Supplementary Information for

Optimal, near-optimal, and robust epidemic control

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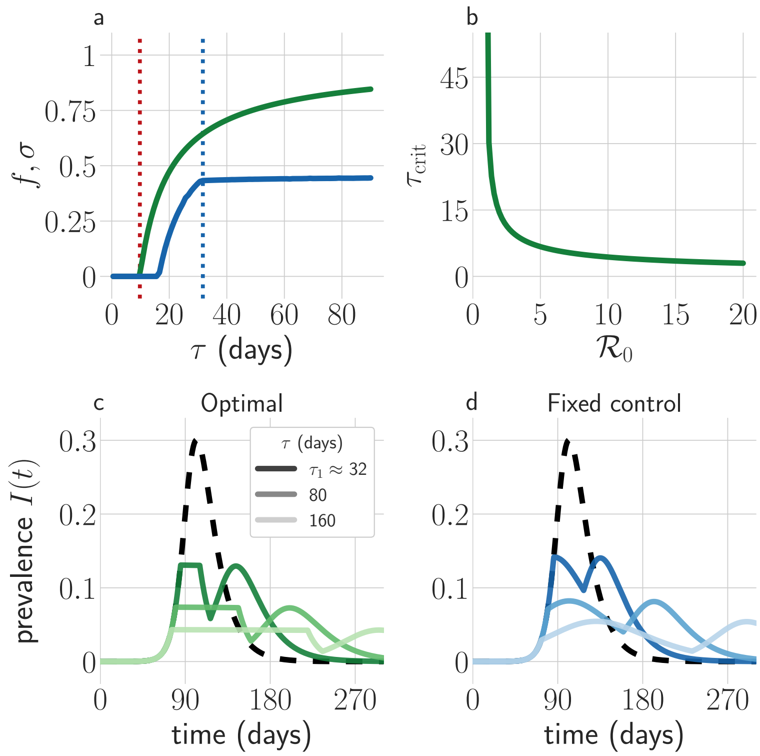
Supplementary Figures



Supplementary Fig. 1 | Effect of τ and $\mathscr{R}_0$ on the optimal and optimized fixed control strategies. a, Optimal maintenance fraction f for the optimal strategy (green line) and optimized strictness σ for the fixed control intervention (blue line) as a function of intervention duration τ. Red dotted vertical line shows the critical value τ_{crit} below which full suppression ($f = 0$) is the globally optimal intervention. Blue dotted vertical line shows τ_1, the point at which $\mathscr{R}_e = 1$ at the start of the optimized fixed control intervention. **b,** Full suppression critical value τ_{crit} as a function of basic reproduction number $\mathscr{R}_0$. **c, d,** Example timecourses for $\tau \geq \tau_1$, under the (**c**) optimal and (**d**) fixed control interventions. Dashed black line shows timecourse in the absence of intervention. Parameters as in Table 1 of the main text unless otherwise stated.

1 Supplementary Note 1: Theorems and Proofs

1.1 Useful notation

We define S_{crit} for an SIR model to be the critical fraction susceptible at which $\mathscr{R}_e = 1$ and $\frac{\mathrm{d}I}{\mathrm{d}t} = 0$ (in the absence of intervention), i.e. $S_{\text{crit}} \equiv \frac{1}{\mathscr{R}_0}$.

1.2 Peak prevalence in an SIR model

Define $I^{\max}(t)$ for an SIR system as the maximum value of $I(x)$ achieved during the interval $x \in [t, \infty)$. Notice that $I^{\max}(0) = I^{\max}$, where $I^{\max}$ is the global maximum value of $I(x)$, which we are seeking to minimize with our intervention.

A known result that is immediate from the original work of Kermack and McKendrick[1] (see Weiss[2] for an explicit derivation) holds that for $S(t) \geq S_{\text{crit}}$:

$$I^{\max}(t) = I(t) + S(t) - \frac{1}{\mathscr{R}_0} \log\Big(S(t)\Big) - \frac{1}{\mathscr{R}_0} + \frac{1}{\mathscr{R}_0} \log\Big(\frac{1}{\mathscr{R}_0}\Big) \tag{5}$$

Remark 1: Continuity of $I^{\max}$. *I^{max} is a continuous function $I^{max} : (0,1]^2 \mapsto [0,1]$ of $v = (S(t), I(t))$ for $v \in [S_{crit}, 1] \times (0,1]$, and therefore also a continuous function of t, $I^{max} : \mathbb{R} \mapsto [0,1]$ for $t \in (-\infty, t_{crit}]$, where t_{crit} is the time such that $S(t_{crit}) = S_{crit}$*

This is immediate from the fact that $I^{\max}(t)$ is a linear combination of univariate functions of $S(t)$ and $I(t)$ that are themselves continuous on $[S_{\text{crit}}, 1]$ and $[0,1]$, respectively. And since $S(t)$ and $I(t)$ are continuous functions of t, $I^{\max}$ is a continuous function of t for $t \in (-\infty, t_{\text{crit}}]$.

Lemma 1: The more fire, the bigger the blaze. *If $0 \leq I_x(t_x) \leq I_y(t_y)$ and $S_x(t_x) = S_y(t_y)$ for two SIR systems x and y with identical parameters $\mathscr{R}_0$ and γ at possibly distinct times $t_x, t_y \in [0, \infty]$ then $I_x^{max}(t_x) \leq I_y^{max}(t_y)$, with equality only if $I_x(t_x) = I_y(t_y)$.*

Proof. There are three cases.

Case 1: $I_x(t_x) = 0$. In this case, $I_x^{\max}(t_x) = 0 \leq I_y(t_y) \leq I_y^{\max}(t_y)$, with equality only if $I_y^{\max}(t_y) = 0$, which can only occur if $I_y(t_y) = 0$.

