Supplementary Material: A Generalized Framework of Antisymmetric Polyspectral Indices for Identifying High-Order Neural Interactions

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1 General case: full derivation

This supplementary section provides the full derivation of the proposed antisymmetric construction in the general case of m observed time series and $m-1$ arbitrary positive frequencies satisfying the polyspectral frequency-balance condition. We then show that the bivariate repeated-frequency case used in the main text follows as a special instance.

Assumptions. Throughout the paper, and thus in this Supplementary Material as well, we assume that: (i) the observed signals are finite instantaneous linear mixtures of latent sources; (ii) the latent sources are zero-mean, real-valued, mutually independent, and strictly stationary; (iii) all expectations are taken in the population, or ensemble, limit; and (iv) the frequency tuples considered below are non-degenerate. By non-degenerate we mean that all frequencies are strictly positive.

No Gaussianity assumption is required.

Notation. Let $\mathbf{x}(t) = \big(x_1(t), \dots, x_m(t)\big)^\top$ be an ordered collection of m observed time series, not necessarily distinct, and let $X_k(f)$ denote the Fourier transform of $x_k(t)$. We write $\mathbf{f} = (f_1, \dots, f_{m-1})$ and define

$$F = \sum_{\ell=1}^{m-1} f_\ell. \tag{1}$$

Thus the product $X_1(f_1)\cdots X_{m-1}(f_{m-1})X_m^*(F)$ has total signed frequency $f_1 + \cdots + f_{m-1} - F = 0$, which is the standard frequency-balance condition for a non-vanishing m-th order polyspectral moment.

We model each observed signal as an instantaneous linear mixture of M latent sources:

$$x_\ell(t) = \sum_{i=1}^{M} a_{\ell i}\, s_i(t), \qquad \ell = 1, \dots, m, \tag{2}$$

where the coefficients $a_{\ell i} \in \mathbb{R}$ are fixed mixing weights. Denote by $S_i(f)$ the Fourier transform of the source $s_i(t)$. By linearity of the Fourier transform, Eq. (2) implies

$$X_\ell(f) = \sum_{i=1}^{M} a_{\ell i}\, S_i(f), \qquad \ell = 1, \dots, m. \tag{3}$$

General m-th order cross-polyspectrum

We define the ordered m-th order cross-polyspectrum associated with the ordered signal tuple $(x_1, \dots, x_m)$ and the frequency tuple $(f_1, \dots, F)$, with F given by Eq. (1), as

$$P^{(m)}_{x_1\cdots x_m}(\mathbf{f}) := \left\langle \prod_{\ell=1}^{m-1} X_\ell(f_\ell)\, X_m^*(F) \right\rangle = \left\langle X_1(f_1)X_2(f_2)\cdots X_{m-1}(f_{m-1})X_m^*(F) \right\rangle. \tag{4}$$

For $m = 3$, this is a bispectral moment. For $m = 4$, it is a trispectral moment. More generally, m denotes the statistical order of the moment.

Substituting Eq. (3) into Eq. (4) gives

$$P^{(m)}_{x_1\cdots x_m}(\mathbf{f}) = \left\langle \left(\sum_{i_1=1}^{M} a_{1i_1} S_{i_1}(f_1)\right)\left(\sum_{i_2=1}^{M} a_{2i_2} S_{i_2}(f_2)\right)\cdots\left(\sum_{i_{m-1}=1}^{M} a_{m-1,i_{m-1}} S_{i_{m-1}}(f_{m-1})\right)\left(\sum_{i_m=1}^{M} a_{mi_m} S_{i_m}(F)\right)^* \right\rangle. \tag{5}$$

Since the mixing coefficients are real and deterministic, the conjugated factor satisfies

$$\left(\sum_{i_m=1}^{M} a_{mi_m} S_{i_m}(F)\right)^* = \sum_{i_m=1}^{M} a_{mi_m} S^*_{i_m}(F).$$

Therefore,

$$P^{(m)}_{x_1\cdots x_m}(\mathbf{f}) = \left\langle \left(\sum_{i_1=1}^{M} a_{1i_1} S_{i_1}(f_1)\right) \cdots \left(\sum_{i_{m-1}=1}^{M} a_{m-1,i_{m-1}} S_{i_{m-1}}(f_{m-1})\right) \left(\sum_{i_m=1}^{M} a_{mi_m} S^*_{i_m}(F)\right) \right\rangle. \tag{6}$$

