Reviewers' comments:

Reviewer #1 (Remarks to the Author):

The manuscript is devoted to the interesting topic of interaction between skyrmions and superconductors. This gained an interest recently as a platform for realizing the Majorana modes. Although the ideas and publications are floating around this manuscript presents an original study, which can be further considered for publication, provided the authors successfully answer the following remarks

1) The author claim to have good reasons to ignore the Rashba spin-orbit coupling, taking a limit where the typical spin-orbit coupling strength is larger than the length of skyrmion. However, I still see a problem as even l_so larger than the length of skyrmion, it will still impose constraints on the so-called s-wave superconductor. Indeed iversion symmetry breaking on the surface will genearte a triplet component of the sc gap beyond the one, induced by the skyrmion structure and it is not clear how stable the Majorana will be in this case.

2) The only answer to the question above can be given within a self-consistent theory calculations. Most of the studies in the literature, including this one, are done without self-consistency. This produces an unsually high expections from the experimental groups, which is actually from the theory side is not really well-grounded. In particular, the present theory assumes transparent limit in the Sk/SC heterostructure. Such a transperency is known to be dangerous especially in this type of heterostructures as there is a mutual interplay of magnetic and sc degrees of freedom and to demonstarte a stability of the effect a self-consistent calculations are required.

3) I find the discussion on the experimentla aspect rather poor. The theory presented has have a validity assumptions with respect to the parameters. As authors correctly pointed out the parameters of either sc system or chiral magnet, hosting skyrmions are typically very well known. This mens they are available. So what the authors should do is to look in the corresponding literature and to show which of the systems could match from the experimental parameters (l_so, exchnage field, skyrmjon radius, cpherence length of sc, and Tc as well as temperatures where skyrmions were obsered at low enough temperatures). In addition the lattice mismacth is an important ingredient to see whether the limit of transparancy, which is required, can really be achieved.

Reviewer #2 (Remarks to the Author):

The authors focus on the interface of an s-wave superconductor and a ferromagnet, hosting a skyrmion. Such heterostructures have been shown to stabilize Majorana bound states and non-trivial transport properties, and are currently being pursed actively.

In their manuscript, the authors show that a skyrmion-like magnetic structure with multiple spin flips will host a multitude of Majorana modes near its edge, and that these topological modes are nearly dispersionless up to the point where topological transition occurs. The authors study a rotationally symmetric setup, which reduces to a 1D probably following Ref. 25, as well as a setup without rotational symmetry, using tight binding calculations. Furthermore, the authors show that the rotationally symmetric setup can be mapped onto 1D Rashba wire, which allow them to derive the boundaries of the topological region. They also show that the edge states grow with the perimeter of the topological region, and confirm their robustness.

I believe the authors proposal is timely and important, and this manuscript can be suitable for publication after a revision aiming at discussing and addressing these points:

- Dzyaloshinskii-Moriya interaction in chiral magnets prefer skyrmions with a single winding and spin flip, and skyrmions with p>1 or q>1 are unstable excitations which quickly decay. This is mentioned in Ref. 44 and arxiv:1904.04177. Skyrmions with higher p are likely to have shorter lifespan. I suggest authors to focus on non-chiral magnetic bubbles instead, which are stabilized by perpendicular magnetic anisotropy, don't have a preferred handedness, and can have high p or q, although tend to be larger in size.

- The authors requirement J=0 outsize of the skyrmion region to avoid delocalization of the edge states is also likely to be an experimental hurdle. I think this should be pointed out before Discussion section.

- The authors say "a correct model for a skyrmion if we choose q=0" and "rotation symmetry of the (q=0) skyrmion". The magnetic texture for q=0 is not a skyrmion but rather a double magnetic domain wall repeated p times, has zero topological charge and is not rotationally symmetric.

- "Deformed skyrmions" typically refers to a skyrmion texture whose shape is deformed, whereas the authors here refer to an exchange field, due to a perfect skyrmion with no deformation, induced in a superconductor with holes in it. Also, "elliptical skyrmion geometry" actually seems to refer to elliptical superconductor geometry; elliptical skyrmions were discussed in PRB 97, 115136 (2018) and behave differently. Can the authors use a different terms which reflect their setup better?