Case 2: $S_x(t_x) = S_y(t_y) < S_{\text{crit}}$. In this case, $I_x^{\max}(t_x) = I_x(t_x)$ and $I_y^{\max}(t_y) = I_y(t_y)$, so our result holds.

Case 3: $S_x(t_x) = S_y(t_y) > S_{\text{crit}}$. In this case, we can apply equation 5. Fixing $S(t) > S_{\text{crit}}$, $I^{\max}(t)$ is an increasing function of $I(t)$ (there is a single, positive $I(t)$ term in the sum), so the result must hold. $\square$

Lemma 2: The more fuel, the bigger the blaze. *If $I_x(t_x) = I_y(t_y) > 0$ and $S_x(t_y) \leq S_y(t_y)$ for two SIR systems x and y with identical parameters $\mathscr{R}_0$ and γ at possibly distinct times $t_x, t_y \in [0, \infty]$ then $I_{t_x}^{max} \leq I_{t_y}^{max}$, with equality only if $S_x(t_x) = S_y(t_y)$ or $S_y(t_y) \leq S_{crit}$*

Proof. There are three cases.

Case 1: $S_y(t_y) \leq S_{\text{crit}}$. If $S_y(t_y) \leq S_{\text{crit}}$, then $S_x(t_x) \leq S_{\text{crit}}$, and neither epidemic will grow after t_x or t_y, respectively. It follows that $I_{t_x}^{\max} = I_x(t_y) = I_{t_y}^{\max} = I_y(t_y)$

Case 2: $S_y(t_y) > S_{\text{crit}}$, $S_x(t_x) < S_{\text{crit}}$. In this case, $I_{t_x}^{\max} = I_x(t_y)$ but $I_{t_y}^{\max} > I_y(t_y)$, since $\frac{\mathrm{d}I_y}{\mathrm{d}t} > 0$ if $I_y(t) > 0$ and $S_y(t) > S_{\text{crit}}$. So we have $I_{t_x}^{\max} < I_{t_y}^{\max}$

Case 3: $S_x(t_x), S_y(t_y) > S_{\text{crit}}$. The result in this case follows immediately from the fact that, fixing $I(t)$, $I^{\max}(t)$ is an increasing function of $S(t)$ when $S(t) > S_{\text{crit}}$. We can see that it is by taking the partial derivative with respect to S of the expression in equation 5:

$$\frac{\partial}{\partial S} I^{\max}(t) = 1 - \frac{1}{\mathscr{R}_0 S} \tag{6}$$

If $S > S_{\text{crit}}$, $\frac{1}{\mathscr{R}_0 S} < 1$, so $\frac{\partial}{\partial S} I^{\max}(t) > 0$, and $I^{\max}(t)$ is an increasing function of $S(t)$. $\square$

1.3 Intervention function

We say that a right-continuous function with finite discontinuity points $b(t) : \mathbb{R} \mapsto [0, 1]$ is an intervention beginning at t_i with duration τ if $b(t) \equiv 1$ for all $t \in (-\infty, t_i) \bigcup (t_i + \tau, \infty)$. The SIR model under such an intervention will then take the form

$$
\begin{aligned}
\frac{dS}{dt} &= -b(t) * \beta S I \\
\frac{dI}{dt} &= b(t) * \beta S I - \gamma I \\
\frac{dR}{dt} &= \gamma I
\end{aligned}
\tag{7}
$$

We wish to show that for every τ there exists an intervention that minimizes $I^{\max}$ and that such an optimal intervention must be identical except at a finite set of times $\{t_j \mid t_j \in (t_i, t_i+\tau)\}$ to one of the form:

$$
b_{\text{opt}}(t) = \begin{cases} \frac{\gamma}{\beta S}, & t \in [t_i, t_i + f\tau) \\ 0, & t \in [t_i + f\tau, t_i + \tau] \end{cases} \tag{8}
$$

For some values of $t_i \in \mathbb{R}$ and $f \in (0,1]$. We divide the proof into a series of lemmas.

Lemma 3: More fire, less fuel. *Let $b_x(t)$ and $b_y(t)$ be two interventions beginning at the same t_i and lasting until $t_f = t_i + \tau$. Denote the infectious and susceptible fractions for $b_x(t)$ and $b_y(t)$, respectively, by $I_x(t), S_x(t)$ and $I_y(t), S_y(t)$. If $I_x(t) \geq I_y(t)$ for all $t \in [t_i, t_f]$, and $I_x(t) > I_y(t)$ for some $t \in (t_i, t_f)$, then $S_x(t_f) < S_y(t_f)$.*

Proof. The difference $\Delta R_{xy}(t) = R_x(t) - R_y(t)$ obeys $\frac{d\Delta R_{xy}}{dt} = \gamma(I_x - I_y)$, which is non-negative for all $t \in [t_i, t_f]$. Therefore $\Delta R_{xy}(t)$ is non-decreasing during any time interval contained in $[t_i, t_f]$. Take $\bar{t} \in (t_i, t_f)$ such that $I_x(\bar{t}) - I_y(\bar{t}) > 0$. Because the intervention $b(t)$ allows only a finite number of discontinuities, the resulting $I(t)$ is continuous. And so there is an ε such that $I_x(t) - I_y(t) > 0$ for all $t \in [\bar{t} - \varepsilon, \bar{t} + \varepsilon]$. Therefore $\Delta R_{xy}(t)$ must be strictly increasing during $[\bar{t} - \varepsilon, \bar{t} + \varepsilon]$.

