Supplementary Note

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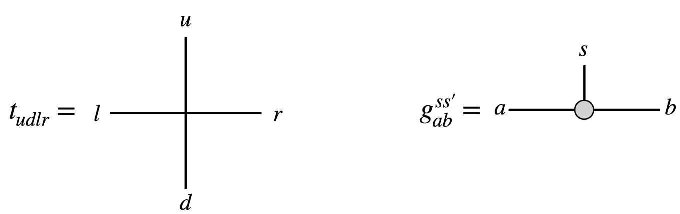
I. TORIC CODE LOOP GAS OPERATOR

A. Gauge Symmetry and Physical-Virtual Relation

The projected entangled pair state (PEPS) representation of the $\mathbb{Z}_2$ toric code (TC) can be efficiently represented as a tensor product with bond dimension $D = 2$:

$$|\psi_{\mathrm{TC}}\rangle = \mathrm{tTr}\prod_{\alpha} T^{\tilde{s}_\alpha}_{u_\alpha d_\alpha l_\alpha r_\alpha} |\tilde{s}_\alpha\rangle. \tag{1}$$

Here, $\tilde{s} = \{s^x, s^y\}$ represents the local physical degrees of freedom. The local tensor $T^{\tilde{s}}_{udlr}$ can be decomposed into two types of smaller tensors, as illustrated below:



(2)

These tensors are defined as:

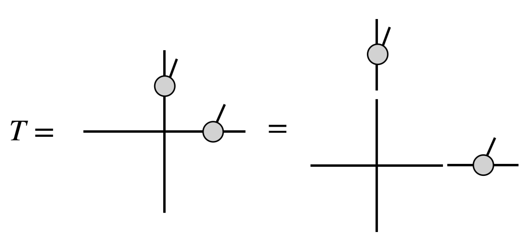
$$\begin{aligned} t_{udlr} &= 1 \quad \text{if and only if } (u - d + r - l) \bmod 2 = 0, \\ &= 0 \quad \text{otherwise,} \end{aligned} \tag{3}$$

and

$$\begin{aligned} g^{s}_{ab} &= 1 \quad \text{if and only if } a = b = s, \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{4}$$

Notably, the index $s = 0$ corresponds to the local physical state $|0\rangle = (1,0)$, while $s = 1$ corresponds to $|1\rangle = (0,1)$.

The local tensor $T^{\tilde{s}}_{udlr}$ can then be expressed as $T^{\tilde{s}}_{udlr} = \sum_{u'r'} t_{u'dlr'} g^{s_x}_{r'r} g^{s_y}_{u'u}$. This relationship can also be represented graphically:



(5)

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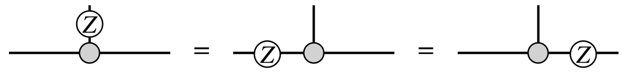
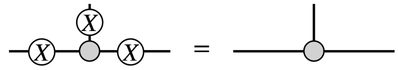
The smaller tensors g^s_{ab} and t_{udlr} exhibit the following local gauge symmetries:

$Z^{ss'}g^{s'}_{ab}$ $\quad$ $Z^{aa'}g^{s}_{a'b}$ $\quad$ $g^{s}_{ab'}Z^{b'b}$



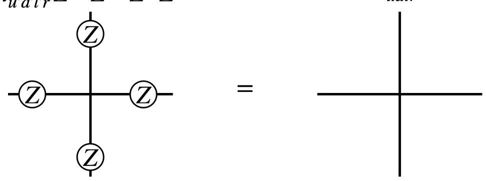
$X^{aa'}g^{s'}_{a'b'}X^{s's}X^{b'b}$ $\quad$ g^{s}_{ab}



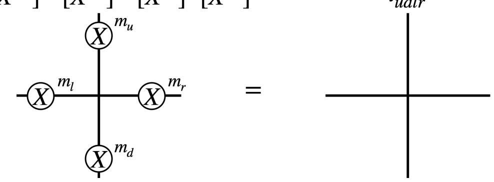
(6)

and

$t_{u'd'l'r'}Z^{u'u}Z^{d'd}Z^{l'l}Z^{r'r}$ $\quad$ t_{udlr}



$t_{u'd'l'r'}[X^{m_u}]^{u'u}[X^{m_d}]^{d'd}[X^{m_l}]^{l'l}[X^{m_r}]^{r'r}$ $\quad$ t_{udlr}

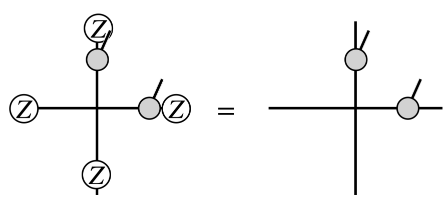


only when $(m_u - m_d + m_r - m_l) \bmod 2 = 0$

(7)

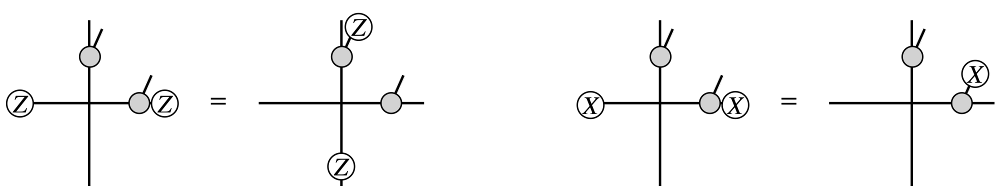
Here, X and Z are the Pauli matrices.

The local gauge symmetries described in Eqs. (6) and (7) are inherited by the local tensor $T^{\tilde{s}}_{udlr}$, resulting in corresponding local gauge symmetries. First, the Z-injective symmetry of the T tensor can be expressed as:



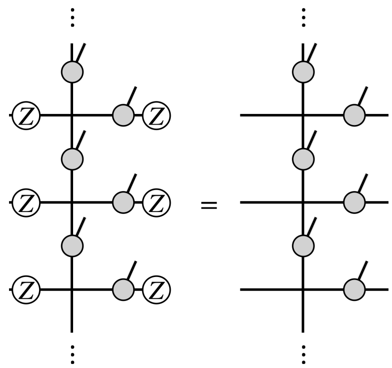
(8)

Additionally, two other useful local gauge symmetries emerge:



(9)

By contracting the vertical indices of the T tensors after arranging them in a column, the $\mathbb{R}$ tensor defined in the main text can be obtained. Using the injective symmetry in Eq. (8), it can be shown that the $\mathbb{C}$ tensor possesses a one-dimensional *global* symmetry at the virtual index level: $(\prod_i Z_i)\,\mathbb{C}\,(\prod_i Z_i) = \mathbb{C}$. This symmetry can be represented graphically as:



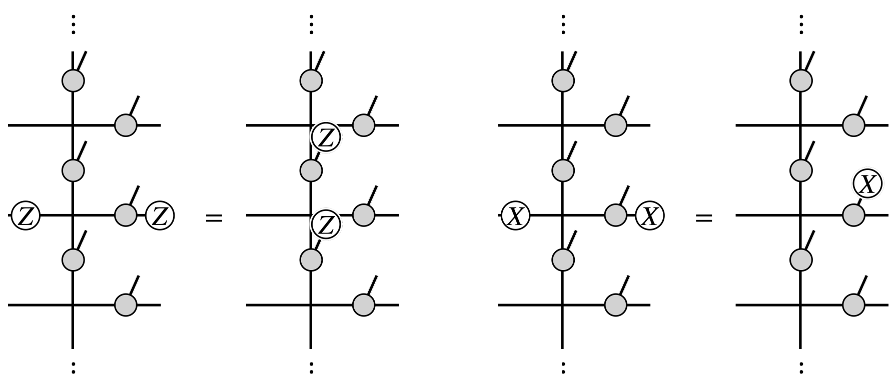
(10)

This global symmetry is directly reflected in the two symmetries $g_{Z\mathbb{I}}$ and g_{ZZ} of the one-dimensional column-to-column transfer matrix $\mathbb{H} = \mathrm{tTr}_{\{s_i\}}\mathbb{C}^{\{s_i\}}\mathbb{C}^{*\{s_i\}}$, such that $g_{Z\mathbb{I}}\mathbb{H}g_{Z\mathbb{I}} = \mathbb{H}$ and $g_{ZZ}\mathbb{H}g_{ZZ} = \mathbb{H}$.

Furthermore, Eq. (9) shows that the local application of X or Z operators on virtual indices is transferred to the physical indices as follows:

$$X_{l_j}\,\mathbb{C}\,X_{r_j} = X_{s_j^x}\mathbb{C}, \qquad Z_{l_j}\,\mathbb{C}\,Z_{r_j} = Z_{s_j^1} Z_{s_{j+1}^1}\mathbb{C}, \tag{11}$$

or equivalently, in graphical form:



(12)

Interestingly, during the contraction of physical indices to construct the column-to-column transfer matrix $\mathbb{H}$, the X and Z operators cancel out. This cancellation gives rise to local symmetries in the transfer matrix, defined as $u_i = Z_i \otimes \bar{Z}_i$ and $v_i = X_i \otimes \bar{X}_i$. Here, operators without a bar act on the ket layer of the transfer matrix, while operators with a bar act on the bra layer.

B. Deformed Toric Code Wavefunction

By applying the filtering operator $D_i(\beta_x, \beta_z) = I_i + \beta_x X_i + \beta_z Z_i$ to each physical degree of freedom, the filtered TC state can be obtained as: $|\mathrm{fTC}(\beta_\mathrm{x}, \beta_\mathrm{z})\rangle = \prod_\mathrm{i} \mathrm{D_i}(\beta_\mathrm{x}, \beta_\mathrm{z})|\mathrm{TC}\rangle$.

The filtering operator locally modifies the $\mathbb{Z}_2$ physical basis as follows:

$$\begin{aligned} |0\rangle \to |\tilde{0}\rangle &\propto \begin{pmatrix} 1+\beta_z \\ \beta_x \end{pmatrix} \\ |1\rangle \to |\tilde{1}\rangle &\propto \begin{pmatrix} \beta_x \\ 1-\beta_z \end{pmatrix} \end{aligned} \tag{13}$$

After filtering, the local states are no longer orthonormal. To simplify the analysis, we set $\langle\tilde{0}|\tilde{0}\rangle = 1$, such that the overlaps between the local states are given by:

$$\langle\tilde{0}|\tilde{1}\rangle = \frac{\beta_x^2 + (\beta_z - 1)^2}{\beta_x^2 + (\beta_z + 1)^2}, \quad \langle\tilde{1}|\tilde{1}\rangle = \frac{2\beta_x}{\beta_x^2 + (\beta_z + 1)^2}. \tag{14}$$

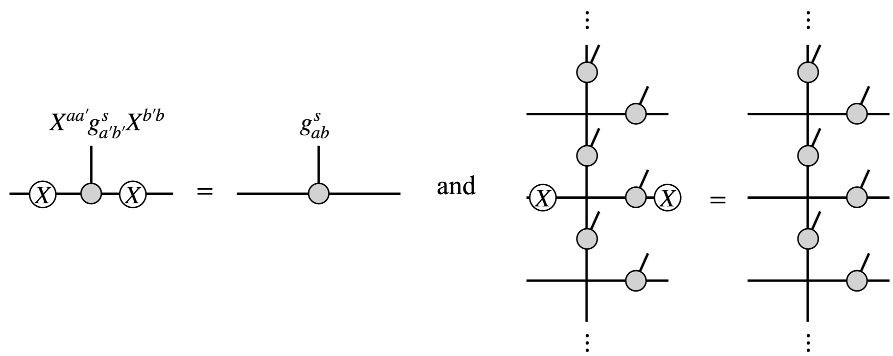
It is important to note that the filtering operator modifies only the smaller tensor g, leaving t unchanged. Consequently, the Z-injectivity property in Eq. (8) remains valid across the entire range of filtering parameters. However, the X local gauge symmetry in Eq. (9) holds only when $\beta_z = 0$, and the Z local gauge symmetry in Eq. (9) holds only when $\beta_x = 0$.

As a result, the column-to-column transfer matrix $\mathbb{H}$ exhibits the local symmetry $u_i = X_i \otimes \bar{X}_i$ only for $\beta_z = 0$, and the local symmetry $v_i = Z_i \otimes \bar{Z}_i$ only for $\beta_x = 0$.

C. Emergent local and global symmetries of transfer matrix

At specific points or along certain lines, the transfer matrix $\mathbb{H}$ exhibits additional local or global symmetries, which are valuable for analyzing the phase diagram of the filtered toric code.

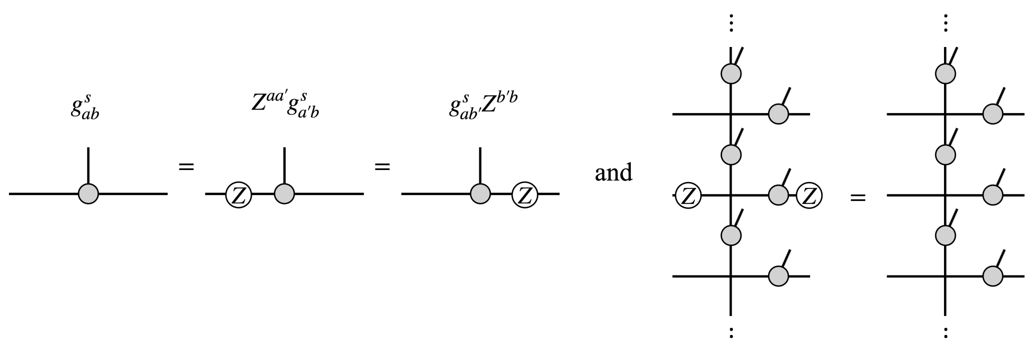
Two special points give rise to additional local symmetries. The first occurs at the point $[\beta_z, \beta_x] = [0, 1]$, where $|\tilde{0}\rangle = |\tilde{1}\rangle$, making the two local physical states indistinguishable. This leads to the following:



(15)

As a result, the transfer matrix attains a local symmetry $m_i^1 = X_i \otimes \bar{\mathbb{I}}_i$ at $[\beta_z, \beta_x] = [0, 1]$.

