

Supplementary Information

Two Scale Black Holes with a Phenomenological Suppression Profile: Extremality, Null Critical Curves, and Test Field Spectra

Anirudh Pradhan, K. Ghaderi, and M. Zeyauddin

Supplementary Methods

Supplementary Data 1 contains the custom computational source code and the machine readable numerical data supporting the article. The modules are executed independently. The script `reproduce_geometry.py` regenerates Figs. 1–3, the energy condition transition table, and their source data. The script `reproduce_spectral.py` regenerates Figs. 4–5, the three method quasinormal mode comparison, the pseudospectral and time domain convergence data, the fitting window records, and the corresponding source data. The script `validation_checks.py` performs independent extremal residual, horizon ordering, null potential stationary point, exterior potential, WKB derivative, and pseudospectral convergence checks. The script `reproduce_identifiability.py` regenerates Fig. 6, the fixed background conditioning grid, and the numerical summary. Shared geometry, effective source, null orbit, WKB, pseudospectral, and time domain routines are collected in `numerics.py`.

Dimensionless units with $G_0M = 1$ are used throughout, and the damping rate is $\gamma = -\text{Im } \omega > 0$. The WKB derivatives are generated analytically by repeated application of $F d/dx$ and independently compared with a symmetric 17 point stencil in the tortoise coordinate. Time domain convergence is evaluated at three spatial resolutions with $\Delta t/\Delta r_* = 1/2$. A separate boundary placement check compares the standard domain $[-100M, 250M]$ with $[-130M, 300M]$. The time domain values in Supplementary Table S1 are componentwise medians of the three late time fitting windows in Supplementary Table S3. The covariance errors in Supplementary Table S3 are conditional errors of each single damped sinusoid fit; the variation across fitting windows is the relevant extraction sensitivity used in the main text.

For the independent frequency domain calculation, the exterior is compactified by $z = 1 - x_+/x$. After extracting the ingoing and outgoing factors

$$e^{i\omega x} z^{-i\omega/F'(x_+)} (1 - z)^{-2i\omega}, \quad (1)$$

the regular residual is collocated at Chebyshev Gauss nodes. The differential equation becomes a quadratic matrix polynomial in ω , which is linearized as a generalized eigenvalue problem. The fundamental branch is tracked continuously from the established Schwarzschild $n = 0$, $\ell_m = 2$ frequency, first along the $q = 0$ Hayward family and then in q at fixed a ; no WKB frequency is used to identify the pseudospectral mode. Supplementary Table S5 shows convergence from 40 to 70 collocation points and reports the relative complex frequency change with respect to $N = 70$.

The archive includes CSV source data for line plots and time domain waveforms, compressed NPZ grids for two dimensional scans, JSON validation summaries, and SHA256 checksums. No observational, experimental, proprietary, or third party dataset is used. The identical code and data archive is publicly available on Zenodo at [10.5281/zenodo.21828841](https://doi.org/10.5281/zenodo.21828841).

Table 1: Three method comparison for the fundamental $n = 0$, $\ell_m = 2$ test field modes. The time domain values are componentwise medians across the three fitting windows in Supplementary Table S3.

Case	Field	$M\omega_{\text{PS}}$	$M\omega_{\text{WKB3}}$	$M\omega_{\text{TD}}$
Schwarzschild	scalar	$0.483644 - 0.096759i$	$0.483211 - 0.096805i$	$0.483297 - 0.096943i$
Schwarzschild	em	$0.457596 - 0.095004i$	$0.457131 - 0.095065i$	$0.457529 - 0.095096i$
A	scalar	$0.499568 - 0.088904i$	$0.498756 - 0.088810i$	$0.499160 - 0.089095i$
A	em	$0.474835 - 0.087072i$	$0.473952 - 0.086960i$	$0.474348 - 0.087221i$
B	scalar	$0.502875 - 0.086539i$	$0.501982 - 0.086435i$	$0.502503 - 0.086835i$
B	em	$0.478508 - 0.084575i$	$0.477529 - 0.084444i$	$0.478023 - 0.084780i$
C	scalar	$0.506526 - 0.084061i$	$0.505584 - 0.083934i$	$0.506197 - 0.084459i$
C	em	$0.482543 - 0.081958i$	$0.481508 - 0.081801i$	$0.482060 - 0.082247i$

Table 2: Time domain grid convergence for the scalar $\ell_m = 2$ waveform fitted over $100 \leq t/M \leq 150$.