Because $\Delta R_{xy}(t)$ never decreases and sometimes increases during $[t_i, t_f]$, it follows that $\Delta R_{xy}(t_i) < \Delta R_{xy}(t_f)$, but $\Delta R_{xy}(t_i) = 0$ so $\Delta R_{xy}(t_f) > 0$ and $R_x(t_f) > R_y(t_f)$. But $I_x(t_f) \geq I_y(t_f)$ and $S = 1 - (I + R)$, so $S_x(t_f) < S_y(t_f)$. □

We now wish to prove that an epidemic left alone—without any intervention—will reach any given level of prevalence faster than it would have if any transmission-reducing intervention had taken place. In other words, an intervention always slows down the epidemic.

Lemma 4: Do nothing to burn faster and brighter. *Let $b_1(t) \equiv 1$ denote the null intervention and take t_{max} such that $I_1(t_{max}) = I_1^{max}$. Let $b_x(t)$ be an intervention that starts at time t_i, has* *duration τ, and satisfies $b_x(t) < 1$ for almost all $t \in [\bar{t}-\varepsilon, \bar{t}+\varepsilon]$ for some $\varepsilon > 0$ and some $\bar{t} < t_{max}$. Then $I_x(t) < I_1(t)$ for all $t \in [\bar{t}, t_{max}]$.*

Proof. Let the time of divergence of the interventions b_1 and b_x be given by $t_d = \inf\{t \in (t_i, t_{\max}) \mid b_x(t) < 1\}$. Note that this infimum must exist by completeness, because it is taken over a bounded and non-empty set. It is easy to see that $b_x(t) = 1$ and that $I_x(t) = I_1(t)$ for all $t < t_d$. Also, by the right continuity of b_x, there exists an interval $(t_d, t_d+\varepsilon)$ such that $I_1 > I_x$ and $b_1\beta S_1 I_1 > b_x \beta S_x I_x$. Suppose there exists a minimal $t_e \in (t_d, t_{\max})$ such that $I_x(t_e) = I_1(t_e)$. We will find a contradiction. Note that $I_x < I_1$ in (t_d, t_e). Because I_1 is monotonic and continuous in $(-\infty, t_{\max})$, we can invert I_1 and define $\hat{t} = I_1^{-1}(I_x(t))$ for all $t \in (-\infty, t_{\max})$.

We can see that $\frac{dI_1}{dt}(\hat{t})$ is the growth rate of the infectious class under the null intervention when $I_1 = I_x(t)$. It follows that $\frac{dI_1}{dt}(\hat{t}) < \frac{dI_x}{dt}(t)$ for some value of t, otherwise I_1 would always dominate I_x, and t_e would not exist. Let $t_{\inf} = \inf\{t \in (t_d, t_e) \mid \frac{dI_1}{dt}(\hat{t}) < \frac{dI_x}{dt}(t)\}$. This implies that $S_1(\hat{t}) \leq S_x(t)$ for some interval including $t_{\inf}$ that can be maximally extended to the left as $(t_{\min}, t_{\inf}]$. By continuity of S, $S_1(\hat{t}_{\min}) = S_x(t_{\min})$.

If $t_{\min} \neq t_d$, then $t_{\min} > \hat{t}_{\min}$, and $R_1(\hat{t}_{\min}) = R_x(t_{\min})$, but this is impossible because by construction $\frac{dI_1}{dt}(\hat{t}) > \frac{dI_x}{dt}(t)$ for all $t \in (t_d, t_{\min})$, and therefore $R_1(\hat{t}_{\min}) < R_x(t_{\min})$.

If $t_{\min} = t_d$, then $t_{\min} = \hat{t}_{\min}$. Also, $R_1(\hat{t}) > R_x(t)$ for all $t \in (t_d, t_{\inf})$. But that is impossible because $\frac{dR_1(\hat{t}_d)}{dt} = \frac{b(t_d)\beta SI - \gamma I}{\beta SI - \gamma I}\gamma I < \gamma I = \frac{dR_x(t_d)}{dt}$, and trivially $R_1(\hat{t}_d) = R_x(t_d)$. $\square$

Lemma 5: Wait, maintain, suppress. *Let b_x be any intervention, and $I_x^{max} = \max\{I_x(t)\}$. There exists an intervention b_y with same beginning and duration as b_x of the form*

$$b_y(t) = \begin{cases} 1 & t \in [t_i, t_i + g\tau) \\ \frac{\gamma}{\beta S}, & t \in [t_i + g\tau, t_i + f\tau) \\ 0, & t \in [t_i + f\tau, t_i + \tau] \end{cases} \tag{9}$$

For some $g < f$ and $g, f \in (0,1]$, such that $I_y^{max} \leq I_x^{max}$.

Proof. Define $I_x^\tau = \max\{I_x(t) \mid t \in [t_i, t_f]\}$. Following Lemma 4, take g such that $t_i + g\tau = I_1^{-1}(I_x^\tau)$. Now take f such that $(1-f)\tau = \frac{1}{\gamma}\log\frac{I_x^\tau}{I_x(t_f)}$. From Lemma 4 it is clear that $I_y(t) \geq I_x(t)$ for $t \in [t_i\tau, t_i + g\tau)$. By construction $I_y(t) = I_x^\tau$ for $t \in [t_i + g\tau, t_i + f\tau)$, and therefore the inequality remains true. Finally, for $t \in [t_i + f\tau, t_i + \tau]$, because $I_y(t)$ decays faster than $I_x(t)$, then if $I_y(t) < I_x(t)$ at any time, then $I_y(t_f) < I_x(t_f)$, but by construction $I_y(t_f) = I_x(t_f)$. By Lemma 3, $S_y(t_f) < S_x(t_f)$, and by Lemma 2, $I_y^{\max} < I_x^{\max}$.