The second special point occurs at $[\beta_z, \beta_x] = [1, 0]$, where $|\tilde{1}\rangle = 0$. In this case, we get:



(16)

Here, the transfer matrix $\mathbb{H}$ achieves a local symmetry $u_i^1 = Z_i \otimes \bar{\mathbb{I}}$ at $[\beta_z, \beta_x] = [1, 0]$.

In contrast, when $\langle\tilde{0}|\tilde{1}\rangle = \langle\tilde{1}|\tilde{1}\rangle$, the transfer matrix $\mathbb{H}$ gains an additional global symmetry, alongside the existing symmetries g_{ZI} and g_{ZZ}. This condition is satisfied along the line:

$$(\beta_x - 1)^2 + (\beta_z - 1)^2 = 1. \tag{17}$$

It is important to note that the line described by Eq. (17) includes the points $[\beta_z, \beta_x] = [1, 0]$ and $[\beta_z, \beta_x] = [0, 1]$.

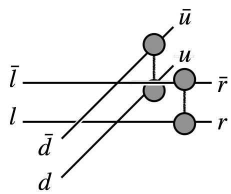
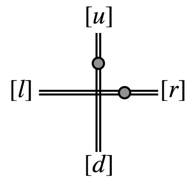
To verify that the transfer matrix $\mathbb{H}$ acquires an additional global symmetry along the line, it is helpful to express the transfer matrix as a tensor product of local tensors:

$$\mathbb{H}^{\{\bar{l}_i\},\{\bar{r}_i\}}_{\{l_i\},\{r_i\}} = \mathrm{tTr}_{u_i,d_i,\bar{u}_i,\bar{d}_i} \mathbb{E}^{\bar{u}_i\bar{l}_i\bar{d}_i\bar{r}_i}_{u_il_id_ir_i}.$$

The local tensor $\mathbb{E}^{\bar{u}_i\bar{l}_i\bar{d}_i\bar{r}_i}_{u_il_id_ir_i}$ is given by:

$$\mathbb{E}^{\bar{u}_i\bar{l}_i\bar{d}_i\bar{r}_i}_{u_il_id_ir_i} = \sum_{s_i} T^{s_i}_{u_il_id_ir_i} (T^*)^{s_i}_{\bar{u}_i\bar{l}_i\bar{d}_i\bar{r}_i},$$

or graphically represented as:

$$\mathbb{E}^{\bar{u}\bar{l}\bar{d}\bar{r}}_{uldr} =$$  $$=$$  (18)

In this expression, the indices with and without bars are grouped, i.e., $[a] = [a, \bar{a}]$. Each of these grouped indices can take four possible values: $[0,\bar{0}]$, $[0,\bar{1}]$, $[1,\bar{0}]$, and $[1,\bar{1}]$.

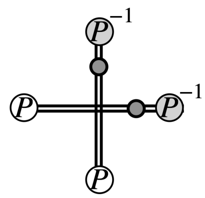
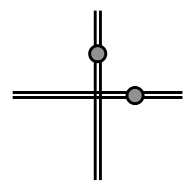
Within this four-dimensional index space, one can define an operator P with the following matrix representation:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{19}$$

This operator permutes the three basis states of the four-dimensional local index space in the sequence: $[0,\bar{1}] \to [1,\bar{0}] \to [1,\bar{1}] \to [0,\bar{1}]$, while leaving $[0,\bar{0}]$ intact.

Upon closer inspection, it becomes clear that the local tensor $\mathbb{E}$ exhibits injective symmetry under P, i.e.,

$$\mathbb{E}_{[u'][l'][d'][r']}(P^{-1})^{[u'][u]}P^{[d'][d]}P^{[l'][l]}(P^{-1})^{[r'][r]} \qquad\qquad \mathbb{E}_{[u'][l'][d'][r']}$$

 $$=$$  (20)

From this injectivity, we deduce that the transfer matrix $\mathbb{H}$ is symmetric under the global symmetry operator $g_P = \prod_i P_i$, meaning that $g_P^{-1}\mathbb{H}g_P = \mathbb{H}$.

By expressing the local order parameter operators as

$$X_i \otimes \bar{I}_i = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad I_i \otimes \bar{X}_i = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{21}$$

it is straightforward to demonstrate the following relations:

$$\begin{aligned} P\,(X_i \otimes \bar{I}_i)\,P^{-1} &= I_i \otimes \bar{X}_i, \\ P\,(I_i \otimes \bar{X}_i)\,P^{-1} &= X_i \otimes \bar{X}_i, \\ P\,(X_i \otimes \bar{X}_i)\,P^{-1} &= I_i \otimes \bar{X}_i. \end{aligned} \tag{22}$$

In other words, along the g_P-symmetric line, the local order parameters $I_i \otimes \bar{X}_i$ and $X_i \otimes \bar{X}_i$ become equivalent. This implies that the corresponding global symmetries g_{ZZ} and $g_{Z\mathbb{I}}$ are either both simultaneously broken or both preserved.

II. KITAEV HONEYCOMB LOOP GAS OPERATOR

A. Gauge Symmetry and Physical-Virtual Relation

The loop gas (LG) operator for the Kitaev honeycomb model is efficiently represented by a tensor product form [?] with a bond dimension $D = 2$: $\hat{Q}_{\mathrm{LG}} = \mathrm{tTr}\prod_\alpha Q^{ss'}_{i_\alpha j_\alpha k_\alpha}|s\rangle\langle s'|$ with a building block tensor

$$Q^{ss'}_{ijk} = \tau_{ijk}[X^{1-i}Y^{1-j}Z^{1-k}]_{ss'}, \tag{23}$$

where X, Y, Z are the Pauli matrices, the virtual indices $i, j, k = 0, 1$, and non-zero elements of τ-tensor are

$$\tau_{000} = -i, \quad \tau_{011} = \tau_{101} = \tau_{110} = 1. \tag{24}$$

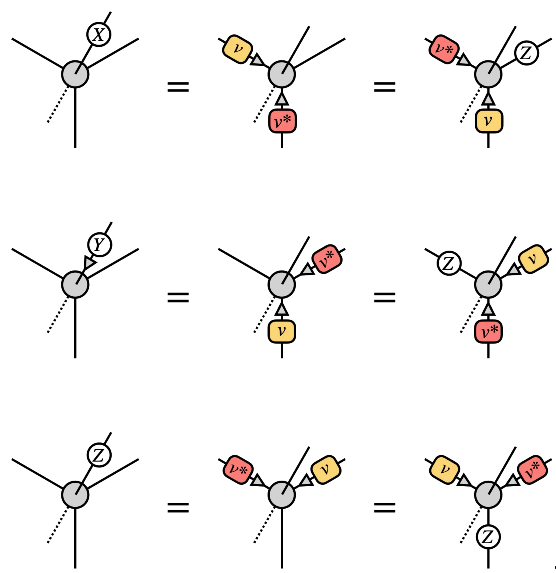
It is important to note that the τ-tensor contains non-zero elements only when the count of index-1 is even. Consequently, regardless of the variational parameter δ, the Q-tensor remains invariant under the action of $Z^{\otimes 3}$ on its virtual indices, i.e., $Z_{ii'}Z_{jj'}Z_{kk'}Q^{ss'}_{i'j'k'} = Q^{ss'}_{ijk}$ or graphically