Case	$\Delta r_*/M$	$\Delta t/M$	$M\omega_R$	$M\gamma$
Schwarzschild	0.12	0.06	0.483302	0.096647
Schwarzschild	0.10	0.05	0.483297	0.096643
Schwarzschild	0.08	0.04	0.483291	0.096640
B	0.12	0.06	0.502187	0.086838
B	0.10	0.05	0.502179	0.086835
B	0.08	0.04	0.502174	0.086832

Table 3: Late time fitting windows. The last two columns are covariance based standard errors conditional on the single damped sinusoid model; they do not include fitting window or method dependence.

Case	Field	t_1/M	t_2/M	$M\omega_R$	$M\gamma$	σ_{ω_R}	σ_γ
Schwarzschild	scalar	100	150	0.483297	0.096643	1.42e-05	1.64e-05
Schwarzschild	scalar	105	155	0.483116	0.096943	2.08e-05	2.41e-05
Schwarzschild	scalar	110	160	0.483947	0.096967	2.12e-05	1.89e-05
Schwarzschild	em	100	150	0.457548	0.095096	1.55e-05	1.71e-05
Schwarzschild	em	105	155	0.457168	0.094884	2.01e-05	1.66e-05
Schwarzschild	em	110	160	0.457529	0.095447	2.03e-05	1.94e-05
A	scalar	100	150	0.498898	0.089095	1.58e-05	1.44e-05
A	scalar	105	155	0.499160	0.089297	1.83e-05	2.08e-05
A	scalar	110	160	0.499705	0.088874	1.17e-05	1.37e-05
A	em	100	150	0.474222	0.087168	1.01e-05	1.19e-05
A	em	105	155	0.474348	0.087221	1.60e-05	1.73e-05
A	em	110	160	0.474948	0.087314	1.48e-05	1.25e-05
B	scalar	100	150	0.502179	0.086835	1.66e-05	1.46e-05
B	scalar	105	155	0.502503	0.086975	1.82e-05	1.95e-05
B	scalar	110	160	0.502953	0.086502	1.01e-05	1.20e-05
B	em	100	150	0.477790	0.084780	1.15e-05	1.30e-05
B	em	105	155	0.478023	0.084791	1.53e-05	1.75e-05
B	em	110	160	0.478617	0.084771	1.27e-05	1.14e-05
C	scalar	100	150	0.505821	0.084459	1.76e-05	1.50e-05
C	scalar	105	155	0.506197	0.084538	1.81e-05	1.82e-05
C	scalar	110	160	0.506545	0.084032	8.69e-06	1.02e-05
C	em	100	150	0.481743	0.082302	1.37e-05	1.46e-05
C	em	105	155	0.482060	0.082247	1.50e-05	1.77e-05
C	em	110	160	0.482634	0.082113	1.06e-05	1.02e-05

Table 4: Independent WKB derivative cross check. The relative difference is $100|\omega_{\text{stencil}} - \omega_{\text{analytic}}|/|\omega_{\text{analytic}}|$.

Case	Field	$M\omega_R^{\text{analytic}}$	$M\gamma^{\text{analytic}}$	$M\omega_R^{\text{stencil}}$	$M\gamma^{\text{stencil}}$	Difference [%]
Schwarzschild	scalar	0.483211	0.096805	0.483211	0.096803	0.00042
Schwarzschild	em	0.457131	0.095065	0.457130	0.095062	0.00071
A	scalar	0.498756	0.088810	0.498757	0.088816	0.00116
A	em	0.473952	0.086960	0.473954	0.086970	0.00197
C	scalar	0.505584	0.083934	0.505581	0.083913	0.00414
C	em	0.481508	0.081801	0.481504	0.081779	0.00463

Table 5: Chebyshev pseudospectral convergence for the validation set. The last column is $|\omega_N - \omega_{70}|/|\omega_{70}|$.