It is possible however that this proposed b_y is not a viable intervention if $S_y(t_y^{\text{crit}}) = S_{\text{crit}}$ for some $t_y^{\text{crit}} \in [t_i + g\tau, t_i + f\tau)$, as this would require b_y to assume values larger than 1. If that is the case we can define $b_{\bar{y}}$, by taking f such that $t_i + f\tau = t_y^{\text{crit}}$. Note that because by the end of the intervention $b_{\bar{y}}$ the infectious class will monotonically decrease because $S_{\bar{y}} \leq S_{\text{crit}}$, this implies that $I_{\bar{y}}^{\max} = I_{\bar{y}}^{\tau} = I_x^{\tau} \leq I_x^{\max}$. □

Theorem 1: Maintain, then suppress. *An intervention $b_{opt}(t)$ such that $I_{opt}^{max} \leq I_x^{max}$ for any intervention $b_x(t)$ must take the form*

$$b_{opt}(t) = \begin{cases} \frac{\gamma}{\beta S}, & t \in [t_i, t_i + f\tau) \\ 0, & t \in [t_i + f\tau, t_i + \tau] \end{cases} \tag{10}$$

for some value of $t_i \in \mathbb{R}$ and $f \in (0,1]$.

Proof. If $b_x(t)$ is an optimal intervention, then Lemma 5 ensures that $b_x(t)$ is of the form given by equation 9. Now consider the strategy $b_{\text{opt}}(t)$ of the form given by equation 10 (main text equation 2) with $t_i^{\text{opt}} = t_i^x + g\tau$ and $f^{\text{opt}} = f^x - g$. This new intervention functions exactly like b_x for the entire duration of b_x, but then b_{opt} is held at 0 for a little longer further reducing I_{opt}. This means that $S_{\text{opt}}(t_f^{\text{opt}}) = S_x(t_f^x)$ and $I_{\text{opt}}(t_f^{\text{opt}}) \leq I_x(t_f^x)$ and therefore by Lemma 1, $I_{\text{opt}}^{\max} \leq I_x^{\max}$. □

Remark 2: The strategy is set at the start. *Because during an intervention b_{opt} the susceptible class is depleted at a constant rate for all $t \in [t_i, t_i + f\tau)$, b_{opt} can be written as*

$$b_{opt}(t) = \begin{cases} \frac{\gamma}{\beta\left(S_{t_i} - I_{t_i}\gamma(t - t_i)\right)}, & t \in [t_i, t_i + f\tau) \\ 0, & t \in [t_i + f\tau, t_i + \tau] \end{cases} \tag{11}$$

For $S_{t_i} = S(t_i)$ and $I_{t_i} = I(t_i)$.

1.4 Results of an optimal intervention

It follows that, given an optimal intervention b of duration τ and depletion fraction f begun at time t_i and ending at $t_f = t_i + \tau$:

$$S(t_f) = S(t_i) - \gamma \tau f I(t_i) \tag{12}$$

$$I(t_f) = I(t_i) \exp\left[-\gamma \tau (1-f)\right] \tag{13}$$

In an SIR system without intervention that begins with a wholly susceptible population, we have the relation:

$$I(S) = 1 - S + \frac{1}{\mathscr{R}_0} \log(S) \tag{14}$$

And so:

$$I(t_i) = 1 - S(t_i) + \frac{1}{\mathscr{R}_0} \log(S(t_i)) \tag{15}$$

And we can likewise write $S(t_f)$ and $I(t_f)$ in terms of $S(t_i)$:

$$S(t_f) = S(t_i) - \gamma \tau f \left(1 - S(t_i) + \frac{1}{\mathscr{R}_0} \log(S(t_i))\right) \tag{16}$$

$$I(t_f) = \left(1 - S(t_i) + \frac{1}{\mathscr{R}_0} \log(S(t_i))\right) \exp\left[-\gamma \tau (1-f)\right] \tag{17}$$

This in turn allows us to express $I^{\max}(t_f)$ purely in terms of $S(t_i)$ and the parameters, though the expression is long:

$$
\begin{aligned}
I^{\max}(t_f) = &\left(1 - S(t_i) + \frac{1}{\mathscr{R}_0}\log(S(t_i))\right)\exp\left[-\gamma\tau(1-f)\right] \\
&+ S(t_i) - \gamma\tau f\left(1 - S(t_i) + \frac{1}{\mathscr{R}_0}\log(S(t_i))\right) \\
&- \frac{1}{\mathscr{R}_0}\log\left(S(t_i) - \gamma\tau f\left(1 - S(t_i) + \frac{1}{\mathscr{R}_0}\log(S(t_i))\right)\right) \\
&- \frac{1}{\mathscr{R}_0} + \frac{1}{\mathscr{R}_0}\log\left(\frac{1}{\mathscr{R}_0}\right)
\end{aligned} \tag{18}
$$

Remark 3: Continuity of $I(t_i), I^{\mathbf{max}}(t_f)$. *Notice that both $I(t_i)$ and $I^{max}(t_f)$ are continuous functions of $S(t_i)$, since they are both linear combinations of continuous functions of $S(t_i)$ and since $S(t_i)$ is continuous in t_i, they are continuous functions of t_i.*

Lemma 6: Don't be late. *Let t_p be the infimum of times such that $I(t) = I^{max}$, and let t_i be the start of an optimal intervention. Then $t_p \geq t_i$. That is, I^{max} cannot occur before the intervention begins for an optimal intervention.*

Proof. Since $b(t) = 1$ for $t < t_i$ $t_p < t_i$ necessarily implies that $S(t_i) < S_{\text{crit}}$, that is, the epidemic is already declining when the intervention begins and will never grow again, regardless of the intervention approach (since $b(t) \leq 1$, we cannot force a declining epidemic to grow). That in turn implies $I^{\max} = I^{\text{peak}}$, which we can with certainty improve upon. So $I^{\max}$ must occur during or after the intervention. □