- Some symbols are used without or before being defined, such as t, m_J. Also, can the authors explicitly write t (which looks like the hopping parameter h\bar^2/2m a^2) in terms of the effective mass?

- Can the authors add an estimate using a realistic set of dimensionful parameters? This will make the manuscript more friendly to experimentalists.

- The authors omit and use \hbar in an inconsistent manner.

- Some relevant references should be mentioned and commented on: PRL 117, 077002 (2016), PRB 97, 115136 (2018), arxiv:1904.04177. The authors may also consider to cite the following reviews: J. Appl. Phys. 124, 240901 (2018) and Solid State Electronics 155 (2019) 93-98.

We thank both referees for their insightful and helpful questions and comments, all of which we addressed by rewriting and expanding the manuscript, as detailed below.

I. SUMMARY OF CHANGES

- In the introduction, magnetic bubbles are first invoked [references 22, 27-34 added].
- In section I.A, the effect and relevance of Rashba spin-orbit, as well as the way we introduce the edge of the system, are discussed in more detail [references 44, 45 added].
- Around Eq.7 in Section I.C, the coplanar $q = 0$ texture is more clearly discussed.
- In the first two paragraphs of section I.D, our definition of edge (either using vanishing exchange or a superconducting island) is implemented more clearly.
- In the discussion section, in order: (1) Self-consistency is discussed in more detail [references 52, 53 added]; (2) New discussion is added about applicability of our results to magnetic bubbles, and significance of material chirality; (3) Discussion of dimensionful ranges for model parameters is rewritten more transparently; (4) New discussion is added about experimental realization of superconductivity, with emphasis on measured parameters and Rashba spin-orbit [references 54-56 added]; (5) Discussion on experimental realization of magnetic texture is expanded, emphasizing measured parameters and predictions of higher winding numbers [references 57-59, 61-63 added]; (6) The problem of interfacing magnetic and superconducting materials is also mentioned.
- Supplementary materials section G is added, showing our new results regarding magnetic bubbles.
- In figure captions, model parameters are clearly defined and appropriate text sections are referred to.
- Minor changes throughout to clearly define parameters and improve clarity of text.

II. RESPONSE TO REFEREE COMMENTS

Reviewer 1 (Remarks to the Author):

1)The author claim to have good reasons to ignore the Rashba spin-orbit coupling, taking a limit where the typical spin-orbit coupling strength is larger than the length of skyrmion. However, I still see a problem as even l_{so} larger than the length of skyrmion, it will still impose constraints on the so-called s-wave superconductor. Indeed iversion symmetry breaking on the surface will genearte a triplet component of the sc gap beyond the one, induced by the skyrmion structure and it is not clear how stable the Majorana will be in this case.

We thank the referee for this precise observation. We agree that Rashba spin-orbit coupling is expected in these setups because of the layering of materials, and we did not discuss it in the original manuscript. Our text only considered the fact that intrinsic spin-orbit coupling is tiny and negligible in typical superconducting materials such as aluminum.

The Rashba spin-orbit coupling may non-trivially interplay with the spin-orbit induced by the skyrmion. In particular, the effective triplet pairings induced by skyrmion and Rashba may change the Majorana physics we found. However, these possibilities would require a separate challenging in-depth study due to the fact that there is no obvious rotational symmetry basis when both skyrmion and Rashba terms are present. In this work we therefore choose to deal only with the regime where Rashba (as well as the intrinsic) spin-orbit is sufficiently weaker than the skyrmion induced one.