(25)

Furthermore, the Q-tensor satisfies remarkable properties, referred to as physical-virtual relations, wherein the action of Pauli matrices on the physical index can be equivalently represented by a unitary transformation on the virtual indices.:

$$\begin{aligned} X_{s\tilde{s}}Q^{\tilde{s}s'}_{ijk} &= v_{jj'}v^*_{kk'}Q^{ss'}_{ij'k'} = v^*_{jj'}v_{kk'}Z_{ii'}Q^{ss'}_{i'j'k'}, \\ Y_{s\tilde{s}}Q^{\tilde{s}s'}_{ijk} &= v_{kk'}v^*_{ii'}Q^{ss'}_{i'jk'} = v^*_{kk'}v_{ii'}Z_{jj'}Q^{ss'}_{i'j'k'}, \\ Z_{s\tilde{s}}Q^{\tilde{s}s'}_{ijk} &= v_{ii'}v^*_{jj'}Q^{ss'}_{i'j'k} = v^*_{ii'}v_{jj'}Z_{kk'}Q^{ss'}_{i'j'k'}, \end{aligned} \tag{26}$$

or graphically



, (27)

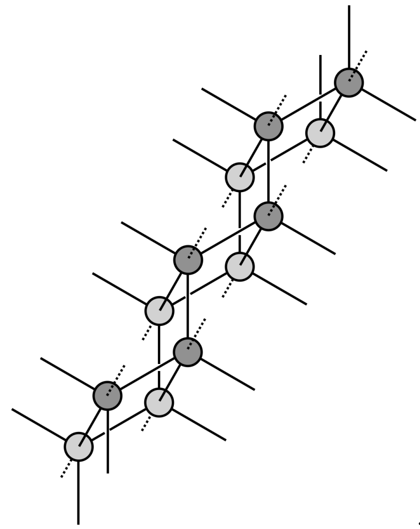
where the unitary matrix

$$v = \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix} = \frac{1}{2}(X - Y) + \frac{i}{2}(X - Y), \tag{28}$$

Here, the direction indicated in the graphical representation corresponds to the direction of matrix multiplication. Specifically, if the direction points outward (inward) from the matrix, then its column (row) index is contracted. For non-symmetric matrices ($M^t \neq M$), the direction of the contraction must be carefully managed to ensure consistency. We omit the direction for symmetric matrices as it becomes redundant. Remarkably, using the first equalities in Eq. (27), it can be shown that the LG operator functions as a projector, mapping any state onto the flux-free sector, expressed as $\hat{W}_p\hat{Q}_{\mathrm{LG}} = \hat{Q}_{\mathrm{LG}}\hat{W}_p = \hat{Q}_{\mathrm{LG}}$ [**?**]. On the other hand, the set of second equalities, which have not been previously recognized, play a pivotal role in the central results of this work alongside the set of first equalities.

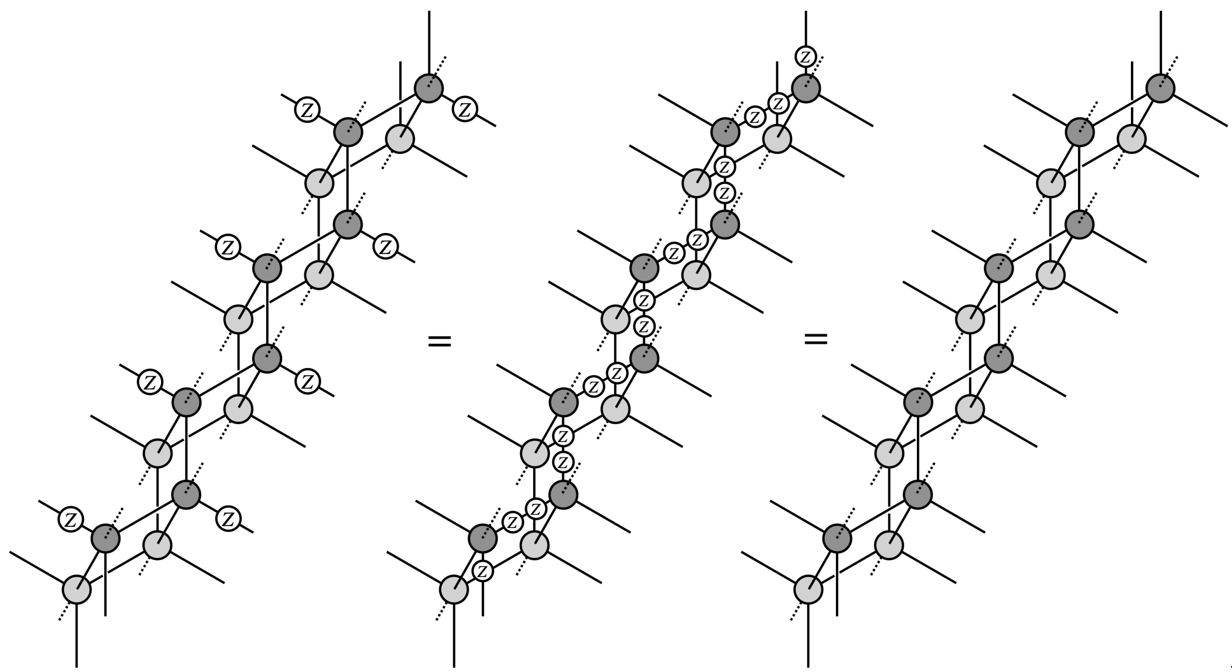
B. Global and Local symmetries of the Transfer Matrix

To discuss the global and local symmetries of the transfer matrix, we begin by considering the double-layer of the LG operator, $\hat{Q}_{\mathrm{LG}}\hat{Q}_{\mathrm{LG}}^{\dagger}$, and its column-to-column operator, as illustrated below:



$$. \tag{29}$$

Using the gauge symmetry of the Q-tensor described in Eq. (25), it is straightforward to show that a global unitary transformation $g_{Z\mathbb{I}} \equiv (Z \otimes \mathbb{I})^{\otimes L}$, acting on the virtual indices in the double-layer operator, leaves it invariant, as illustrated below:

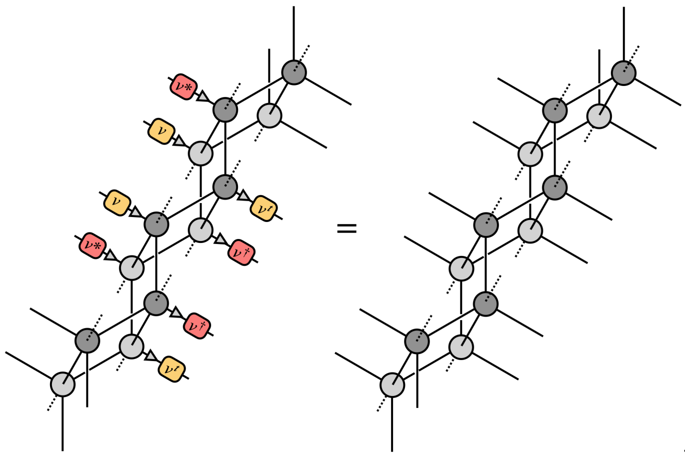


$$. \tag{30}$$

Similarly, another global operation, $g_{ZZ} \equiv (Z \otimes Z)^{\otimes L}$, also leaves the double-layer operator invariant. Consequently, the transfer matrix $\mathbb{H}$ of a quantum state $|\psi\rangle = \hat{Q}_{\mathrm{LG}}|\varphi\rangle$ possesses the global $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry generated by $g_{Z\mathbb{I}}$ and g_{ZZ}, provided that the state $|\varphi\rangle$ is trivial. More specifically, $|\varphi\rangle$ can be represented by a PEPS in which the tensor does not exhibit gauge symmetry, regardless of whether it is a product state or an entangled state.