Corollary 1: Start with fuel. *It is immediate that $S(t_i) > S_{crit}$.*

Corollary 2: Peak early. *Since $I(t) \leq I(t_i)$ for $t \in [t_i, t_i + \tau]$ during an optimal intervention, if I^{max} occurs during the intervention, $I^{max} = I(t_i)$*

Theorem 2: Twin Peaks. *Let $b_x(t)$ be an optimal intervention. Then $I_x(t) \leq I_x^{max}$ for all $t \in (-\infty, t_f]$, with equality for $t \in [t_i, t_i + \tau f]$, and furthermore $I_x(t_p) = I_x^{max}$ for some $t_p \in [t_f, \infty)$ with $t_p = t_f$ only if $f = 1$.*

That is, if $0 < f < 1$, there will be a plateau during the intervention followed by a peak of equal height that occurs strictly after the intervention finishes. If $f = 0$, there will be two peaks of equal height, one at the start of the intervention and one strictly after it finishes, and if $f = 1$ there will be a plateau during the intervention with no subsequent peak.

Proof. Let b_x be an optimal intervention of the form given by equation 10. By Lemma 6, $I_x^{\max}$ must occur during or after the intervention. First let us assume that the $I_x^{\max}$ occurs during the intervention and is never again attained after the intervention. If $f = 1$, this implies that the whole intervention is a plateau and no further peaks occur. For $f < 1$ we can build a new intervention b_y of the same form as b_x but that starts at $t_i - \varepsilon$ rather than at t_i. From the continuity of $I^{\max}$ in t_i for any intervention, it follows that for some ε small enough, $I_y^{\max}(t_f)$ (the post-intervention maximum of $I(t)$) must still be smaller than $I_y^{\tau} = I(t_i - \varepsilon)$ (the maximum during the intervention), but because $I(t_i - \varepsilon) < I(t_i)$, corollary 2 implies that our new b_y outperforms b_x, which contradicts the optimality of b_x.

Now let us assume that $I_x^{\max}$ occurs after the intervention and is larger than any value of I_x during the intervention. If $f > 0$, we can once again build a new intervention function b_y

$$b_y(t) = \begin{cases} 1 & t \in [t_i, t_i + \varepsilon) \\ \frac{\gamma}{\beta S}, & t \in [t_i + \varepsilon, t_i + f_y \tau) \\ 0, & t \in [t_i + f_y \tau, t_i + \tau] \end{cases} \tag{19}$$

With f_y chosen such that $(1 - f_y)\tau = \frac{1}{\gamma} \log \frac{I_x(t_i+\varepsilon)}{I_x(t_f)}$. For sufficiently small ε, $I_y(t_i + \varepsilon) < I_x^{\max}$. Also, following Lemma 5, $I_y^{\max} < I_x^{\max}$ and therefore b_y outperforms b_x, which once again contradicts the optimality of b_x.

Finally, if $f = 0$, we build yet another b_y of the same form as b_x but that starts at $t_i + \varepsilon$ rather than at t_i. Because intervention b_y starts out with a smaller susceptible fraction than intervention b_x, and an increase in the infected fraction that is smaller than the susceptible fraction decrease, it follows from equation 5 that $I_y^{\max} < I_x^{\max}$ and therefore b_y outperforms b_x, which once again contradicts the optimality of b_x. □

Corollary 3: The longer the intervention, the earlier it starts. *If $t_i^{opt}(\tau)$ is the starting time of the optimal intervention with duration τ, let $S_i^{opt}(\tau) = S(t_i^{opt}(\tau))$. Then $S_i^{opt}(\tau)$ is a non-decreasing function of τ.*

Proof. Trivially a longer optimal intervention must outperform a shorter optimal intervention. But by Theorem 2 $I_i^{\text{opt}} = I^{\max}$ for any optimal intervention, which implies that $I_i^{\text{opt}}(\tau)$ is a non-increasing function of τ and $S_i^{\text{opt}}(\tau)$ is a non-decreasing function of τ. □

Corollary 4: If you don't have much gunpowder, don't shoot until you see the whites of their eyes. *Given the optimal intervention of duration* τ, b^{τ}_{opt}, *denote the initial time of that intervention by* t^{τ}_i. *As* $\tau \to 0$, $t^{\tau}_i \to t_{crit}$.

Proof. From equation 18 we can see that $I^{\max}_{\text{opt}}$ is a continuous function of τ, and that as $\tau \to 0$, the effect of intervention defined as $I^{\max}_1 - I^{\max}_{\text{opt}} \to 0$. We know from Theorem 2 that $I_{\text{opt}}(t_i) = I^{\max}_{\text{opt}}$, and therefore, as $\tau \to 0$, $I_{\text{opt}}(t_i) \to I^{max}_1$, which implies $t^{\tau}_i \to t_{\text{crit}}$. □

Corollary 5: No need to burn all the fuel. *After an optimal intervention,* $S(t_f) \geq S_{crit}$, *with equality if and only if* $f = 1$.