Ideally we would argue for experimental relevance of this regime by comparing the estimated skyrmion-induced spin-orbit term (and hence, indirectly, the skyrmion-induced triplet pairing) to a precise measured value of Rashba spin-orbit coupling (and hence, indirectly, the Rashba-induced triplet pairing) in a typical material that could serve as a superconducting layer. Such precise measurements of Rashba are hard to come by. We found an estimate of $E_{so}/\Delta_0 = 5\%$ for the ratio of energy scale E_{so} of spin-orbit scattering to the energy scale Δ_0 of s-wave pairing in thin-film aluminum [Meservey and P. M. Tedrow, Physics Reports, 238, 4, 173-243 (1994), Table 1, p.194]. The skyrmion-induced spin-orbit energy scale $E_{so} = (\pi^2/2m) \cdot (p^2/R^2)$ [cf. derivation of wire model, eq.3] can without problem reach $5\%\Delta_0$ or several times higher for reasonable values of the parameters p (radial winding, $\lesssim 10$) and R (skyrmion radius, $\approx 10\,\text{nm} - 100\,\text{nm}$) and Δ_0 ($\approx$ meV) for given microscopic electron lengthscale l_F (Fermi length, $\approx 0.5\,\text{nm} - 1\,\text{nm}$). We checked that combining with a reasonable range for J (Zeeman exchange field magnitude, $\approx 10\,\text{meV}$) our results remain relevant, for example, the effective induced p-wave gap remains appreciable.

In the revised manuscript we made the discussion of ranges of parameters more detailed and transparent, including the discussion of Rashba spin-orbit and its measurement.

2) The only answer to the question above can be given within a self-consistent theory calculations. Most of the studies in the literature, including this one, are done without self-consistency. This produces an unsually high expections from the experimental groups, which is actually from the theory side is not really well-grounded. In particular, the present theory assumes transparent limit in the Sk/SC heterostructure. Such a transparency is known to be dangerous especially in this type of heterostructures as there is a mutual interplay of magnetic and sc degrees of freedom and to demonstarte a stability of the effect a self-consistent calculations are required.

We agree with the referee that most studies dealing with proximitized superconductivity are done without self-consistent calculations. However, we believe that in our case a self-consistent calculation would not change our main conclusions. First, we assumed that our skyrmion arises in a magnetic insulator which guarantees that the mutual interplay between the magnetic insulator and the superconductor is weak. Second, assuming rotation symmetry, we mapped our initial 2D Hamiltonian to a set of 1D Hamiltonians which are close variants of 1D Hamiltonians describing topological semiconducting wires. It turns out that self-consistent calculations have been made for this class of Hamiltonians e.g. in K. Bjornsson et al., Phys. Rev. B 92, 214501 (2015) or in M. H. Christensen et al., Phys. Rev. B 94, 144509 (2016) . A self-consistent calculation does not change qualitatively the physics in this case but may affect the spectrum especially near the topological phase transition where the gap is small. In our case, this may for example slightly shift the value of m_J^* defined in Eq. (6) at which the gap closes.

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As far as experimental realisations are concerned we already proposed to focus on the two magnetic insulators Cu_2OSeO_3 and $BaFe_{12-x-0.05}Sc_xMg_{0.05}O_{19}$ with $x = 1.6$. In particular, we can quote [K. Everschor-Sitte *et al.* , Journal of Applied Physics 124, 240901 (2018)] that 100 nm-thick Cu_2OSeO_3 films host skyrmions of radius 25 nm at temperatures ranging from a few Kelvins up to 57K. There is a sizeable electronic gap of 2.5 eV at 15K [R. B. Versteeg *et al.* , Phys. Rev. B 94, 094409 (2016)]. The lattice constant is 8.925 $\mathring{A}$ [P. Y. Portnichenko *et al.* , Nature Communications 7, 10725 (2016)]. All these parameters are within the ranges for which our results are relevant, as seen in our reply above. Finally, we expect that the strength of exchange field will depend on the achieved interfacing between the two materials, so a theoretical prediction is basically impossible.
As far as the superconducting part of the setup is concerned, we propose to focus on aluminium, which is a known superconductor and has negligible atomic spin-orbit coupling. Disordered thin films of aluminium have a critical temperature of the order of 3K with a coherence length of around 50 nm [R. Meservey and P. M. Tedrow, Journal of Applied Physics 42, 51 (1971) and K. Steinberg *et al.* , Phys. Rev. B 77, 214517 (2008)]. Again this is within our theoretically relevant range.
We note that in $BaFe_{12-x-0.05}Sc_xMg_{0.05}O_{19}$ ($x = 1.6$) the skyrmions are larger, but could be within the upper limit of tens-of-nanometers range we consider for this parameter.
In the revised manuscript we expanded the discussion of materials and added the citations.