Expanding the product of sums gives

$$P^{(m)}_{x_1\cdots x_m}(\mathbf{f}) = \sum_{i_1=1}^{M}\sum_{i_2=1}^{M}\cdots\sum_{i_m=1}^{M} a_{1i_1}a_{2i_2}\cdots a_{m-1,i_{m-1}}a_{mi_m}\left\langle S_{i_1}(f_1)\,S_{i_2}(f_2)\cdots S_{i_{m-1}}(f_{m-1})\,S^*_{i_m}(F)\right\rangle. \tag{7}$$

Equivalently,

$$P^{(m)}_{x_1\cdots x_m}(\mathbf{f}) = \sum_{i_1,\ldots,i_m=1}^{M}\left(\prod_{\ell=1}^{m} a_{\ell i_\ell}\right)\left\langle S_{i_1}(f_1)\,S_{i_2}(f_2)\cdots S_{i_{m-1}}(f_{m-1})\,S^*_{i_m}(F)\right\rangle. \tag{8}$$

We now show explicitly which terms survive under the null model of mutually independent latent sources.

Fix an index tuple $(i_1,\ldots,i_m)$. Let $\mathscr{M} = \{i_1,\ldots,i_m\}$ be the set of distinct source labels appearing in this tuple. For each source label $r \in \mathscr{M}$, define the set of factor positions associated with source r: $I_r = \{\ell \in \{1,\ldots,m\} : i_\ell = r\}$. The corresponding spectral factor in Eq. (8) can then be grouped source by source. Since the sources are mutually independent, the expectation factorizes over the distinct source labels:

$$\left\langle S_{i_1}(f_1)\cdots S_{i_{m-1}}(f_{m-1})S^*_{i_m}(F)\right\rangle = \prod_{r\in\mathscr{M}}\left\langle \prod_{\substack{\ell\in I_r\\ \ell<m}} S_r(f_\ell)\prod_{\substack{\ell\in I_r\\ \ell=m}} S^*_r(F)\right\rangle, \tag{9}$$

where the products over empty index sets are understood to be equal to one. Because there is only one conjugated factor in the original moment, namely the factor at frequency F, at most one of the source-specific blocks in Eq. (9) can contain a conjugated Fourier coefficient. All other blocks contain only non-conjugated coefficients at strictly positive frequencies.

For a zero-mean strictly stationary process, a spectral moment can be nonzero only if the signed sum of the frequencies appearing in that moment is zero. Hence, for each source-specific block in Eq. (9), the following source-wise balance condition must hold:

$$\sum_{\substack{\ell\in I_r\\ \ell<m}} f_\ell - \mathbf{1}_{\{m\in I_r\}}F = 0,$$

where $\mathbf{1}_{\{m\in I_r\}}$ is equal to one if the conjugated factor belongs to the block associated with source r, and zero otherwise.

There are two cases.

First, suppose that $m \notin I_r$. Then the block associated with source r contains only non-conjugated Fourier coefficients at strictly positive frequencies. Since $r \in \mathscr{M}$, the set I_r is non-empty, and therefore

$$\sum_{\ell\in I_r} f_\ell > 0.$$

Thus the source-wise balance condition cannot be satisfied, and the corresponding source-specific moment vanishes.

Second, suppose that $m \in I_r$. Then the block associated with source r contains the conjugated coefficient $S^*_r(F)$, and its signed frequency sum is $\sum_{\substack{\ell\in I_r\\ \ell<m}} f_\ell - F$. Using Eq. (1), this signed sum can be rewritten as

$$\sum_{\substack{\ell\in I_r\\ \ell<m}} f_\ell - \sum_{\ell=1}^{m-1} f_\ell = -\sum_{\substack{\ell<m\\ \ell\notin I_r}} f_\ell.$$

Since all frequencies $f_1, \dots, f_{m-1}$ are strictly positive, this expression is zero if and only if

$$\{1, \dots, m-1\} \subseteq I_r.$$

In words, the only source-specific block that contains the conjugated coefficient can be frequency-balanced only if it also contains all $m-1$ positive-frequency coefficients.