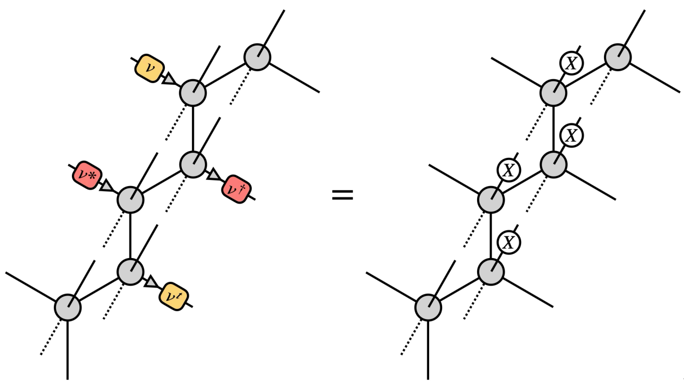
Now, we show that the column-to-column operator is invariant under a local unitary transformation $\boldsymbol{v}_i \equiv (v_{i-1}^{*} \otimes v_{i-1}^{t})(v_i \otimes v_i^{\dagger})$

as illustrated below;



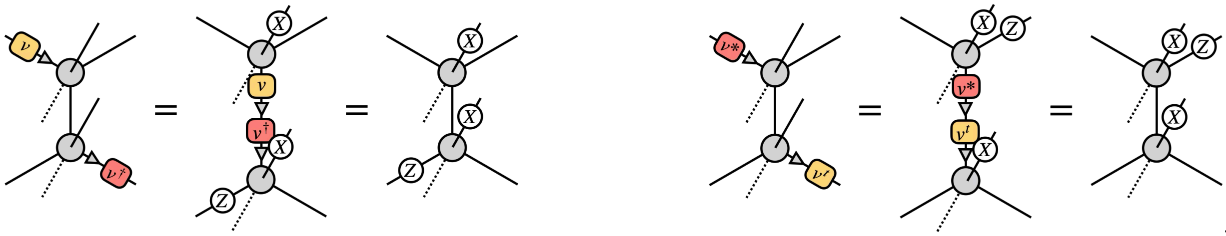
$$.\tag{31}$$

The local unitary transformation $v_{i-1}^{*}\, v_i$ acting on the virtual indices of the LG operator is equivalent to the action of a Pauli matrix on the corresponding physical indices, as illustrated below:



$$,\tag{32}$$

where we use Eq. (27) to derive the following two identities:



$$.\tag{33}$$

Therefore, the double-layer operator is symmetric under the action of the $\boldsymbol{v}_i$ operator, as the Pauli X matrices acting on the physical indices cancel out during the contraction of physical indices.

C. Variational Ansatz

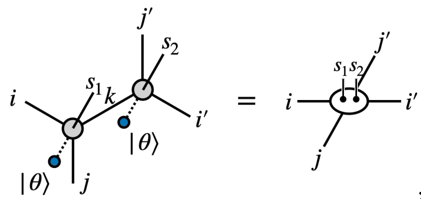
The variational ansatz we consider is as follows:

$$|\mathrm{LG}(\theta)\rangle = \hat{Q}_{\mathrm{LG}} |\theta\rangle^{\otimes N}, \tag{34}$$

where the product state $|\theta\rangle^{\otimes N} \equiv \prod_{n=1}^{N} |\theta\rangle_n$ with $|\theta\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\pi/4}\sin\frac{\theta}{2}|1\rangle$. By contracting tensors in a unit-cell,

$$T^{s_1 s_2}_{i j i' j'} = \sum_{s_1', s_2'} \sum_{k} Q^{s_1 s_1'}_{ijk} Q^{s_2 s_2'}_{i' j' k} \langle s_1' | \theta \rangle \langle s_2' | \theta \rangle, \tag{35}$$

or graphically



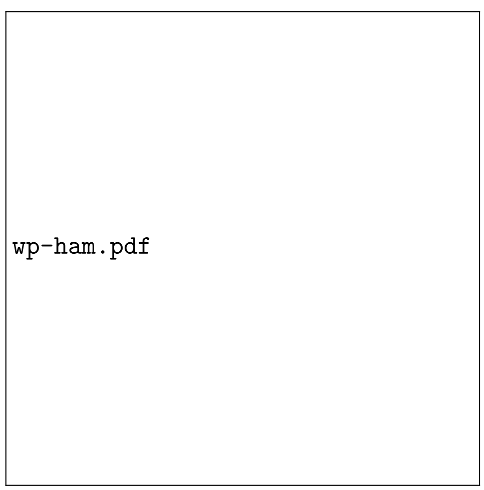
, (36)

the network transforms into a square lattice.

III. LG ANSATZ AT $\theta = 0$: GROUND STATE OF WEN PLAQUETTE MODEL

A. PEPS and Wilson loop operator

At $\theta = 0$, the ansatz becomes the exact ground state of the Kitaev honeycomb model in the strongly anisotropic limit ($J_z \gg J_x, J_y$), where the model maps onto the Wen plaquette (WP) model:



(37)

At $\theta = 0$, the product state $|\theta\rangle \otimes |\theta\rangle$ in Eq. (36) reduces to

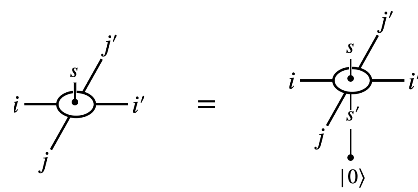
$$|0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Consequently, the effective action of physical operators on this state is projected and simplified as follows:

$$
\begin{aligned}
Z\otimes Z &= \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 1 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \to \begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}, &\quad Z\otimes I &= \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 1 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix} \to \begin{pmatrix} -1 & 0\\ 0 & 1\end{pmatrix} \equiv -X,\\
I\otimes Z &= \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix} \to \begin{pmatrix} -1 & 0\\ 0 & 1\end{pmatrix} \equiv -X, &\quad X\otimes X &= \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix} \to \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix} \equiv Z,\\
Y\otimes Y &= \begin{pmatrix} 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\end{pmatrix} \to \begin{pmatrix} 0 & -1\\ -1 & 0\end{pmatrix} \equiv -Z, &\quad X\otimes Y &= \begin{pmatrix} 0 & 0 & 0 & -i\\ 0 & 0 & -i & 0\\ 0 & i & 0 & 0\\ i & 0 & 0 & 0\end{pmatrix} \to \begin{pmatrix} 0 & -i\\ i & 0\end{pmatrix} \equiv Y,\\
Y\otimes X &= \begin{pmatrix} 0 & 0 & 0 & -i\\ 0 & 0 & i & 0\\ 0 & -i & 0 & 0\\ i & 0 & 0 & 0\end{pmatrix} \to \begin{pmatrix} 0 & -i\\ i & 0\end{pmatrix} \equiv Y. & &
\end{aligned} \tag{38}
$$