Reviewer 2 (Remarks to the Author):

The authors focus on the interface of an s-wave superconductor and a ferromagnet, hosting a skyrmion. Such heterostructures have been shown to stabilize Majorana bound states and non-trivial transport properties, and are currently being pursed actively. In their manuscript, the authors show that a skyrmion-like magnetic structure with multiple spin flips will host a multitude of Majorana modes near its edge, and that these topological modes are nearly dispersionless up to the point where topological transition occurs. The authors study a rotationally symmetric setup, which reduces to a 1D probably following Ref. 25, as well as a setup without rotational symmetry, using tight binding calculations. Furthermore, the authors show that the rotationally symmetric setup can be mapped onto 1D Rashba wire, which allow them to derive the boundaries of the topological region. They also show that the edge states grow with the perimeter of the topological region, and confirm their robustness. I believe the authors proposal is timely and important, and this manuscript can be suitable for publication after a revision aiming at discussing and addressing these points:

- Dzyaloshinskii-Moriya interaction in chiral magnets prefer skyrmions with a single winding and spin flip, and skyrmions with $p > 1$ or $q > 1$ are unstable excitations which quickly decay. This is mentioned in Ref. 44 and arxiv:1904.04177. Skyrmions with higher p are likely to have shorter lifespan. I suggest authors to focus on non-chiral magnetic bubbles instead, which are stabilized by perpendicular magnetic anisotropy, don't have a preferred handedness, and can have high p or q, although tend to be larger in size.

We thank the referee for pointing out the useful connection to magnetic bubbles. Indeed the search for higher winding numbers in textures might well profit from including non-chiral materials and magnetic bubbles. Indeed it was numerically shown that $q = 2$ skyrmions can be stabilized in non-chiral magnets [A. O. Leonov and M. Mostovoy, Nat. Commun. 6, 8275 (2015)].
We agree that generally the stabilizing mechanism for bubbles and skyrmions is different, and chirality of the material plays a role. For our work, however, the key concern is the spatial profile of the exchange term caused by the texture. We therefore focus on comparing the real-space shape of the bubble texture compared to the skyrmion texture. In that regard, literature offers various defining distinctions between skyrmions and bubbles. First, in terms of topology of their shape, some works note that skyrmions and bubbles are mathematically the same[]. Second, in terms of chirality the referee further seems to suggest that bubbles are non-chiral textures, as opposed to the chiral skyrmion textures we studied. Yet, we find that all isolated magnetization textures observed in the literature do break mirror symmetry in plane perpendicular to the plane of the texture, and hence they are all by mathematical definition chiral textures. Of course, in a non-chiral host material (no preferred handedness), there should be multiple copies of the texture randomly mixed between left- and right-oriented, see for example Fig.3 in review [N. Nagaosa and Y. Tokura, Nature nanotechnology, 8, 899 (2013)]. Having this in mind, we point out that handedness of skyrmions was defined in this review by the parameter γ, which influences strongly the spatial profile of the skyrmion texture. This handedness γ is independent from any winding numbers of the skyrmion, and in our model for the exchange interaction, such a γ can be completely removed by a unitary transformation, not affecting the spectrum or localization properties of wavefunctions. Therefore, there may be modifications of our skyrmion model that may involve handedness, but we believe these are not crucial for a distinction from a magnetic bubble.
Consequently, we find that for our model the most important spatial feature of bubbles is that they are essentially annulus-shaped domains of uniform polarization separated by ring-shaped domain walls [**? ? ? ?**], see Fig. 1a. We therefore contrast such a straightforward model of a bubble with our model of the skyrmion in Fig. 1b,c. In Fig. 1b the skyrmion shows smooth oscillation between out-of-plane (blue line is z-component) and in-plane magnetization (orange line). In contrast, the bubble, Fig. 1c, has magnetization roughly out-of-plane and constant on concentric annuli (blue line is z-component), while at the ring-shaped domain-walls between these annuli there is in-plane winding of magnetization (orange line is in-plane component).