Proof. By Theorem 2, $I(t_i) = I^{\max}(t_f) = I^{\max}$. If $f < 1$, we must have $I(t_f) < I(t_i) = I^{\max}$, so in order for the epidemic to reach $I^{\max}(t_f) = I^{\max}$, we must have $S(t_f) > S_{\text{crit}}$. If $f = 1$, then $I(t_f) = I(t_i) = I^{\max}$, so then must have $S(t_f) = S_{\text{crit}}$, otherwise the epidemic would grow to a peak above $I(t_i) = I(t_f)$, which would be a contradiction of Theorem 2. □

Corollary 6: Putting out existing fire can only do so much. *Consider a full suppression intervention of duration* τ *defined by* $b_0(t) = 0$ *for all* $t \in [t_i, t_i + \tau]$. *For every* τ *there is a* t_i *that minimizes* I^{max}_0. *Consider these optimized full suppression interventions. Then, as* $\tau \to \infty$, *the maximum infectious prevalence* $I^{max}_0 \to \frac{1}{2} + \frac{1}{2\mathscr{R}_0}\left(\log\left(\frac{1}{\mathscr{R}_0}\right) - 1\right)$. *In other words, full suppression interventions have a limit in how much they can reduce* I^{max}.

Proof. From the proof of Theorem 2 for $f = 0$, it follows that full suppression interventions have an optimal start time t_i, and that $I_0(t_i) = I^{\max}_0$. Also, because no new infections occur during a full suppression intervention, $S_0(t_i) = S_0(t_f)$ so substituting into equation 5

$$I^{\max}_0(t_i) = I_0(t_f) + S_0(t_i) - \frac{1}{\mathscr{R}_0}\log\Big(S_0(t_i)\Big) - \frac{1}{\mathscr{R}_0} + \frac{1}{\mathscr{R}_0}\log\left(\frac{1}{\mathscr{R}_0}\right) \tag{20}$$

We the further use equation 14 to substitute $S_0(t_i)$ and take $\tau \to \infty$ such that $I_0(t_f) \to 0$ which finally yields

$$I^{\max}_0 = \frac{1}{2} + \frac{1}{2\mathscr{R}_0}\left(\log\left(\frac{1}{\mathscr{R}_0}\right) - 1\right) \tag{21}$$

□

Corollary 7: But when in doubt, put out the fire. *Let b_0 be the optimized full suppression intervention with duration τ and starting time t_i^0. For a full suppression intervention $b_{\hat{0}}$ that also has duration τ but that has a starting time $t_i^{\hat{0}} \in [t_i^0, t_{crit}]$, the infected fraction peak is $I_{\hat{0}}^{max} = I_{\hat{0}}(t_i^{\hat{0}})$, moreover no intervention starting at that same time can attain a lower peak.*

Proof. It suffices to show that delaying a full suppression intervention by an infinitesimal ε diminishes the secondary peak. From equation 25 with $f = 0$, it is clear that

$$
\begin{aligned}
\frac{\partial}{\partial t_i} I^{\max}(t_f) &= -\frac{S'(t_i)}{\mathscr{R}_0 S_i} + e^{-\gamma\tau} I'(t_i) + S'(t_i) \\
\frac{\partial}{\partial t_i} I^{\max}(t_f) &= \gamma I_i + e^{-\gamma\tau}(\beta I_i S_i - \gamma I_i) - \beta I_i S_i \\
\frac{\partial}{\partial t_i} I^{\max}(t_f) &= (e^{-\gamma\tau} - 1)(I'(t_i))
\end{aligned}
\tag{22}
$$

But $I'(t_i) > 0$ for $t_i < t_{\text{crit}}$ and $(e^{-\gamma\tau} - 1) < 0$ which means that delaying the full suppression decreases the post-intervention peak. Trivially, no intervention can attain a global $I^{\max}$ value lower than its initial prevalence $I(t_i)$, which concludes the proof. $\square$

Theorem 3: Twin peaks for fixed control interventions. *Let b_x be an optimized fixed control intervention of duration τ that starts at some time t_i and has strictness σ. That is, b_x outperforms any other fixed control intervention of duration τ regardless of their t_i or σ. Then $I_x = I_x^{max}$ at exactly two time points, one during the intervention ($t \in [t_i, t_i + \tau]$) and one after the intervention ($t \in (t_i + \tau, \infty)$).*

Proof. Because during the intervention and after the intervention, the time course is that of an SIR (albeit with a modified $\mathscr{R}_0$ during the intervention), it is clear that there can be at most one local I maximum during the intervention and at most one after the intervention.

First let us assume that the infectious peak during the intervention is higher than any peak after the intervention. By continuity it is possible to construct a new fixed control intervention that starts a little earlier and is slightly stricter such that the peak during the intervention is reduced but the peak after the intervention is still below the intervention peak. This new intervention outperforms b_x which contradicts the optimality of b_x.

Now let us assume that the infectious peak during the intervention is lower than any peak after the intervention. We will show that a small change in the t_i of the intervention will lower the post-intervention peak. It suffices to show that $\frac{\partial}{\partial t_i} I^{\max} \neq 0$. For a fixed control intervention we have

$$\frac{\partial}{\partial t_i} I^{\max} = \frac{(1-\sigma)}{\sigma \mathscr{R}_0 S_x(t_f)} * \frac{\partial}{\partial t_i} S_x(t_f) \tag{23}$$

Which when set to 0 implies $\frac{\partial}{\partial t_i} S_x(t_f) = 0$. This in turn can be substituted into the equation that relates I to S under a fixed control intervention

$$I(t_f) = -S(t_f) + \frac{1}{\sigma \mathscr{R}_0} \log\big(S(t_f)\big) + I(t_i) + S(t_i) - \frac{1}{\mathscr{R}_0} \log(S(t_i)) \tag{24}$$

Which leads to $\frac{\partial}{\partial t_i} I_x(t_f) = 0$ and finally $\frac{\partial}{\partial t_i} R_x(t_f) = 0$ but this is impossible because the fraction of recovered individuals by the end of a fixed control intervention must decrease as the intervention starts earlier. This completes the proof.

