Supplementary Note 1

The dynamics for the ground state magnetization is derived in a semi-classical approximation from the Liouville equation for the density matrix. ρ_{ij} denotes the component of the density matrix for states with magnetic quantum numbers $m = i, j$. A dash indicates Zeeman sublevels in the excited state. The experiment is performed on a $F = 2 \rightarrow F' = 3$ transition, but the model applies to a $F = 1 \rightarrow F' = 2$ transition, which keeps the essential properties of a $F \rightarrow F' = F + 1$ transition as well as the important dipole and potentially quadrupole multipole components. We take the quantization axis along the wavevector of the pump beam. The light fields is then expressed in circular components by

$$\mathbf{E}(t) = \frac{1}{2} \sum_{q=\pm 1} (-1)^q E_q(t) \hat{\mathbf{e}}_{-\mathbf{q}} e^{i\omega t} + \text{c.c.}\,. \tag{1}$$

Following the spirit of Ref. [1], we consider these approximations:

- The populations and coherences of the excited state are adiabatically eliminated since their decays are faster than the dynamics of the ground state variables, neglecting terms of higher order than first order in Rabi frequency divided by detuning.
- Afterwards, excited state populations are neglected as the pump rate is kept low. Hence the total population remains in the ground state and is constant, i.e. $\rho_{-1-1} + \rho_{00} + \rho_{11} = 1$.
- Optical coherences are adiabatically eliminated.
- In order to describe the frequency of the Larmor precession in the ground state correctly, we use the Landé g-factor, $g_F = 0.5$, of the $F = 2$ ground state for the Larmor frequencies

$$\Omega_{x,y,z}/\Gamma_2 = 0.23 \times B_{x,y,z}/\text{G}, \tag{2}$$

  where $\Gamma_2 = \Gamma/2$ is the relaxation constant of the optical coherences (see below).
- To calculate the change in detuning, the Landé of the excited state is assumed also to be 0.5 for simplicity. This only provides a minor correction to the detuning properties of linear and nonlinear Faraday effect.
- In the Raman transition pump rates, detuning changes due to the B_z field are neglected.

The pump rates for the $\sigma_\pm$-fields on the stretched state transitions $m_1 \rightarrow m_{2'}$ and $m_{-1} \rightarrow m_{-2'}$ are obtained via the Rabi-frequencies as

$$P_+ = \frac{1}{\Gamma_2} \frac{|\Omega_+|^2}{1+\left(\bar{\Delta} - \frac{\Omega_z}{\Gamma_2}\right)^2}, \;\; P_- = \frac{1}{\Gamma_2} \frac{|\Omega_-|^2}{1+\left(\bar{\Delta} + \frac{\Omega_z}{\Gamma_2}\right)^2}, \tag{3}$$

where the detuning is written in units of coherence decay rate Γ_2 as $\bar{\Delta} = \delta/\Gamma_2$. Sum and difference pump rates are conveniently defined as

$$S = P_+ + P_-, \;\; D = P_+ - P_-. \tag{4}$$

The Raman transitions pump rates driving $\Delta m = 2$-coherence are given by

$$P_{\Lambda+} = \frac{1}{\Gamma_2} \frac{1}{1+\bar{\Delta}^2} \left(\Omega_+^* \Omega_- + \Omega_+ \Omega_-^*\right), \;\; P_{\Lambda-} = \frac{i}{\Gamma_2} \frac{1}{1+\bar{\Delta}^2} \left(\Omega_+^* \Omega_- - \Omega_+ \Omega_-^*\right). \tag{5}$$

The pump rates are related to the experimentally measured intensities of the circularly polarized components like

$$P_\pm = 2\Gamma_2 \frac{I_\pm}{I_{\text{sat}}} \frac{1}{1 + \left(\bar{\Delta} \mp \frac{\Omega_z}{\Gamma_2}\right)^2}, \tag{6}$$

where $2\Gamma_2 = 2\pi \times 6.066$ MHz (= Γ denoting the linewidth (full width at half height) used for normalization in the main article) and $I_{\text{sat}} = 1.669$ mW/cm^2 for circular light probing the $F = 2 \rightarrow F' = 3$ transition of the D_2 line of ^{87}Rb for all atoms in the stretched state $|m| = 2$ [2]. The total input power is $I_{\text{in}} = I_+ + I_-$.