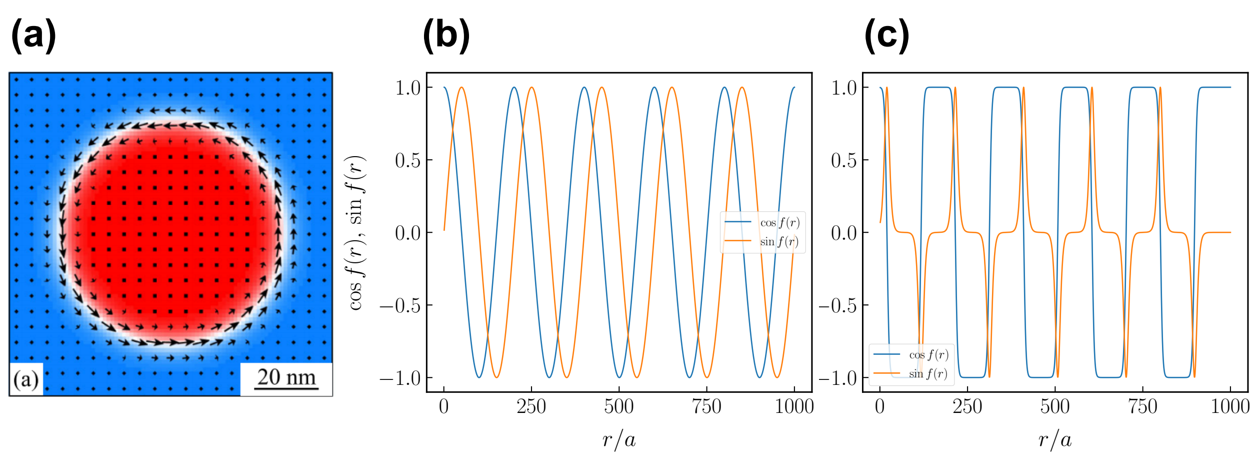

FIG. 1: **(a)** Real-space profile of a bubble, taken from I. Makhfudz, B. Krüger, and O. Tchernyshyov, Phys. Rev. Lett. 109, 217201 (2012). Our model of $p = 10$ **(b)** skyrmion and **(c)** bubble profiles. For the bubble the width of a domain wall $w/a = 5$ (defined in text).

For sake of completeness, we remind that the general texture profile is $\{n_x, n_y, n_z\} \equiv \{\sin(f(r)) * \cos(\theta), \sin(f(r)) * \sin(\theta), \cos(f(r))\}$, with r the radial coordinate and θ the polar angle, and the explicit profile for the bubble we choose as:

$$f(r) = \pi \sum_i g(r - r_0 - r_i, w) \tag{1}$$

where $g(x, w) = (1 + \exp(-x/w))^{-1}$ is an inverted Fermi function ($g(x) = n_F(-x)$), the $r_0 = 4w$ is a global offset, and $r_i = i(L - pw/2)/p$ is the position of the i^{th} domain wall, while w is the radial width of the domain-wall.

We show in Fig. 2 the result of our radial tight-binding computations with the bubble profile of Fig. 1c. The momentum-resolved excitation spectrum, Fig. 2a, shows that the nearly-flat band and the effective gap survive; the wavefunction amplitude of the zero-energy mode (its partner is not shown) at angular momentum zero shows that the Majorana wavefunctions at the center and at the edge remain localized, although their detailed spatial profile has changed significantly. In conclusion, if the bubbles uniform magnetization domains are not too large in real space

