Supplementary material for: A class of entanglement witnesses and a realignment-like criterion

Vahid Jannesary[1], Vahid Karimipour[1], and Dariusz Chruściński[2*]

[1]Department of Physics, Sharif University of Technology, Tehran 14588, Iran
[2]Institute of Physics, Faculty of Physics, Astronomy and Informatics, Nicolaus Copernicus University, Grudziadzka 5/7, 87-100 Torun, Poland
[*]darch@fizyka.umk.pl

ABSTRACT

1 Proof of positivity of eq. (24)

One finds for normalized $\phi \in H_{d_1}$ and $\psi \in H_{d_2}$:

$$\langle\psi|\Phi(|\phi\rangle\langle\phi|)|\psi\rangle = 1 - \sum_{\beta=0}^{d_2^2-1} \langle\phi|\tilde{\Gamma}_\beta|\phi\rangle\langle\psi|\Omega_\beta|\psi\rangle,$$

where

$$\tilde{\Gamma}_\beta = \sum_{\alpha=0}^{d_1^2-1} O_{\beta\alpha}\Gamma_\alpha.$$

Now, using Cauchy-Schwarz inequality one obtains

$$\sum_{\beta=0}^{d_2^2-1} \langle\phi\tilde{\Gamma}_\beta|\phi\rangle\langle\psi\Omega_\beta|\psi\rangle \leq \sqrt{\sum_{\beta=0}^{d_2^2-1} |\langle\phi|\tilde{\Gamma}_\beta|\phi\rangle|^2}\sqrt{\sum_{\beta=0}^{d_2^2-1} |\langle\psi|\Omega_\beta|\psi\rangle|^2}.$$

Now, recalling

$$\sum_{\alpha=0}^{d_1^2-1} \Gamma_\alpha^{\mathrm{T}} \otimes \Gamma_\alpha = d_1 P_{d_1}^+, \quad \sum_{\beta=0}^{d_2^2-1} \Omega_\beta^{\mathrm{T}} \otimes \Omega_\beta = d_2 P_{d_2}^+,$$

where $P_{d_1}^+$ and $P_{d_2}^+$ stand for maximally entangled state in $H_{d_1} \otimes H_{d_1}$ and $H_{d_1} \otimes H_{d_2}$, respectively, one finds

$$\sum_{\beta=0}^{d_2^2-1} |\langle\phi|\tilde{\Gamma}_\beta|\phi\rangle|^2 \leq 1, \quad \sum_{\beta=0}^{d_2^2-1} |\langle\psi|\Omega_\beta|\psi\rangle|^2 = 1,$$

and hence

$$\langle\psi|\Phi(|\phi\rangle\langle\phi|)|\psi\rangle \geq 0,$$

which proves positivity of the map Φ.

2 An alternative construction of Entanglement Witnesses

Here, without resorting to positive maps, we construct an Entanglement Witness W in a direct manner. We use an alternative definition of an Entanglement Witness as an operator which is block-positive but not entirely positive. A block-positive operator W is one which satisfies $\langle\alpha\otimes\langle\beta|W|\alpha\rangle\otimes\beta\rangle\geq 0$ for all product states $|\alpha\rangle\otimes|\beta\rangle$, but is not entirely positive and has negative eigenvalues on the full Hilbert space.

Sufficient condition for block-positivity: To this end, we first apply Mehta's lemma to derive sufficient conditions for block-positivity of W. We will later show that for detecting the entanglement witness of a given state, unital entanglement witnesses (i.e. those which correspond to unital postive map) act better than any other witnesses. So from (**??**) we take a unital W to be

$$W=R_{00}\Gamma_0\otimes\Omega_0+\mathbf{s}\cdot\boldsymbol{\Gamma}\otimes\Omega_0+\Gamma_0\otimes\mathbf{t}\cdot\boldsymbol{\Omega}+\Lambda_{\beta,\alpha}\Gamma_\alpha^T\otimes\Omega_\beta.$$

