c)_{\text{MSE}}}{(R_c)_{\text{MSE}}} \right) \times 100$
0.1	37.57198	0.96339	36.82057	0.9613	1.99%
0.2	37.92882	0.96721	37.10537	0.96543	2.17%
0.3	38.29993	0.97040	37.3928	0.96957	2.37%
0.4	38.68617	0.97358	37.68277	0.97371	2.59%

Table S1. Comparison of values obtained from STGM and MSE for different values of $\mathcal{K}$ and for $Bi = 10$.

Bi	$(R_c)_{\text{STGM}}$	a_c	$(R_c)_{\text{MSE}}$	a_c	Percentage relative error in STGM $\left(\frac{(R_c)_{\text{STGM}} - (R_c)_{\text{MS}}}{(R_c)_{\text{MSE}}} \right) \times 100$
1	33.54001	0.81922	31.10401	0.843202	7.26%
5	36.32803	0.92934	35.21734	0.93297	3.06%
10	37.57120	0.96117	36.82057	0.96129	1.99%
30	38.96615	0.99230	38.41441	0.98549	1.41%
50	39.34326	0.99618	38.81302	0.99121	1.34%
100	39.65652	1.00190	39.13480	0.99600	1.31%

Table S2. Comparison of values obtained from STGM and MSE for different values of Bi and for $\mathcal{K} = 0.1$.

The Tables S1 and S2 represent the critical values of a and R obtained from two different methods (STGM and MSE) incorporated in our study. It is clear that the values obtained from MSE give accurate results compared to STGM. The values of a_c obtained from both the methods are very close for all cases with a small difference in the third or fourth decimal place. However, for an increasing value of $\mathcal{K}$, the relative difference between the two methods increases, and hence we can say that STGM may slightly overpredict. However, the relative difference is large for $Bi = 1$, and it gradually decreases later, as Bi increases.

Appendix D: Principle of exchange of stabilities

To verify principle of exchange of stabilities, the linearised form of Eqs. (18) – (20) is considered:

$$\frac{\partial \mathcal{U}}{\partial \mathcal{X}} + \frac{\partial \mathcal{W}}{\partial \mathcal{Z}} = 0, \quad (70)$$

$$\frac{1}{Va} \frac{\partial}{\partial \tau} (\nabla^2 \mathcal{W}) - \bar{R} \frac{\partial^2 \Theta}{\partial \mathcal{X}^2} + \nabla^2 \mathcal{W} = 0, \quad (71)$$

$$\frac{\partial \Theta}{\partial \tau} - B_{eff} \mathcal{W} = \nabla^2 \Theta. \quad (72)$$

Using the normal mode analysis, the solution of Eqs. (70) – (72) can be taken as

$$\mathcal{U}(\mathcal{X}, \mathcal{Z}) = -e^{(\sigma + i\omega)\tau} \sin(\pi a x) W'(\mathcal{Z}), \quad (73)$$

$$\mathcal{W}(\mathcal{X}, \mathcal{Z}) = e^{(\sigma + i\omega)\tau} \sin(\pi a x) W(\mathcal{Z}), \quad (74)$$

$$\Theta(\mathcal{X}, \mathcal{Z}) = B_{eff} e^{(\sigma + i\omega)\tau} \sin(\pi a x) \theta(\mathcal{Z}), \quad (75)$$

where σ is the growth rate and ω is the frequency of oscillations.

The assumption of marginal stability is adopted and hence $\sigma = 0$. Substituting Eq.(73) – (75) into Eqs.(70) – (72) yields:

$$\left(\frac{d^2}{d\mathcal{Z}^2} - \pi^2 a^2 \right) W + R a^2 \theta = -\frac{i\omega}{Va} \left(\frac{d^2}{d\mathcal{Z}^2} - \pi^2 a^2 \right) W, \quad (76)$$

$$\left(\frac{d^2}{d\mathcal{Z}^2} - \pi^2 a^2 \right) \theta + W = i\omega \theta. \quad (77)$$