Therefore, a nonzero contribution is possible only if all factors belong to the same source label:

$$i_1 = i_2 = \cdots = i_m =: i.$$

All non-diagonal source-index tuples vanish in the population limit.

Consequently, Eq. (8) reduces to the diagonal contribution

$$P^{(m)}_{x_1 \cdots x_m}(\mathbf{f}) = \sum_{i=1}^{M} \left(\prod_{\ell=1}^{m} a_{\ell i} \right) \left\langle S_i(f_1)\, S_i(f_2) \cdots S_i(f_{m-1})\, S_i^*(F) \right\rangle. \tag{10}$$

Antisymmetric contrast. To remove the contribution induced purely by instantaneous linear mixing of independent sources, we compare the ordered polyspectrum in Eq. (4) with the polyspectrum obtained by exchanging the first and last observed signals while keeping the same ordered frequency tuple. We define

$$P^{(m)}_{[x_1|\cdots|x_m]}(\mathbf{f}) := P^{(m)}_{x_1 \cdots x_m}(\mathbf{f}) - P^{(m)}_{x_m x_2 \cdots x_{m-1} x_1}(\mathbf{f}), \tag{11}$$

where

$$P^{(m)}_{x_m x_2 \cdots x_{m-1} x_1}(\mathbf{f}) := \left\langle X_m(f_1)\, X_2(f_2) \cdots X_{m-1}(f_{m-1})\, X_1^*(F) \right\rangle. \tag{12}$$

Notice that the exchange is between the observed signals occupying the first and the conjugated output positions. The frequencies themselves are not permuted.

We now expand Eq. (12). Using the mixing model,

$$X_m(f_1) = \sum_{i_1=1}^{M} a_{m i_1} S_{i_1}(f_1),$$

$$X_\ell(f_\ell) = \sum_{i_\ell=1}^{M} a_{\ell i_\ell} S_{i_\ell}(f_\ell), \qquad \ell = 2, \dots, m-1,$$

and

$$X_1^*(F) = \sum_{i_m=1}^{M} a_{1 i_m} S_{i_m}^*(F).$$

Therefore,

$$P^{(m)}_{x_m x_2 \cdots x_{m-1} x_1}(\mathbf{f}) = \sum_{i_1, \dots, i_m=1}^{M} a_{m i_1} \left(\prod_{\ell=2}^{m-1} a_{\ell i_\ell} \right) a_{1 i_m} \left\langle S_{i_1}(f_1)\, S_{i_2}(f_2) \cdots S_{i_{m-1}}(f_{m-1})\, S_{i_m}^*(F) \right\rangle. \tag{13}$$

By the same independence and frequency-balance argument used above, the only surviving terms are again those satisfying

$$i_1 = i_2 = \cdots = i_m =: i.$$

Thus Eq. (13) reduces to

$$P^{(m)}_{x_m x_2 \cdots x_{m-1} x_1}(\mathbf{f}) = \sum_{i=1}^{M} a_{mi} \left(\prod_{\ell=2}^{m-1} a_{\ell i} \right) a_{1i} \left\langle S_i(f_1)\, S_i(f_2) \cdots S_i(f_{m-1})\, S_i^*(F) \right\rangle. \tag{14}$$

Since

$$a_{mi} \left(\prod_{\ell=2}^{m-1} a_{\ell i} \right) a_{1i} = \prod_{\ell=1}^{m} a_{\ell i},$$

we have

$$P^{(m)}_{x_m x_2 \cdots x_{m-1} x_1}(\mathbf{f}) = \sum_{i=1}^{M} \left(\prod_{\ell=1}^{m} a_{\ell i} \right) \left\langle S_i(f_1)\, S_i(f_2) \cdots S_i(f_{m-1})\, S_i^*(F) \right\rangle. \tag{15}$$

Comparing Eq. (15) with Eq. (10), we obtain

$$P^{(m)}_{x_m x_2 \cdots x_{m-1} x_1}(\mathbf{f}) = P^{(m)}_{x_1 \cdots x_m}(\mathbf{f})$$

under the null model. Substituting this identity into Eq. (11) gives

$$P^{(m)}_{[x_1|\cdots|x_m]}(\mathbf{f}) = 0. \tag{16}$$

This is the key robustness property of the proposed construction.