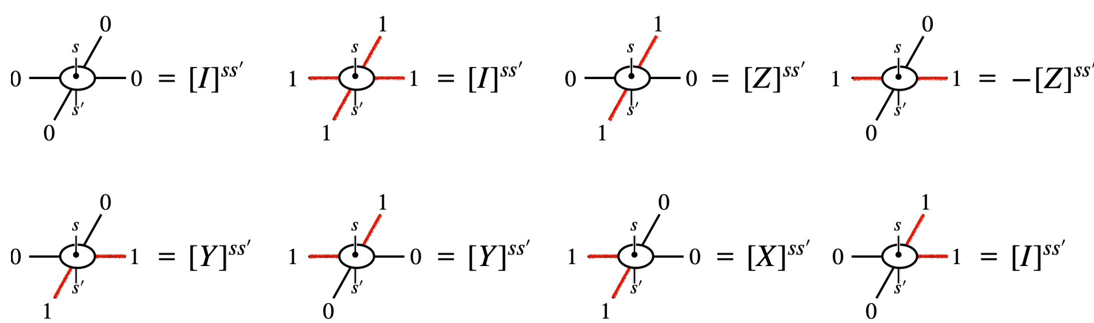
Accordingly, the local state is projected as $|0\rangle \otimes |0\rangle \to |0\rangle$.

In this projected Hilbert space, the local tensor in Eq. (36) can be understood as applying the following tensor to the local state $|0\rangle$:



(39)

Here, the applied tensor is defined as



(40)

On the other hand, one of the ground states of Eq. 37 is given by

$$
|\psi_{\mathrm{WP}}\rangle = \prod_i (1+F_i)\,|0\rangle, \tag{41}
$$

which can be expressed in the loop-gas picture as

$$
|\psi_{\mathrm{WP}}\rangle = \sum_c |c\rangle, \tag{42}
$$

where the sum runs over all possible *contractible* closed loop configurations $|c\rangle$. When a loop passes through a site, the corresponding physical operator is applied to the local state $|0\rangle$, as illustrated in Fig. 1.

The PEPS state $|\psi_{\mathrm{PEPS}}\rangle$, constructed from Eq. (40), can also be interpreted in terms of loop configurations, where the virtual index value 0 corresponds to an empty site and 1 to an occupied loop segment. In contrast to $|\psi_{\mathrm{WP}}\rangle$, the PEPS state additionally includes *non-contractible* closed loops, represented by the Wilson loop operators W_Z^x and W_Z^y. That is,

$$
|\psi_{\mathrm{PEPS}}\rangle = \left(1+W_Z^x+W_Z^y+W_Z^xW_Z^y\right)|\psi_{\mathrm{WP}}\rangle. \tag{43}
$$

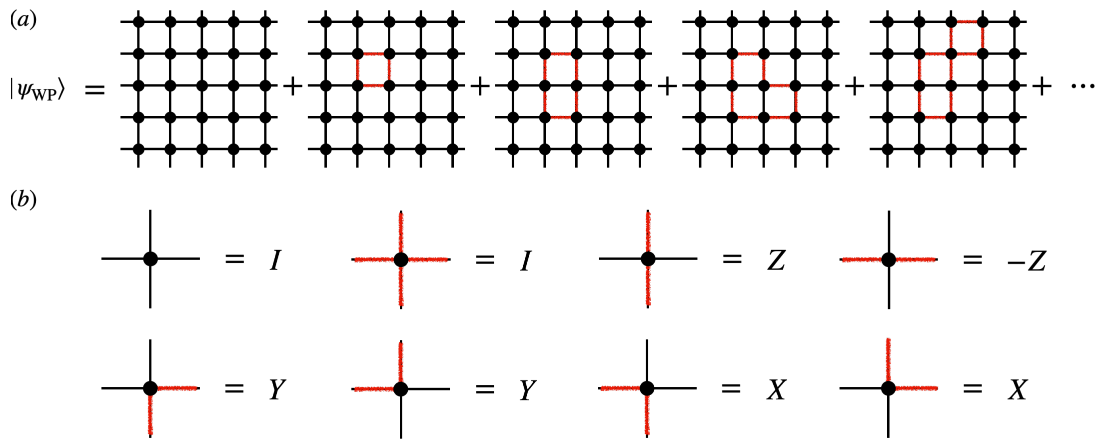


FIG. 1. (a) Graphical illustration of the loop-gas picture for $|\psi_{\mathrm{WP}}\rangle$ in Eq. (42). The state is an equal-weight superposition of all possible *contractible* closed loop configurations. (b) Examples of loop segments and the corresponding physical operators applied to the local state $|0\rangle$.

However, each Wilson loop operator W_Z^a $(a = x, y)$ commutes with the projector in Eq. (41) and leaves $|\psi_{\mathrm{WP}}\rangle$ invariant:

$$W_Z^a\,|\psi_{\mathrm{WP}}\rangle = W_Z^a \prod_p (1+F_p)\,|0\rangle^{\otimes N} \tag{44}$$

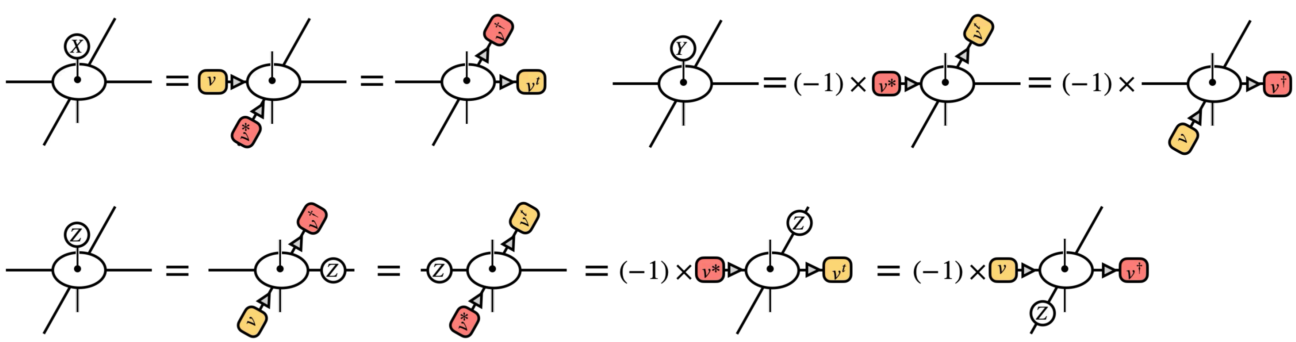
$$= \prod_p (1+F_p)\,W_Z^a|0\rangle^{\otimes N} \tag{45}$$

$$= \prod_p (1+F_p)\,|0\rangle^{\otimes N} = |\psi_{\mathrm{WP}}\rangle, \tag{46}$$

where the last equality follows from the fact that $|0\rangle^{\otimes N}$ is an eigenstate of W_Z^a with eigenvalue 1. As a result, we conclude that $|\psi_{\mathrm{PEPS}}\rangle \sim |\psi_{\mathrm{WP}}\rangle$, and the state is an eigenstate of the Wilson loop operators W_Z^a.