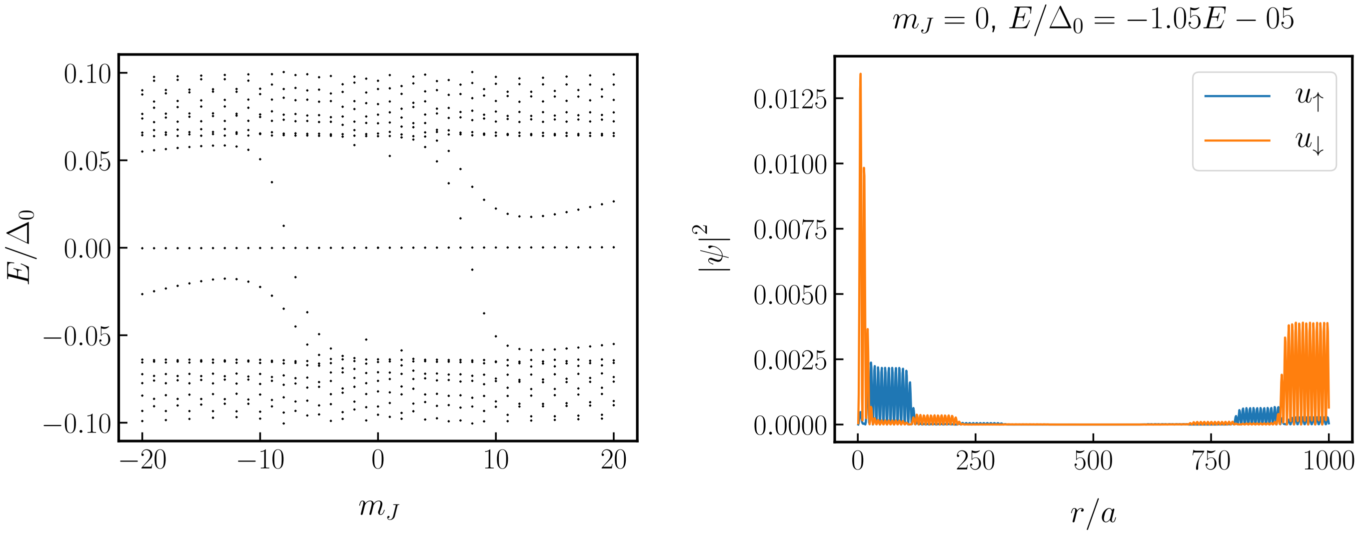

FIG. 2: Numerical diagonalization results for the parameters $p = 10$, $L = 1000$, $q = 2$, $J/t = 0.2$, $\Delta_0/t = 0.1$, $\mu/t = 0$ and a DW width $w/a = 5$. **Left:** Spectrum. **Right:** Density of one of the Majorana zero-modes.

(radial width of the domains in above calculation is 100 sites, corresponding to lengthscale of nm to 10 nm), our results are robust. If the domains increase, one expects to reach a regime where the ferromagnetic nature of the domains dominates, and the effective gap closes.

We have added a supplementary section for our bubble model calculations. We expanded the main text to discuss non-chiral materials, magnetic bubbles and their distinction from skyrmions, and the relevance of our results for bubble systems.

- The authors requirement J=0 outsize of the skyrmion region to avoid delocalization of the edge states is also likely to be an experimental hurdle. I think this should be pointed out before Discussion section.

We have included this comment in the main text, at the point where we introduce the concept of "edge" of our system. We then explain that the most natural realization of the edge is that the superconductor is deposited as an island, defining the edge on top of the underlying texture with non-zero J.

- The authors say "a correct model for a skyrmion if we choose $q = 0$" and "rotation symmetry of the ($q = 0$) skyrmion". The magnetic texture for $q = 0$ is not a skyrmion but rather a double magnetic domain wall repeated p times, has zero topological charge and is not rotationally symmetric.

We agree that using the name "skyrmion" for the $q = 0$ texture is too technical and might be misleading. We however note that the $q = 0$ texture does not depend on the polar angle θ and can therefore be fully reduced to a radial problem for each angular momentum channel m_J, with angular momentum operator being $-i\partial_\theta$; it is thus correct, although maybe too formal, to call it "rotationally symmetric". Nevertheless, we fully accept the referee's point, so we accordingly replaced the name "$q = 0$ skyrmion" by "coplanar texture", and emphasized the special nature of this particular texture.