Normalization. To obtain a bounded, dimensionless quantity, we define for each observed signal the m-th order amplitude norm

$$Q^{(m)}_{x_\ell}(f) := \left\langle |X_\ell(f)|^m \right\rangle^{1/m}, \qquad \ell = 1, \ldots, m. \tag{17}$$

We now derive the normalization bound explicitly. Consider

$$P^{(m)}_{x_1 \cdots x_m}(\mathbf{f}) = \left\langle X_1(f_1) \cdots X_{m-1}(f_{m-1}) X_m^*(F) \right\rangle.$$

Taking the modulus gives

$$\begin{aligned} \left| P^{(m)}_{x_1 \cdots x_m}(\mathbf{f}) \right| &= \left| \left\langle X_1(f_1) \cdots X_{m-1}(f_{m-1}) X_m^*(F) \right\rangle \right| \\ &\le \left\langle |X_1(f_1) \cdots X_{m-1}(f_{m-1}) X_m^*(F)| \right\rangle \\ &= \left\langle |X_1(f_1)| \cdots |X_{m-1}(f_{m-1})|\, |X_m(F)| \right\rangle. \end{aligned} \tag{18}$$

Applying Hölder's inequality with m factors and exponents all equal to m, so that

$$\frac{1}{m} + \cdots + \frac{1}{m} = 1,$$

we obtain

$$\begin{aligned} \left| P^{(m)}_{x_1 \cdots x_m}(\mathbf{f}) \right| &\le \left\langle |X_1(f_1)|^m \right\rangle^{1/m} \cdots \left\langle |X_{m-1}(f_{m-1})|^m \right\rangle^{1/m} \left\langle |X_m(F)|^m \right\rangle^{1/m} \\ &= Q^{(m)}_{x_1}(f_1)\, Q^{(m)}_{x_2}(f_2) \cdots Q^{(m)}_{x_{m-1}}(f_{m-1})\, Q^{(m)}_{x_m}(F). \end{aligned} \tag{19}$$

Similarly, for the swapped polyspectrum, the same argument yields

$$\left| P^{(m)}_{x_m x_2 \cdots x_{m-1} x_1}(\mathbf{f}) \right| \le Q^{(m)}_{x_m}(f_1)\, Q^{(m)}_{x_2}(f_2) \cdots Q^{(m)}_{x_{m-1}}(f_{m-1})\, Q^{(m)}_{x_1}(F). \tag{20}$$

We therefore define the normalized antisymmetric cross-polycoherency as

$$\Gamma^{(m)}_{[x_1|\cdots|x_m]}(\mathbf{f}) := \frac{P^{(m)}_{[x_1|\cdots|x_m]}(\mathbf{f})}{D^{(m)}_{[x_1|\cdots|x_m]}(\mathbf{f})}, \tag{21}$$

where

$$\begin{aligned} D^{(m)}_{[x_1|\cdots|x_m]}(\mathbf{f}) := &\; Q^{(m)}_{x_1}(f_1)\, Q^{(m)}_{x_2}(f_2) \cdots Q^{(m)}_{x_{m-1}}(f_{m-1})\, Q^{(m)}_{x_m}(F) \\ &+ Q^{(m)}_{x_m}(f_1)\, Q^{(m)}_{x_2}(f_2) \cdots Q^{(m)}_{x_{m-1}}(f_{m-1})\, Q^{(m)}_{x_1}(F), \end{aligned} \tag{22}$$

which is assumed to be non-zero. By construction, this denominator is the sum of the two Hölder bounds for the two ordered polyspectra entering the antisymmetric contrast.