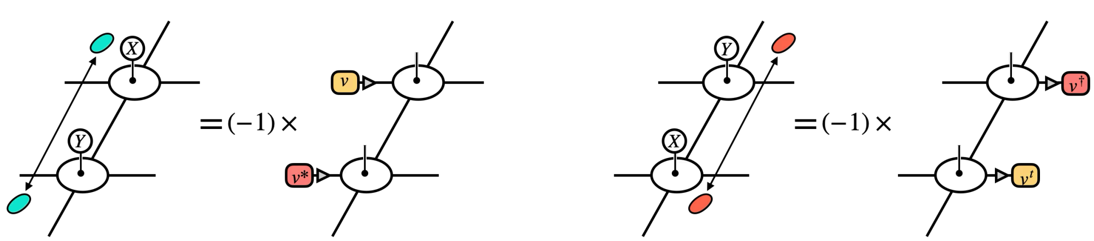
B. Symmetries and Anyons

The action of Pauli operators on the physical legs of the tensor in Eq. (40) can be equivalently represented by the application of unitaries on the virtual legs:



(47)

Using Eq. (47), one can show that anyon pair creation operations can also be represented as the application of unitaries on the virtual bonds:



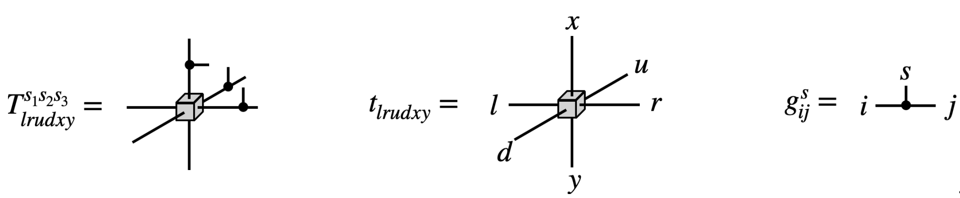
(48)

Based on this correspondence, applying $\boldsymbol{v}_i \equiv (v^*_{i-1} \otimes v^t_{i-1})(v_i \otimes v^\dagger_i)$ to the transfer matrix, as defined in Eq. (32), is equivalent to creating a pair of anyons on both the ket and bra layers of the transfer matrix.

IV. 3d TORIC CODE AND X-CUBE STATES

A. PEPS

The 3d toric code (3TC) and X-cube (XC) states can be efficiently represented by three-dimensional PEPS on the cubic lattice with the following building block tensors:

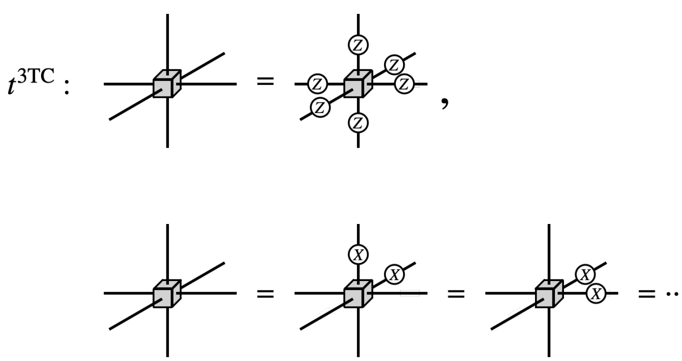


, (49)

where g-tensor is the same as the one defined in the 2d toric code state in Eq. (4) for both 3TC and XC states. On the other hand, the definition of t-tensor for each state is the following[**?**]:

$$t^{\mathrm{3TC}}_{lrudxy} = \begin{cases} 1, & \text{if} \quad l+r+u+d+x+y = 0 \bmod 2 \\ 0, & \text{otherwise} \end{cases} \quad , \quad t^{\mathrm{XC}}_{lrudxy} = \begin{cases} 1, & \text{if} \quad \begin{cases} l+r+u+d = 0 \bmod 2 \\ l+r+x+y = 0 \bmod 2 \\ u+d+x+y = 0 \bmod 2 \end{cases} \\ 0, & \text{otherwise} \end{cases} \quad . \tag{50}$$

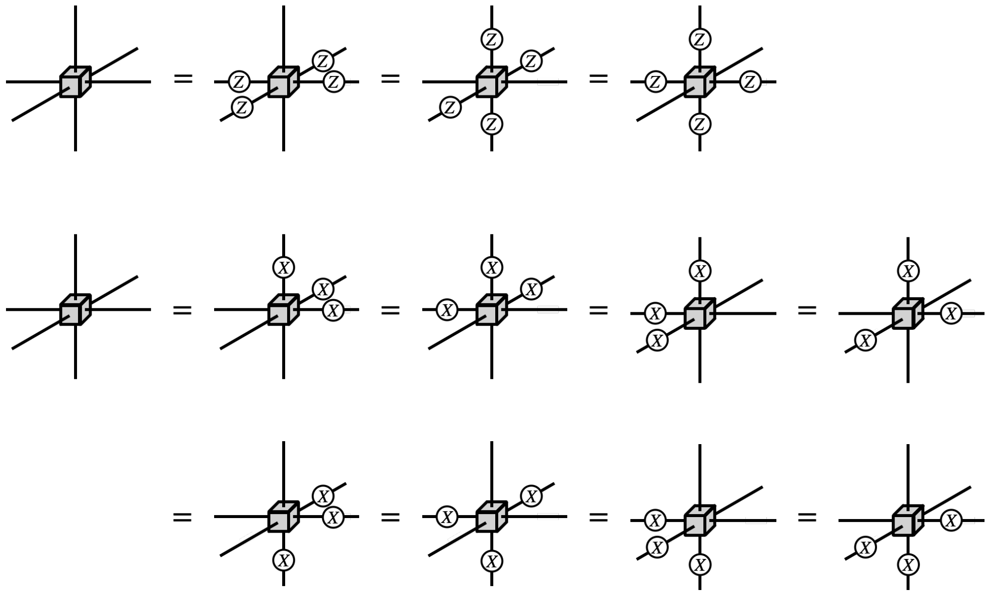
Using the definition above, one can easily show that the t-tensor for the 3TC state is invariant under the action of the Pauli Z on all virtual indices and the action of X on a pair of virtual indices, as graphically illustrated below:



(51)

On the other hand, the one for the XC state is invariant under the following transformations:

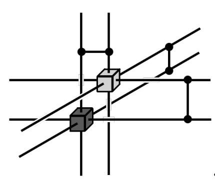
t^{XC} :



(52)