Case	Field	N	$M\omega_R$	$M\gamma$	Relative change
Schwarzschild	scalar	40	0.483643872210	0.096758775979	2.33e-11
Schwarzschild	scalar	50	0.483643872211	0.096758775977	1.82e-11
Schwarzschild	scalar	60	0.483643872206	0.096758775976	2.54e-11
Schwarzschild	scalar	70	0.483643872217	0.096758775970	0.00e+00
Schwarzschild	em	40	0.457595511629	0.095004425819	2.76e-11
Schwarzschild	em	50	0.457595511631	0.095004425818	2.45e-11
Schwarzschild	em	60	0.457595511635	0.095004425818	1.90e-11
Schwarzschild	em	70	0.457595511640	0.095004425825	0.00e+00
A	scalar	40	0.499568231331	0.088903565336	5.74e-11
A	scalar	50	0.499568231332	0.088903565327	5.38e-11
A	scalar	60	0.499568231317	0.088903565316	2.91e-11
A	scalar	70	0.499568231305	0.088903565324	0.00e+00
A	em	40	0.474834601882	0.087072257790	5.85e-11
A	em	50	0.474834601887	0.087072257783	5.68e-11
A	em	60	0.474834601882	0.087072257764	3.68e-11
A	em	70	0.474834601865	0.087072257768	0.00e+00
B	scalar	40	0.502874653218	0.086539110903	2.82e-11
B	scalar	50	0.502874653207	0.086539110908	4.70e-11
B	scalar	60	0.502874653220	0.086539110924	3.28e-11
B	scalar	70	0.502874653230	0.086539110911	0.00e+00
B	em	40	0.478508357838	0.084574580050	3.32e-11
B	em	50	0.478508357827	0.084574580047	5.39e-11
B	em	60	0.478508357826	0.084574580066	4.07e-11
B	em	70	0.478508357846	0.084574580065	0.00e+00
C	scalar	40	0.506526458736	0.084060847847	6.16e-11
C	scalar	50	0.506526458739	0.084060847848	6.00e-11
C	scalar	60	0.506526458743	0.084060847848	6.13e-11
C	scalar	70	0.506526458737	0.084060847879	0.00e+00
C	em	40	0.482543189479	0.081958272253	5.73e-11
C	em	50	0.482543189480	0.081958272255	5.66e-11
C	em	60	0.482543189483	0.081958272257	5.83e-11
C	em	70	0.482543189458	0.081958272271	0.00e+00

Table 6: Time domain boundary placement check for the scalar $\ell_m = 2$ waveform fitted over $100 \leq t/M \leq 150$. The last column is the relative complex frequency change from the standard domain.

Case	r_*^{left}/M	r_*^{right}/M	$M\omega_R$	$M\gamma$	Relative change
Schwarzschild	-100	250	0.483296815231	0.096643246359	0.00e+00
Schwarzschild	-130	300	0.483296815230	0.096643246358	8.24e-13
B	-100	250	0.502178565433	0.086835307300	0.00e+00
B	-130	300	0.502178565434	0.086835307300	8.80e-13

Table 7: Energy condition transition bounds in the surveyed domain $0.55 \leq a \leq 0.72$, $0.05 \leq \zeta \leq 0.90$.

Quantity	Range or bound
$\max(x_{\text{NEC}}/x_-)$	0.591094
x_{SEC}/x_-	0.681807–0.992878
x_{DEC}/x_+	0.586572–0.863940

Supplementary Data 1 file inventory

Category	Files and contents
Core numerical code	<code>numerics.py</code> , <code>reproduce_geometry.py</code> , <code>reproduce_spectral.py</code> , <code>reproduce_identifiability.py</code> , and <code>validation_checks.py</code> .
Geometry and source data	<code>source_fig01a_lapse_profiles.csv</code> , <code>source_fig01b_extremal_branches.csv</code> , <code>source_fig01c_horizon_radii.csv</code> , <code>source_fig01d_black_hole_domain.npz</code> , <code>source_fig02a_central_lapse_scaling.csv</code> , <code>source_fig02b_curvature_scales.csv</code> , <code>source_fig02c_effective_source.csv</code> , <code>source_fig02d_nec_restoration_grid.npz</code> , <code>table_energy_condition_radii.csv</code> , <code>energy_condition_summary.json</code> , <code>source_fig03a_stationary_radii.csv</code> , and <code>source_fig03bcd_critical_curve_observables.csv</code> .
Spectral source data	<code>source_fig04ab_potential_barriers.csv</code> , <code>source_fig04c_ringdown_fit.csv</code> , and <code>table_spectral_scan.csv</code> .
Reproduced figures	<code>fig01_geometry_extremality.pdf/png</code> , <code>fig02_core_effective_matter.pdf/png</code> , <code>fig03_critical_curve.pdf/png</code> , <code>fig04_qnm_validation.pdf/png</code> , <code>fig05_spectral_response.pdf/png</code> , and <code>fig06_identifiability.pdf/png</code> .
Validation tables	<code>table_qnm_validation.csv</code> , <code>table_pseudospectral_convergence.csv</code> , <code>table_pseudospectral_crosscheck.csv</code> , <code>table_td_convergence.csv</code> , <code>table_td_boundary_check.csv</code> , <code>table_td_fit_windows.csv</code> , and <code>table_wkb_derivative_crosscheck.csv</code> .
Conditioning and summaries	<code>identifiability_grid.npz</code> , <code>numerical_summary.json</code> , and <code>validation_checks.json</code> .
Environment, citation, and integrity	<code>requirements.txt</code> , <code>README.md</code> , <code>CITATION.cff</code> , <code>ZENODO_METADATA.txt</code> , <code>verify_checksums.py</code> , and <code>SHA256SUMS.txt</code> .

The checksum manifest covers every distributed file except the manifest itself.