The boundary condition to solve Eqs. (76) – (77) is

$$W(0) = 0, \theta(0) + \mathcal{K} \theta(1) = 0, \quad (78)$$

$$W(1) = 0, D\theta(1) + Bi \theta(1) = 0. \quad (79)$$

Eqs. (76) – (77) are solved subject to boundary condition (78) – (79) using the STGM. To obtain R using STGM, it is assumed that $W(\mathcal{Z}) = \mathcal{A} W_1(\mathcal{Z})$ and $\theta(\mathcal{Z}) = B \theta_1(\mathcal{Z})$, where $W_1(\mathcal{Z})$ and $\theta_1(\mathcal{Z})$ are trial functions, taken as:

$$W_1(\mathcal{Z}) = \mathcal{Z}^2 - \mathcal{Z}, \quad (80)$$

$$\theta_1(\mathcal{Z}) = \mathcal{Z}^2 - \left(\frac{2 + 2\mathcal{K} + Bi}{1 + \mathcal{K} + Bi} \right) \mathcal{Z} + \frac{\mathcal{K}}{1 + \mathcal{K} + Bi}, \quad (81)$$

where $\mathcal{A}$, $\mathcal{B}$ are constants. Note that Eqs. (80) – (81) satisfy the boundary condition (78) – (79).

Substituting Eqs. (80) – (81) into Eqs. (76) – (77), and multiplying Eq. (76) by W_1 and Eq. (77) by θ_1 and integration with $\mathcal{Z}$ between $[0,1]$ yields:

$$\left(\frac{i\omega}{Va} + 1 \right) (I_1 - \pi^2 a^2 I_2) \mathcal{A} + R \pi^2 a^2 I_3 \mathcal{B} = 0, \quad (82)$$

$$I_3 \mathcal{A} + (I_6 - \pi^2 a^2 - i\omega I_4) \mathcal{B} = 0, \quad (83)$$

where

$$\begin{aligned} I_1 &= \int_{\mathcal{Z}=0}^{\mathcal{Z}=1} W_1''(\mathcal{Z}) W_1(\mathcal{Z}) d\mathcal{Z}, & I_2 &= \int_{\mathcal{Z}=0}^{\mathcal{Z}=1} W_1^2(\mathcal{Z}) d\mathcal{Z}, \\ I_3 &= \int_{\mathcal{Z}=0}^{\mathcal{Z}=1} W_1(\mathcal{Z}) \theta_1(\mathcal{Z}) d\mathcal{Z}, & I_4 &= \int_{\mathcal{Z}=0}^{\mathcal{Z}=1} \theta_1^2(\mathcal{Z}) d\mathcal{Z}, \\ I_6 &= \int_{\mathcal{Z}=0}^{\mathcal{Z}=1} \theta_1(\mathcal{Z}) \theta_1''(\mathcal{Z}) d\mathcal{Z}. \end{aligned}$$

To solve Eqs. (82) – (83) for a nontrivial solution, the solution is sought in the form

$$\begin{vmatrix} \left(\frac{i\omega}{Va} + 1 \right) (I_1 - \pi^2 a^2 I_2) & \pi^2 a^2 I_3 \\ I_3 & (I_6 - \pi^2 a^2 - i\omega I_4) \end{vmatrix} = 0. \quad (84)$$

Solving for R gives

$$R = \frac{(\pi^2 a^2 I_2 - I_1)(\pi^2 a^2 I_4 - I_6)}{\pi^2 a^2 I_3^2} - \frac{\omega^2 (\pi^2 a^2 I_2 - I_1) I_4}{Va \pi^2 a^2 I_3^2} + i\omega \frac{(\pi^2 a^2 I_2 - I_1)}{\pi^2 a^2 I_3^2} \left\{ I_4 + \frac{\pi^2 a^2 I_4 - I_6}{Va} \right\}. \quad (85)$$

Eq. (85) is the expression for oscillatory Rayleigh number.

Since R is real then either $\omega = 0$ or $(\pi^2 a^2 I_2 - I_1) \left\{ I_4 + \frac{\pi^2 a^2 I_4 - I_6}{Va} \right\} = 0$. But it is observed from the computation that $(\pi^2 a^2 I_2 - I_1) \left\{ I_4 + \frac{\pi^2 a^2 I_4 - I_6}{Va} \right\} \neq 0$ and hence $\omega = 0$. Hence the principle of exchange of stabilities is valid in this case and we take $\omega = 0$ to obtain the stationary Rayleigh number as

$$R = \frac{(\pi^2 a^2 I_2 - I_1)(\pi^2 a^2 I_4 - I_6)}{\pi^2 a^2 I_3^2}. \quad (86)$$

Eq. (86) is the expression for stationary Rayleigh number.