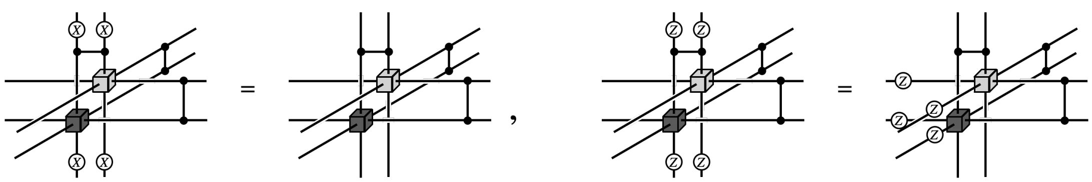
B. Global, local symmetries and idempotence of transfer matrices

Now, we construct the transfer matrices (TM) of the norm of the 3TC and XC states, which can be represented by a two-dimensional projected entangled-pair operator (PEPO) consisting of a building block tensor illustrated as:



, (53)

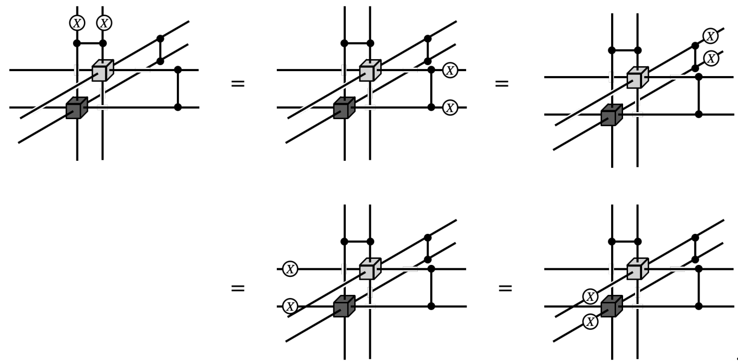
where the vertical indices remain open, and all other indices are contracted to form the PEPO. It is straightforward to show that the TMs, $\mathbb{H}^{3\mathrm{TC}}$ and $\mathbb{H}^{\mathrm{XC}}$, are invariant under the global unitary transformations $U_{ZZ} = \prod_i Z_i \otimes \overline{Z}_i$ and $U_{ZI} = \prod_i Z_i \otimes \overline{I}_i$ using Eqs. (6), (51) and (52). It is a simple generalization of Eq. (10). Furthermore, one can show that both TMs, $\mathbb{H}^{3\mathrm{TC}}$ and $\mathbb{H}^{\mathrm{XC}}$, also commute with local unitary transformations $u_i^z = Z_i \otimes \overline{Z}_i$ and $u_i^x = X_i \otimes \overline{X}_i$. Similarly, using Eqs. (6), (51), and (52), one can derive the following identities:



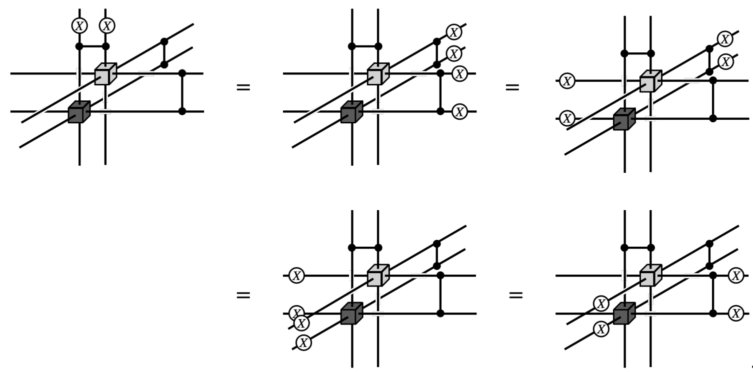
(54)

In the second identity, the Pauli Z appears to remain but cancels out when the tensors are contracted with neighboring ones due to Eq. (6). In particular, the second identity, or the local symmetry u_i^z, remains valid even after the Z-filtering operation is applied.

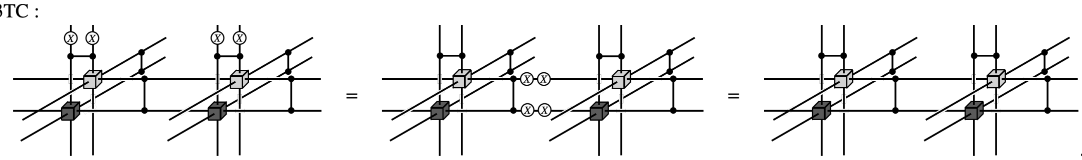
Using Eqs. (6), (51) and (52), one can further derive the following equalities for the 3TC state

3TC :  , (55)

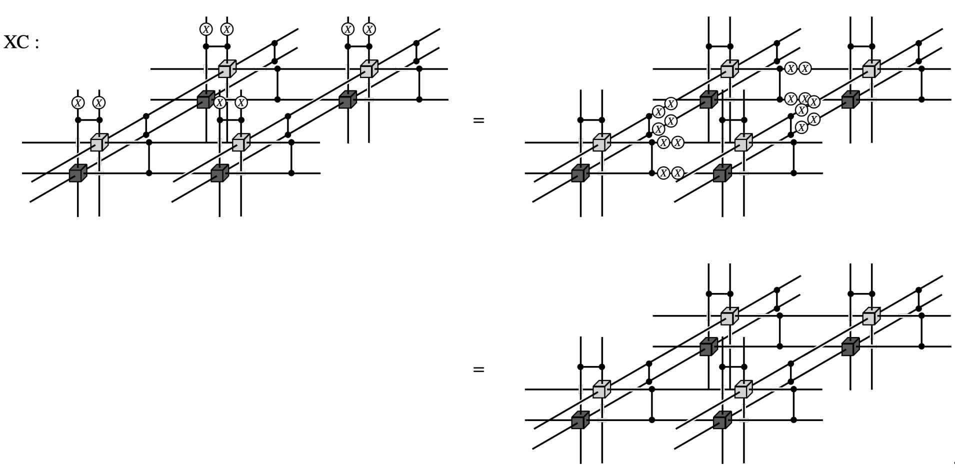
and for the XC state

XC :  . (56)

Using above equalities, one can verify that $(X_i \otimes \overline{X}_i)(X_j \otimes \overline{X}_j)\mathbb{H}^{3\mathrm{TC}} = \mathbb{H}^{3\mathrm{TC}}$ for all neighboring i, j as graphically illustrated below:

3TC :  , (57)

and $\prod_{i\in\Box}(X_i \otimes \overline{X}_i)\mathbb{H}^{\mathrm{XC}} = \mathbb{H}^{\mathrm{XC}}$ for all elementary plaquettes $\Box$ on the square lattice:

XC :  . (58)

These indicate that both TMs are idempotent such that

$$\mathbb{H}^{3\mathrm{TC}} \cong \prod_{\langle ij\rangle}\left[1 + (X_i \otimes \overline{X}_i)(X_j \otimes \overline{X}_j)\right], \quad \mathbb{H}^{\mathrm{XC}} \cong \prod_{\Box}\left[1 + \prod_{i\in\Box}(X_i \otimes \overline{X}_i)\right]. \tag{59}$$