- "Deformed skyrmions" typically refers to a skyrmion texture whose shape is deformed, whereas the authors here refer to an exchange field, due to a perfect skyrmion with no deformation, induced in a superconductor with holes in it. Also, "elliptical skyrmion geometry" actually seems to refer to elliptical superconductor geometry; elliptical skyrmions were discussed in PRB 97, 115136 (2018) and behave differently. Can the authors use a different terms which reflect their setup better?

We thank the referee for pointing out this potential source of confusion. We corrected it by replacing "deformed skyrmion" by "system with deformed edge". We also emphasized that the texture is not deformed locally. Instead, the edge could be provided by setting the exchange to zero outside the edge, which is experimentally hard; more naturally, the deformed edge of the system naturally occurs for example if the heterostructure is made by depositing a finite superconducting island whose boundary is the effective edge of our model system. We added this discussion in the appropriate places of main text.

Let us also mention that in Güngördü *et al.* , Phys. Rev. B 97, 115136, 2018 suggested by the referee, the authors investigate elongated magnetic skyrmions in a very different limit than us. Their skyrmions have extreme aspect ratios whereas we focused on small deformations of edge shape. We expect that had we continued to increase our aspect ratio, the Majorana modes would have hybridized pairwise to finally reach the limit obtained by Güngördü *et al.* . We have added this reference and comment.

- Some symbols are used without or before being defined, such as t, m_J. Also, can the authors explicitly write t (which looks like the hopping parameter $\hbar^2/2ma^2$) in terms of the effective mass?

We corrected the definitions of parameters, and commented on the connection between hopping energy t and effective electron mass m. The definitions are now highlighted in figure captions, and definitions appear in the text before referring to those figures. We also carefully refer to appropriate model definition in Methods sections. However, we reserve the right to use t as the symbol for hopping parameter in most of text, as is standard in tight-binding modeling.

- Can the authors add an estimate using a realistic set of dimensionful parameters? This will make the manuscript more friendly to experimentalists.

We have expanded and clarified the discussion of model parameters, emphasizing the constraints on their values and the corresponding ranges of dimensionful values. We also compared these dimensionful values to some measurements of parameters in materials.

- The authors omit and use $\hbar$ in an inconsistent manner.

We fixed this inconsistency.

- Some relevant references should be mentioned and commented on: PRL 117, 077002 (2016), PRB 97, 115136 (2018), arxiv:1904.04177. The authors may also consider to cite the following reviews: J. Appl. Phys. 124, 240901 (2018) and Solid State Electronics 155 (2019) 93-98.

We expanded the introduction to better represent: 1) ways to engineer topological states using magnetism (citing PRL 117, 077002), 2) the role of intrinsic and Rashba spin-orbit coupling and possible occurrence of vortices (citing arxiv:1904.04177). We also included the JAP review, as well as PRB 97, 115136 (2018) in the introduction.

REVIEWERS' COMMENTS:

Reviewer #1 (Remarks to the Author):

The authors have answered satisfactory my comments and I believe the manuscript can be now accepted for publictaion.

Reviewer #2 (Remarks to the Author):

I find the authors response and revision satisfactory, and recommend the manuscript for publication in Nature Communications Physics.

I have one optional recommendation for the authors:
- The the domain wall size in magnetic bubbles is determined by (square root of) the ratio of magnetic exchange to uniaxial magnetic anisotropy strength, which can be used for making estimates.

REVIEWERS' COMMENTS:

We are pleased that both referees recommended publication of our revised manuscript.
We implemented the optional suggestion of referee #2:

Reviewer #2 (Remarks to the Author):

I have one optional recommendation for the authors:

- The the domain wall size in magnetic bubbles is determined by (square root of) the ratio of magnetic exchange to uniaxial magnetic anisotropy strength, which can be used for making estimates.

We thank the referee for mentioning this theoretical estimate. We have used it with typical material values (in Supp. Refs. 5 and 6) , namely, picoJoules per square meter for the exchange and megaJoules per cubic meter for the anisotropy, to find a domain wall sizerange of 1-10nm. This is consistent with our numerical model for the bubble in the last Supp. Note, to which we also added a few sentences of this discussion.