□

1.5 First-order conditions for t_i, f

Applying equations 12, 13, the partial derivatives of $I^{\max}(t_f)$ with respect to f and t_i are given by:

$$\begin{aligned} \frac{\partial}{\partial f} I^{\max}(t_f) &= \frac{(\gamma\tau I_i)}{\mathscr{R}_0 \Big(S_i - \gamma\tau f I_i\Big)} + \gamma\tau I_i e^{-(1-f)\gamma\tau} - \gamma\tau I_i \\ \frac{\partial}{\partial t_i} I^{\max}(t_f) &= -\frac{S'(t_i) - f\gamma\tau I'(t_i)}{\mathscr{R}_0 \Big(S_i - f\gamma\tau I_i\Big)} + e^{-(1-f)\gamma\tau} I'(t_i) - f\gamma\tau I'(t_i) + S'(t_i) \end{aligned} \tag{25}$$

Substituting the values of $\frac{\mathrm{d}S}{\mathrm{d}t}, \frac{\mathrm{d}I}{\mathrm{d}t}$:

$$\begin{aligned} \frac{\partial}{\partial t_i} I^{\max}(t_f) = &-\frac{(-\beta S_i I_i) - f\gamma\tau(\beta S_i I_i - \gamma I_i)}{\mathscr{R}_0 (S_i - f\gamma\tau I_i)} \\ &+ e^{-(1-f)\gamma\tau}(\beta S_i I_i - \gamma I_i) \\ &- f\gamma\tau(\beta S_i I_i - \gamma I_i) - \beta S_i I_i \end{aligned} \tag{26}$$

Setting equal to zero and simplifying:

$$\frac{\partial}{\partial f} I^{\max}(t_f) = \frac{1}{\mathscr{R}_0 \Big(S_i - f\gamma\tau I_i \Big)} + \gamma\tau e^{-(1-f)\gamma\tau} - 1 = 0 \tag{27}$$

$$\begin{aligned} \frac{\partial}{\partial t_i} I^{\max}(t_f) = -\frac{(-\beta S_i) - f\gamma\tau\,(\beta S_i - \gamma)}{\mathscr{R}_0\,(S_i - f\gamma\tau I_i)} + e^{-(1-f)\gamma\tau}\,(\beta S_i - \gamma) \\ - f\gamma\tau\,(\beta S_i - \gamma) - \beta S_i = 0 \end{aligned} \tag{28}$$

While these first-order conditions do not yield a simple closed form for the optimal f and t_i, we use them in the proofs above and below to establish properties of the optimal strategy.

Lemma 7: If you have lots of gunpowder, shoot early. *For an optimal intervention b_{opt}^{τ} acting on a SIR with full susceptibility as its initial condition, as $\tau \to \infty$, $S_{opt}^{\tau}(t_i) \to 1$; that is, we intervene almost immediately, when almost all the population is initially susceptible and a long intervention is possible.*

Proof. Define an auxiliary intervention b_x^{τ} with $\tau > \frac{1-S_{crit}}{\gamma I^{max}}$ such that

$$b_x^{\tau}(t) = \begin{cases} \frac{\gamma}{\beta S}, & \text{if } t > t_x \text{ and } S > S_{crit} \\ 1, & \text{elsewhere} \end{cases} \tag{29}$$

with $t_x = I_1^{-1}(\frac{1-S_{\text{crit}}}{\gamma\tau})$. It is clear that such an intervention has a duration of τ or less, and that by the end of such an intervention $S_x \leq S_{\text{crit}}$. Therefore $I_x^{\max} = I_x(t_x) = \frac{1-S_{\text{crit}}}{\gamma\tau}$, but by definition $I_{\text{opt}}^{\max} \leq I_x^{\max}$. Combining with Corollary 2, $I_{\text{opt}}^{\tau}(t_i) \leq \frac{1-S_{\text{crit}}}{\gamma\tau}$ so as $\tau \to \infty$ both $I_{\text{opt}}^{\tau}(t_i) \to 0$ and $S_{\text{opt}}^{\tau}(t_i) \to 1$. □

Theorem 4: Sometimes maintain, always suppress. *For an SIR with full susceptibility as its initial condition, it is the case that any optimal intervention with positive duration as defined by equation 10 has $f < 1$. In other words, the optimal intervention for an emerging pathogen ($S(0) \to 1$, $I(0) \to 0$) always has a total suppression phase. Also, for any $\mathscr{R}_0 \in (1,\infty)$ there is a τ_{crit} such that the optimal intervention is always a full suppression intervention ($f = 0$) if $\tau < \tau_{crit}$. Moreover, $\lim_{\mathscr{R}_0 \to 1} \tau_{crit} = \infty$ and $\lim_{\mathscr{R}_0 \to \infty} \tau_{crit} = 0$.*

Proof. Suppose $f = 1$. From equation 27,

$$\frac{1}{\mathscr{R}_0\left(S_i - \gamma\tau I_i\right)} + \gamma\tau - 1 = 0 \tag{30}$$

But following corollary 5, $S_i - \gamma\tau I_i = S_{\text{crit}}$, which substituting in the previous equation implies that $\gamma\tau = 0$, which is impossible. Therefore f cannot be 1.

Now let us investigate if $f = 0$ can be a local minimum. Once again, from the first order condition expressed in equation 27 we require that

$$\begin{aligned} S_i &= \frac{1}{\mathscr{R}_0 x} \\ x &= 1 - e^{-\gamma\tau} \end{aligned} \tag{31}$$

The second order condition for an extremum to be a minimum is given by

$$\frac{\partial^2}{\partial f^2} I^{\max} = I_i(\gamma\tau)^2 e^{-\gamma\tau(1-f)} + \frac{(\gamma\tau I_i)^2}{\mathscr{R}_0(S_i - \gamma\tau f I_i)^2} > 0 \tag{32}$$

Note that the condition is always satisfied, and therefore any extremum is always a minimum. This also ensures that any minimum must be unique and that $I^{\max}$ grows monotonically as f moves away from f^{opt}.