Block-positivity means that for any vector $|v\rangle\in H_{d_2}$, the following matrix should be positive

$$B\equiv\langle v|W|v\rangle=(\frac{R_{00}}{\sqrt{d_2}}+\langle v|\mathbf{t}\cdot\boldsymbol{\Omega}|v\rangle)\Gamma_0+\frac{1}{\sqrt{d_2}}\mathbf{s}\cdot\boldsymbol{\Gamma}+\Lambda_{\beta,\alpha}\Gamma_\alpha^T\langle v|\Omega_\beta|v\rangle.$$

Using the notation $q_\beta:=\langle v|\Omega_\beta|v\rangle$, and $\mathbf{q}=(q_1,q_2,\cdots q_{d_2})$ we have

$$\mathrm{Tr}(B)=\sqrt{d_1}(\frac{R_{00}}{\sqrt{d_2}}+\mathbf{t}\cdot\mathbf{q}),\tag{1}$$

and

$$\mathrm{Tr}(B^2)=(\frac{R_{00}}{\sqrt{d_2}}+\mathbf{t}\cdot\mathbf{q})^2+\|\frac{\mathbf{s}}{\sqrt{d_2}}+\Lambda\mathbf{q}\|^2$$

where we have used the relations $\mathrm{Tr}(\Gamma_\alpha)=0,\quad \mathrm{Tr}(\Gamma_\alpha\Gamma_\beta)=\delta_{\alpha\beta}$. We now resort to the Mehta's Lemma and demand that the following condition $\mathrm{Tr}(B^2)\leq\frac{\mathrm{Tr}(B)^2}{d_1-1}$ holds for all the vectors $|v\rangle$. With a little rearrangement, this leads to

$$\|\frac{\mathbf{s}}{\sqrt{d_2}}+\Lambda\mathbf{q}\|^2\leq\frac{(\frac{R_{00}}{\sqrt{d_2}}+\mathbf{t}\cdot\mathbf{q})^2}{d_1-1}\forall\,\mathbf{q}.$$

or

$$\|\frac{\mathbf{s}}{\sqrt{d_2}}+\Lambda\mathbf{q}\|\leq\frac{(\frac{R_{00}}{\sqrt{d_2}}+\mathbf{t}\cdot\mathbf{q})}{\sqrt{d_1-1}}\qquad\forall\,\mathbf{q}.\tag{2}$$

To strengthen this sufficient condition, we replace the left hand side of this inequality with a larger term and the right hand side with a smaller one to arrive at

$$\frac{\|\mathbf{s}\|}{\sqrt{d_2}}+\|\Lambda\|_\infty\|\mathbf{q}\|\leq\frac{\frac{R_{00}}{\sqrt{d_2}}-\|\mathbf{t}\|\|\mathbf{q}\|}{\sqrt{d_1-1}}.\tag{3}$$

Inserting the value $\|\mathbf{q}\|=\sqrt{1-\frac{1}{d_2}}$ and simplifying, we arrive at the condition (**??**), namely

$$\sqrt{d_2-1}\|\mathbf{t}\|+\sqrt{d_1-1}\|\mathbf{s}\|+\sqrt{(d_1-1)(d_2-1)}\|\Lambda\|_\infty\leq R_{00},\tag{4}$$

Just for completeness, we remind the reader of how $\|\mathbf{q}\|$ is calculated. Using the summation convention for repeated indices, we first note that

$$\mathbf{q}.\mathbf{q}=\langle v|\Omega_\mu|v\rangle\langle v|\Omega_\mu|v\rangle=\langle v,v|\Omega_\mu\otimes\Omega_\mu|v,v\rangle.$$

Using the identity

$$\frac{1}{d}I+\Omega_\mu\otimes\Omega_\mu=P,\tag{5}$$

where P is the permutation operator $P|\mu,\nu\rangle=|\nu,\mu\rangle$, we find that

$$\mathbf{q}.\mathbf{q}=\langle v,v|P-\frac{1}{d_2}I|v,v\rangle=1-\frac{1}{d_2}.$$