Indeed, using the triangle inequality,

$$\begin{aligned} \left| P^{(m)}_{[x_1|\cdots|x_m]}(\mathbf{f}) \right| &= \left| P^{(m)}_{x_1 \cdots x_m}(\mathbf{f}) - P^{(m)}_{x_m x_2 \cdots x_{m-1} x_1}(\mathbf{f}) \right| \\ &\le \left| P^{(m)}_{x_1 \cdots x_m}(\mathbf{f}) \right| + \left| P^{(m)}_{x_m x_2 \cdots x_{m-1} x_1}(\mathbf{f}) \right|. \end{aligned} \tag{23}$$

Therefore, we get $\left| P^{(m)}_{[x_1|\cdots|x_m]}(\mathbf{f}) \right| \le D^{(m)}_{[x_1|\cdots|x_m]}(\mathbf{f})$, and thus, $\left| \Gamma^{(m)}_{[x_1|\cdots|x_m]}(\mathbf{f}) \right| \le 1$.

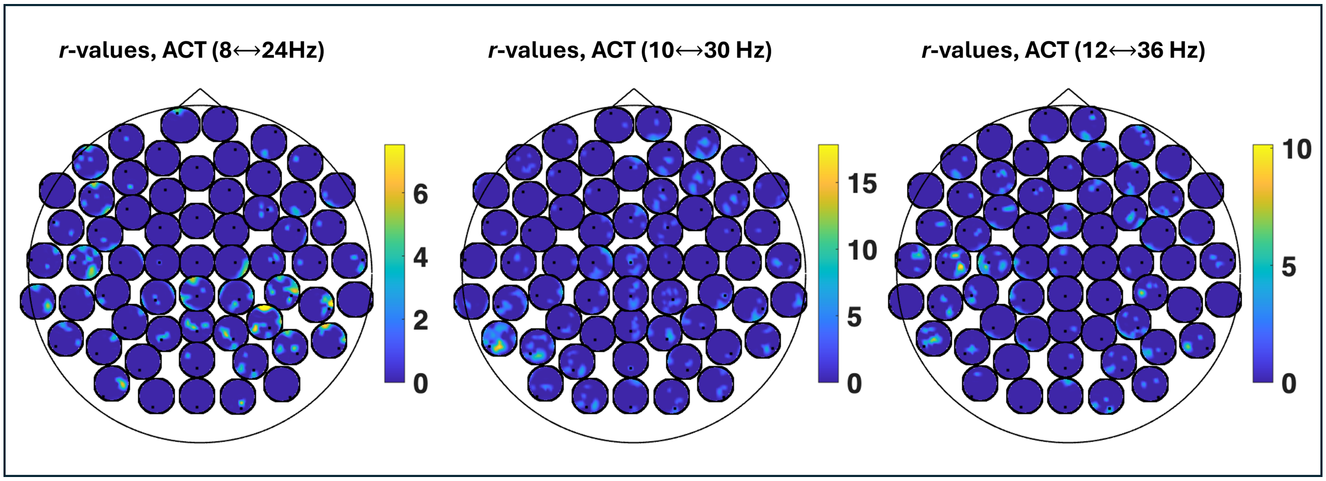

Figure S1. Exploratory single-subject sensor-level ACT r-value maps for harmonic cross-frequency interactions at $8 \leftrightarrow 24$ Hz, $10 \leftrightarrow 30$ Hz, and $12 \leftrightarrow 36$ Hz. Only values surviving an uncorrected threshold of $p < 0.05$ are displayed. Maps are shown as seed-based ACT topographies for visualization purposes only. The r-value color scales differ across frequencies; therefore, direct color-based comparisons across panels should be made only with reference to the corresponding color bars. A confirmatory statistical assessment would require appropriate multiple-comparison correction and validation in larger cohorts.

2 Single-subject connectome for ACT

Supplementary Figure S1 shows exploratory single-subject sensor-level ACT r-value maps for three harmonic cross-frequency configurations: $8 \leftrightarrow 24$ Hz, $10 \leftrightarrow 30$ Hz, and $12 \leftrightarrow 36$ Hz. The input frequencies were chosen within the alpha range and spaced by 2 Hz to provide a compact exploratory sampling of nearby harmonic 1:3 interactions while avoiding a dense frequency grid. For each frequency pair, the figure displays seed-based topographies in which supra-threshold ACT values are shown after applying an uncorrected threshold of $p < 0.05$. Across the three frequency configurations, ACT values appear spatially sparse and distributed rather than dominated by a uniform local spread around all seeds.