From Theorem 2 we have $I_i = I^{\max}$. Substituting this and equations 31 and 14 into equation 18, we get an implicit function for x (and therefore for τ) such that $f = 0$ is a minimum:

$$0 = x(\log(x) + x\log(\mathscr{R}_0 x) - \mathscr{R}_0 x) + 1 \tag{33}$$

We now show that $\mathscr{R}_0 \in (1,\infty)$ implies $x \in (0,1)$ which in turn implies a τ_{crit} that satisfies the relation.

From equation 33 it is easy to see that as $\mathscr{R}_0 \to 1$, then $x \to 1$, and as $\mathscr{R}_0 \to \infty$, then $x \to 0$. So showing that $x(\mathscr{R}_0)$ is a continuous, decreasing function of $\mathscr{R}_0$ on $\mathscr{R}_0 \in (1,\infty)$ will establish the existence of τ_{crit}.

By implicit differentiation we have that

$$\frac{\partial x}{\partial \mathscr{R}_0} = \frac{x^3\left(\frac{\mathscr{R}_0 - 1}{\mathscr{R}_0}\right)}{x + x^2 - x\log(x) - 2} \tag{34}$$

For positive x the numerator is positive and the denominator is always negative if $x \in (0,1)$. Therefore, for $\mathscr{R}_0 \in (1,\infty)$ there exists $x \in (0,1)$ that satisfies equation 33.

To complete the proof, it suffices to show that f^{opt} is an increasing function of τ when $f = 0$, as this implies that $f^{\text{opt}}(\tau_{crit}) = 0$ and cannot go above 0 for $\tau < \tau_{\text{crit}}$. We start by letting f^{opt}, S_i^{opt}, and I_i^{opt} be functions of τ, applying equation 27, implicitly differentiating with respect to τ, and setting $f = 0$. After a bit of algebra we obtain

$$\left[\tau e^{-\gamma\tau} + \frac{\tau I_i^{\text{opt}}}{\mathscr{R}_0 (S_i^{\text{opt}})^2}\right] * \frac{\partial}{\partial \tau} f^{\text{opt}} = e^{-\gamma\tau} + \frac{1}{\mathscr{R}_0 \gamma (S_i^{\text{opt}})^2} * \frac{\partial}{\partial \tau} S_i^{\text{opt}} \tag{35}$$

Note that $\frac{\partial}{\partial \tau} f^{\text{opt}}$ is being multiplied by a strictly positive quantity on the left-hand side of the equation. On the right-hand side of the equation there is a positive quantity ($e^{-\gamma\tau}$) added to $\frac{\partial}{\partial \tau} S_i^{\text{opt}}$ multiplied by another positive quantity ($\frac{1}{\mathscr{R}_0 (S_i^{\text{opt}})^2}$), but by corollary 3, $\frac{\partial}{\partial \tau} S_i^{\text{opt}} > 0$. It follows that $\frac{\partial}{\partial \tau} f^{\text{opt}} > 0$ when $f^{\text{opt}} = 0$ which completes the proof.

□

1.6 A general classification of interventions

From equation 5 it is clear that there are two fundamental methods of reducing the peak of an epidemic: depleting the infected fraction and depleting the susceptible fraction. We have shown that different interventions achieve peak reduction with different combinations of those methods. Our optimal intervention, for example, is characterized by a pure susceptible depletion phase followed by a pure infected depletion phase. Full suppression interventions, in contrast, operate solely by depleting the infected fraction. We observe that interventions can be classified in terms of how much they rely on depleting the susceptible fraction versus depleting the infected fraction.

The effect of an intervention can be understood as the infectious peak if no intervention were to take place minus the infectious peak given the intervention. In a more formal notation, an intervention b_x has an effect $I^{\max} - I_x^{\max} = I^{\max}(I_x(t_i), S_x(t_i)) - I^{\max}(I_x(t_f), S_x(t_f)) = \Delta_x(t_f)$. By applying equation 5 we obtain

$$\begin{aligned}\Delta_x(t_f) &= -\Big[I_x(t_f) - I_x(t_i)\Big] - \Big[G\Big(S_x(t_f)\Big) - G\Big(S_x(t_i)\Big)\Big] \\ G(S) &= S - \frac{1}{\mathscr{R}_0}\log(S)\end{aligned} \tag{36}$$

Then if $-\Big[I_x(t_f) - I_x(t_i)\Big] > -\Big[G\Big(S_x(t_f)\Big) - G\Big(S_x(t_i)\Big)\Big]$ we can say that the intervention b_x overall relies more on infected depletion, whereas if the opposite is true we can say that it relies more on susceptible depletion. Moreover, by applying the fundamental theorem of calculus, we obtain

$$\int_{t_i}^{t_f} \Delta_x'(t)dt = -\int_{t_i}^{t_f} I_x'(t) + S_x'(t)\Big(1 - \frac{1}{\mathscr{R}_0 S_x(t)}\Big)dt \tag{37}$$

Which allows us to look at a certain time $t \in [t_i, t_f]$ and say that if

$$I_x'(t) < S_x'(t)\Big(1 - \frac{1}{\mathscr{R}_0 S_x(t)}\Big) \tag{38}$$

Then at that moment t, the intervention b_x acts more by depleting the infected fraction than by depleting the susceptible fraction. The condition can be simplified to

$$b_x(t)\Big(2\mathscr{R}_0 S_x(t) - 1\Big) < 1 \tag{39}$$

Supplementary References

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