Defining the ground state variables as

$$\begin{aligned}
u &= \rho_{-11} + \rho_{1-1}, \\
v &= i(\rho_{-11} - \rho_{1-1}), \\
w &= \rho_{11} - \rho_{-1-1}, \\
X &= \rho_{11} + \rho_{-1-1} - 2\rho_{00}, \\
y_1 &= \rho_{-10} + \rho_{0-1}, \\
z_1 &= \rho_{01} + \rho_{10}, \\
y_2 &= i(\rho_{-10} - \rho_{0-1}), \\
z_2 &= i(\rho_{01} - \rho_{10}),
\end{aligned} \tag{7}$$

results in the equations of motion

$$\dot{u} = -\Gamma_c u + \left(2\Omega_z + \frac{5}{6}D\bar{\Delta}\right) v + \frac{1}{6}P_{\Lambda-}\bar{\Delta}w - \frac{1}{9}P_{\Lambda+}X + \frac{5}{18}P_{\Lambda+} - \frac{\Omega_x}{\sqrt{2}}(z_2 - y_2) + \frac{\Omega_y}{\sqrt{2}}(z_1 - y_1), \tag{8a}$$

$$\dot{v} = -\Gamma_c v - \left(2\Omega_z + \frac{5}{6}D\bar{\Delta}\right) u + \frac{1}{6}P_{\Lambda+}\bar{\Delta}w + \frac{1}{9}P_{\Lambda-}X - \frac{5}{18}P_{\Lambda-} + \frac{\Omega_x}{\sqrt{2}}(z_1 - y_1) + \frac{\Omega_y}{\sqrt{2}}(z_2 - y_2), \tag{8b}$$

$$\dot{w} = -\Gamma_w w - \frac{1}{6}P_{\Lambda-}\bar{\Delta}u - \frac{1}{6}P_{\Lambda+}\bar{\Delta}v - \frac{1}{9}DX + \frac{5}{18}D - \frac{\Omega_x}{\sqrt{2}}(y_2 + z_2) - \frac{\Omega_y}{\sqrt{2}}(y_1 + z_1), \tag{8c}$$

$$\dot{X} = -\Gamma_X X - \frac{1}{3}P_{\Lambda+}u + \frac{1}{3}P_{\Lambda-}v + \frac{1}{3}Dw + \frac{5}{18}S + 3\frac{\Omega_x}{\sqrt{2}}(y_2 - z_2) + 3\frac{\Omega_y}{\sqrt{2}}(y_1 - z_1), \tag{8d}$$

$$\dot{y_1} = -\Gamma_y y_1 + (\Omega_z + \bar{\Delta}D_y)y_2 + \left(\frac{P'_-}{6} + \frac{1}{12}\left(\bar{\Delta}P_{\Lambda-} - P_{\Lambda+}\right)\right) z_1 \tag{8e}$$

$$+ \left(\frac{\bar{\Delta}P'_-}{6} + \frac{1}{12}\left(\bar{\Delta}P_{\Lambda+} + P_{\Lambda-}\right)\right) z_2 + \frac{\Omega_x}{\sqrt{2}}v + \frac{\Omega_y}{\sqrt{2}}(w - X + u), \tag{8f}$$

$$\dot{y_2} = -\Gamma_y y_2 - (\Omega_z + \bar{\Delta}D_y)y_1 - \left(\frac{\bar{\Delta}P'_-}{6} - \frac{1}{12}\left(\bar{\Delta}P_{\Lambda+} + P_{\Lambda-}\right)\right) z_1 \tag{8g}$$

$$+ \left(\frac{P'_-}{6} + \frac{1}{12}\left(P_{\Lambda+} - \bar{\Delta}P_{\Lambda-}\right)\right) z_2 + \frac{\Omega_x}{\sqrt{2}}(w - X - u) + \frac{\Omega_y}{\sqrt{2}}v, \tag{8h}$$

$$\dot{z_1} = -\Gamma_z z_1 + (\Omega_z + \bar{\Delta}D_z)z_2 + \left(\frac{P'_+}{6} - \frac{1}{12}\left(\bar{\Delta}P_{\Lambda-} + P_{\Lambda+}\right)\right) y_1 \tag{8i}$$

$$- \left(\frac{\bar{\Delta}P'_+}{6} + \frac{1}{12}\left(\bar{\Delta}P_{\Lambda+} - P_{\Lambda-}\right)\right) y_2 - \frac{\Omega_x}{\sqrt{2}}v + \frac{\Omega_y}{\sqrt{2}}(w + X - u), \tag{8j}$$

$$\dot{z_2} = -\Gamma_z z_2 - (\Omega_z + \bar{\Delta}D_z)z_1 + \left(\frac{\bar{\Delta}P'_+}{6} - \frac{1}{12}\left(\bar{\Delta}P_{\Lambda+} - P_{\Lambda-}\right)\right) y_1 \tag{8k}$$

$$+ \left(\frac{P'_+}{6} + \frac{1}{12}\left(\bar{\Delta}P_{\Lambda-} + P_{\Lambda+}\right)\right) y_2 + \frac{\Omega_x}{\sqrt{2}}(w + X + u) - \frac{\Omega_y}{\sqrt{2}}v\,. \tag{8l}$$

The decay rates of the atomic variables are

$$\Gamma_w = r + \frac{1}{6}(P_+ + P_-) \tag{9}$$

$$\Gamma_X = r + \frac{11}{18}(P_+ + P_-) \tag{10}$$

$$\Gamma_c = r + \frac{7}{6}(P_+ + P_-) - \frac{1}{3\Gamma_2}\frac{1}{1 + \bar{\Delta}^2}(|\Omega_+|^2 + |\Omega_-|^2) \tag{11}$$

$$\Gamma_y = r + P'_+ + \frac{7}{12}P'_- \quad \text{with } P'_\pm = \frac{1}{\Gamma_2}\frac{|\Omega_\pm|^2}{1 + \left(\bar{\Delta} \mp \frac{2\Omega_z}{\Gamma_2}\right)^2} \tag{12}$$

$$\Gamma_z = r + \frac{7}{12}P'_+ + P'_-, \tag{13}$$

where r is an effective decay rate of the Zeeman ground state population and coherences. The difference pump rates in the light-shift terms for y_1, y_2, z_1, z_2 are

$$D_y = P'_+ - \tfrac{7}{12}P'_-, \;\; D_z = \tfrac{7}{12}P'_+ - P'_-, \tag{14}$$

The numerical simulations are performed in the representation of Eqs. (8), but it is useful to note that these equations can be rewritten in terms of irreducible tensor components of rank 1 describing orientation or magnetic dipoles and components of rank 2 describing alignment components or magnetic quadrupoles, respectively, and that the magnetic field provides only a coupling within the dipole, respectively quadrupole, components, but not between tensors of different rank. A magnetic dipole vector is defined by

$$m_x = \frac{1}{\sqrt{2}}(y_1 + z_1) \tag{15}$$

$$m_y = -\frac{1}{\sqrt{2}}(y_2 + z_2) \tag{16}$$

$$m_z = w, \tag{17}$$

resulting in

$$\dot{\mathbf{m}} = \boldsymbol{\Omega} \times \mathbf{m} \tag{18}$$

for the coherent evolution of the magnetization in the external magnetic field. Neglecting the contributions from the rank 2 components, solving for the m_x, m_y components (Eqs. (8e)-(8l) in steady state (formally an adiabatic elimination) and inserting this into Eq. (8c) results in an effective equation for the $m_z = w$ component given as Eq. (5) in the main text.

From the residual thermal motion the ground state decay rate can be estimated to be

$$r = \frac{2\bar{v}_{\rm trans}}{\Lambda}, \tag{19}$$

where $\Lambda/2$ denotes a characteristic distance, here half the pattern period. $\bar{v}_{\rm trans}$ is a characteristic transverse velocity and is obtained from integrating the most probable thermal velocity $\bar{v}$ over polar angles by

$$\bar{v}_{\rm trans} = \frac{2}{\pi}\,\bar{v} = \frac{2}{\pi}\sqrt{\frac{8k_BT}{\pi m}} \tag{20}$$

with k_B denoting Boltzman's constant, T temperature and m the mass of a Rb atom. For $T = 120$ μK, $\Lambda \approx 50$ μm (Strathclyde experiment), $r \approx 4.4 \times 10^3 {\rm s}^{-1}$, for $T = 200$ μk, $\Lambda \approx 100$ μm (Nice experiment), $r \approx 2.8 \times 10^3 {\rm s}^{-1}$. This rate constant counteracts the magnetic ordering and can be thought of arising from the fluctuations induced by an effective temperature of the spins due to the residual motion.

The optical response of the medium is determined by the polarization components $\mathcal{P}_\pm$

$$\mathbf{P}(z,t) = \frac{1}{2}\sum_{q=\pm}(-1)^q \mathcal{P}_q(z,t)\hat{\mathbf{e}}_{-\mathbf{q}} e^{i\omega t - ikz} + \text{c.c.} \tag{21}$$

given by

$$\mathcal{P}_\pm = \varepsilon_0 \chi_\pm \left(\left(1 \pm \frac{3}{4}w + \frac{1}{20}X \right) E_\pm + \frac{3}{20}(u \mp iv)E_\mp \right), \tag{22}$$

where the linear susceptibility is given by

$$\chi_\pm = -\frac{b_0\Gamma_2}{kL}\frac{i\Gamma_2 + (\delta \mp \Omega_z)}{\Gamma_2^2 + (\delta \mp \Omega_z)^2}. \tag{23}$$

The total linear phase shift across the atomic sample is given by

$$\phi_\pm = \frac{b_0}{2}\frac{(\delta \mp \Omega_z)}{\Gamma_2}\frac{\Gamma_2^2}{\Gamma_2^2 + (\delta \mp \Omega_z)^2}. \tag{24}$$

Supplementary References

[1] Mitschke, F., Deserno, R., Lange, W. & Mlynek, J. Magnetically induced optical self-pulsing in a nonlinear resonator. *Phys. Rev. A* **33**, 3219–3231 (1986).

[2] Steck, D. A. Rubidium 87 D Line Data. Report (2010). Available online at http://steck.us/alkalidata (revision 2.1.4, 23 December 2010).